A Wadge hierarchy for second countable spaces

Reducibility via relatively continuous relations

Yann Pequignot

Université de Lausanne

Université Paris Diderot

Séminaire général de logique
Paris — 7 avril, 2014

Classify Definable subsets of topological spaces

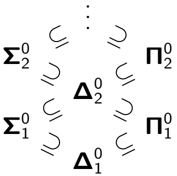
X a 2^{nd} countable T_0 topological space:

- A countable basis of open sets,
- Two points which have same neighbourhoods are equal.

Borel sets are naturally classified according to their definition

$$\begin{aligned}
\mathbf{\Sigma}^0_1(X) &= \{O \subseteq X \mid X \text{ is open}\},\\
\mathbf{\Sigma}^0_2(X) &= \left\{\bigcup_{i\in\omega} B_i \;\middle|\; B_i \text{ Boolean combination of open sets}\right\},\\
\mathbf{\Pi}^0_\alpha(X) &= \left\{A^{\complement} \;\middle|\; A \in \mathbf{\Sigma}^0_\alpha(X)\right\},\\
\mathbf{\Sigma}^0_\alpha(X) &= \left\{\bigcup_{i\in\omega} P_i \;\middle|\; P_i \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)\right\}, \quad \text{for } \alpha > 2.
\end{aligned}$$

All these classes are *pointclasses*, i.e. if $f : X \to X$ is continuous and $A \in \Gamma$ then $f^{-1}(A) \in \Gamma$.



Wadge reducibility

Let X be a topological space, $A, B \subseteq X$.
A is Wadge reducible to B, or $A \leq_W B$,
if there is a continuous function $f : X \to X$
that reduces A to B, i.e. such that
$f^{-1}(B) = A$ or equivalently

$$\forall x \in X \quad (x \in A \longleftrightarrow f(x) \in B).$$

Bill Wadge

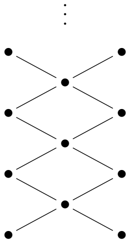
The idea is that the continuous function f reduces
the *membership question* for A to the membership question for B.

- The identity on X is continous, and
- continuous functions compose, so

Wadge reducibility is a *quasiorder* on subsets of X that refines the Baire hierarchy. Pointclasses are simply initial segments of $\leq_W$.

Hierarchies?

On Polish 0-dimensional spaces,
Wagde reducibility $\leq_W$ yields a
nice and well understood hierarchy,
by results of Wadge, Martin, Monk,
Louveau, Duparc and others.



Thanks to a game theoretic
formulation of the reduction.

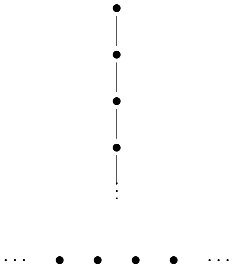
"The Wadge hierarchy is the ultimate analysis of $\mathcal{P}(\omega^\omega)$ in terms of topological complexity..."
Andretta and Louveau.

Applications to Computer Science:
Duparc, Finkel, Ressayre...

And Automata Theory in particular:
Perrin, Pin, Facchini...

Outside Polish 0-dimensional spaces

On non 0-dimensional spaces, the Wadge reducibility $\leq_W$ yields no hierarchy in general, by results of Schlicht, Hertling, Ikegami, Tanaka, Grigorieff, Selivanov and others.



There can be very few continuous maps...

Recall that $D_2(\mathbf{\Sigma}^0_1) = \{U \cap V^\complement \mid U, V \in \mathbf{\Sigma}^0_1\}$.

Theorem (Ikegami, 2010)

The poset $(\mathcal{P}(\omega), \subseteq_{fin})$ embeds into $(D_2(\mathbf{\Sigma}^0_1(\mathbb{R})), \leq_W)$.

Notice that $(\mathcal{P}(\omega), \subseteq_{\mathrm{fin}})$ contains descending chains and antichains of size $\aleph_1$.

Theorem (Schlicht, 2012)

In every non 0-dimensional metric space, there exists a $\leq_W$-antichain of size $2^{\aleph_0}$ among the $D_2(\mathbf{\Sigma}^0_1)$.

Consider more general reductions

Reduction by continuous functions yield a nice hierarchy of subsets of Polish 0-dimensional spaces, but not in arbitrary spaces, so we may ask:

> *Is the continuous function the right notion of reduction to study topological complexity in a more general context?*

Motto Ros, Schlicht and Selivanov have considered

- Reducibility by *reasonably* discontinuous functions.

