Embeddability on functions: Order and Chaos

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Ordering functions

- One way to understand objects consists of ordering them.
- For sets $A, B \subseteq \omega^\omega$, continuous reducibility (Wadge qo):

$$A \leqslant_W B \quad \longleftrightarrow \quad \exists f : \omega^\omega \to \omega^\omega \text{ continuous such that } \\ \forall x \in \omega^\omega (x \in A \leftrightarrow f(x) \in B).$$

- For equivalence relations E, F on ω^ω, Borel reducibility:

$$E \leqslant_B F \quad \longleftrightarrow \quad \exists f : \omega^\omega \to \omega^\omega \text{ Borel such that } \\ \forall x, y \in \omega^\omega (x \, E \, y \leftrightarrow f(x) \, F \, f(y)).$$

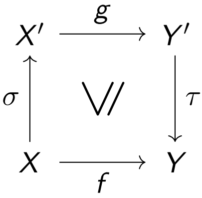
- What about functions?

All spaces considered are Polish zero-dimensional spaces, denoted by variables X, Y,...

Continuous reducibility on functions

Definition (Hertling-Weihrauch, Carroy)

Say that $f : X \to Y$ *reduces to* $g : X' \to Y'$ if there are $\sigma : X \to X'$ continuous and $\tau : \mathrm{im}(g \circ \sigma) \to Y$ continuous such that $f = \tau \circ g \circ \sigma$.



Theorem (Carroy, 2012)

Continuous reducibility is a well-order on continuous functions with compact domains.

A *quasi-order* (qo) is a reflexive and transitive binary relation.

A *well-quasi-order* (wqo) is a well-founded qo with no infinite antichain.

Conjecture (Carroy)

Continuous reducibility is a wqo on continuous functions.

Question: Is there any infinite antichain for continuous reducibility among Baire class 1 functions?

Topological embeddability on functions

Definition

Say that $f : X \to Y$ *embeds into* $g : X' \to Y'$ if there are embeddings $\sigma : X \to X'$ and $\tau : \operatorname{im} f \to Y'$ such that $\tau \circ f = g \circ \sigma$.

$$\begin{array}{ccc} X' & \xrightarrow{\;g\;} & Y' \\ \sigma \big\uparrow & \sqcup\!\!| & \big\uparrow \tau \\ X & \xrightarrow[\;f\;]{} & Y \end{array}$$

- Embeddability is finer than reducibility: $f \sqsubseteq g \;\to\; f \leqslant q$.
- The projection $p : \omega^\omega \times \omega^\omega \to \omega^\omega$ is a maximum for continuous functions: $f : X \to Y$ is continuous iff $f \sqsubseteq p$.
- The two discontinuous functions

$$\begin{aligned} d_0 : \omega + 1 &\longrightarrow 2 \\ \omega &\longmapsto 0 \\ n &\longmapsto 1 \end{aligned} \qquad\qquad \begin{aligned} d_1 : \omega + 1 &\longrightarrow \omega \\ \omega &\longmapsto 0 \\ n &\longmapsto n+1 \end{aligned}$$

form a 2-element *basis* for discontinuous functions: $f : X \to Y$ is discontinuous iff $d_0 \sqsubseteq f$ or $d_1 \sqsubseteq f$.

Topological embeddability on functions, continued.

Theorem

The following classes admits a minimum under embeddability:

1. *(Solecki, 98') The class of Baire class* 1 *functions that are not* σ*-continuous.*
2. *(Zapletal, 04') The Borel functions that are not* σ*-continuous.*
3. *(Carroy-Miller, 17') The class of Baire class* 1 *functions that are not* F_σ*-to-one.*

Theorem (Carroy-Miller, 17')

The following classes admits a finite basis under embeddability:

1. *The Borel functions that are not in the first Baire class.*
2. *The Borel functions that are not* σ*-continuous with closed witnesses.*

Conjecture, $\alpha > 1$:

The Borel functions that are not Baire class α admit a finite basis.

Our main theorem: Order and Chaos

For X compact, $C(X, Y)$ denotes the space of continuous functions $X \to Y$ with the topology of uniform convergence.

Proposition (Carroy, P., Vidnyánszky)

If X, Y are Polish and X is compact, then embeddability is an analytic quasi-order on $C(X, Y)$.

An analytic qo Q on a Polish space Z is *analytic complete* if it Borel reduces every analytic qo on any Polish space.

Theorem (Carroy, P., Vidnyánszky)

Suppose that X, Y are Polish zero-dimensional and X is compact. Then exactly one of the following holds:

1. *embeddability on $C(X, Y)$ is an analytic complete quasi-order,*
2. *embeddability on $C(X, Y)$ is a well-quasi-order.*

Moreover 1 holds exactly when X has infinitely many non-isolated points and Y is not discrete. For instance for $C(2^\omega, 2^\omega)$.

Chaos

Let $\mathbb{G}$ denote the Polish space of (simple) graphs with vertex set $\mathbb{N}$. For $G, H \in \mathbb{G}$ let

$G \leqslant_i H \quad \longleftrightarrow \quad$ there is an injective homomorphism from G to H.

Theorem (Louveau-Rosendal)

The qo $\leqslant_i$ on $\mathbb{G}$ is an analytic complete quasi-order.

Theorem (Carroy, P., Vidnyánszky)

There is a continuous function $\mathbb{G} \to C(\omega^2+1, \omega+1)$, $G \mapsto f^G$ that reduces $\leqslant_i$ to $\sqsubseteq$:

$$G \leqslant_i H \quad \longleftrightarrow \quad f^G \sqsubseteq f^H.$$

So embeddability on $C(\omega^2+1, \omega+1)$ is an analytic complete qo.

Order

We use the *better-quasi-orders* (bqo) introduced by Nash-Williams.
We have: well-order $\rightarrow$ better-quasi-order $\rightarrow$ well-quasi-order.
Let $\mathbb{Q}$ be the space of rationals, $(P, \leqslant_P)$ a quasi-order.
Let $P^{\mathbb{Q}}$ be the set of maps $l : \mathbb{Q} \rightarrow P$ quasi-ordered by

$$l_0 \leqslant l_1 \quad \longleftrightarrow \quad \text{there is a topological embedding } \tau : \mathbb{Q} \rightarrow \mathbb{Q} \text{ such that } l_0(q) \leqslant_P l_1(\tau(q)) \text{ for all } q \in \mathbb{Q}.$$

Theorem (van Engelen-Miller-Steel)

If P is bqo, then $P^{\mathbb{Q}}$ is bqo.

Theorem (van Engelen-Miller-Steel, Carroy)

The Polish 0-dimensional spaces with embeddability are bqo.

Proposition (Carroy, P., Vidnyánszky)

The locally constant maps are bqo under embeddability.

On Baire class α functions

- Recall that the projection $p : \omega^\omega \times \omega^\omega \to \omega^\omega$ is a maximum for continuous functions for embeddability.
- So in particular, $C(\omega^\omega, \omega^\omega)$ admits a maximum for embeddability.

In contrast,

Theorem (Carroy, P., Vidnyánszky)

Let X be uncountable and $|Y| \geqslant 2$. For every $\alpha \geqslant 1$, there exists no maximal Baire class α function : $X \to Y$ for embeddability.

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