We propose to weaken the second fundamental concept at stake, namely the concept of function:

- reducibility by *relatively continuous relations*.

Reductions

Fix sets X, Y, and subsets $A \subseteq X$, $B \subseteq Y$.
A *reduction* of A to B is a function $f : X \to Y$ such that

$$\forall x \in X \ (x \in A \leftrightarrow f(x) \in B).$$

A *total relation* from X to Y is a relation $R \subseteq X \times Y$ with $\forall x \in X \ \exists y \in Y \ R(x, y)$, in symbols $R : X \rightrightarrows Y$.

Definition

A *reduction* of A to B is a total relation $R : X \rightrightarrows Y$ such that

$$\forall x \in X \ \forall y \in Y \ \Big(R(x, y) \to (x \in A \leftrightarrow y \in B)\Big),$$

or equivalently

$$\forall x \in X \ \Big(x \in A \wedge R(x) \subseteq B\Big) \vee \Big(x \in A^{\complement} \wedge R(x) \subseteq B^{\complement}\Big)$$

where $R(x) = \{y \in Y : R(x, y)\}$.

Reductions, basic properties

Basic Properties

Let $A \subseteq X$, $B \subseteq Y$, $C \subseteq Z$, and $R : X \rightrightarrows Y$, $T : Y \rightrightarrows Z$:

- *If R reduces A to B and T reduces B to C, then*

$$T \circ R = \{(x, z) : \exists y \in Y \; R(x, y) \wedge T(y, z)\}$$

reduces A to C.

Let $\mathcal{R}$ be a class of total relations from X to X with

1. the identity on X belongs $\mathcal{R}$,
2. $\mathcal{R}$ is closed under composition.

For $A, B \subseteq X$,

$$A \; \mathcal{R}\text{-reducible to } B \quad \longleftrightarrow \quad \exists R \in \mathcal{R} \quad R \text{ reduces } A \text{ to } B$$

This defines a quasi-order $\leq_{\mathcal{R}}$ on subsets of X.

Reductions, basic properties

$$\forall x \in X \; \forall y \in Y \; \Big(S(x,y) \to (x \in A \leftrightarrow y \in B)\Big).$$

Basic Properties

Let $A \subseteq X$, $B \subseteq Y$, $R, S : X \rightrightarrows Y$:

- *If $R \subseteq S$ and S reduces A to B, then R also reduces A to B.*

Let $\mathcal{R}$ be a class of total relations from X to X with

1. the identity on X belongs $\mathcal{R}$,
2. $\mathcal{R}$ is closed under composition.

Let $\overline{\mathcal{R}} = \{S : X \rightrightarrows X : \exists R \in \mathcal{R} \quad R \subseteq S\}$, then for any $A, B \subseteq X$,

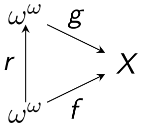
$$A \; \mathcal{R}\text{-reducible to } B \quad \longleftrightarrow \quad A \; \overline{\mathcal{R}}\text{-reducible to } B$$

So as far as reducibility is concerned, we can always consider upward closed classes of total relations.

Admissible representations

Let $f, g :\subseteq \omega^\omega \to X$ be partial maps.
Say f *continuously reduces* to g, $f \leq_C g$, if

$$\exists \text{ a continuous } r : \operatorname{dom} f \to \operatorname{dom} g$$
$$\forall \alpha \in \operatorname{dom} f \quad f(\alpha) = g \circ r(\alpha).$$



Proposition (Kreitz, Weihrauch, Schröder)

Let X be 2nd countable T_0. There exists a partial map $\rho :\subseteq \omega^\omega \to X$ such that

- *ρ is continuous (and surjective),*
- *ρ is $\leq_C$-greatest among continuous partial functions, i.e. $\forall$ continuous $f :\subseteq \omega^\omega \to X$, $f \leq_C \rho$.*

Such a map is called an admissible representation of X.

This concept is fundamental to the approach to computable analysis called Type-2 Theory of Effectivity.

Examples of admissible representations

1. Let $\mathbb{Q} = \{q_n \mid n \in \omega\}$. A sequence $(x_k)_{k\in\omega}$ is said to be *rapidly Cauchy* if $i < j \to d(x_i, x_j) \le 2^{-i}$.
   The Cauchy representation $\sigma_{\mathbb{R}} :\subseteq \omega^\omega \to \mathbb{R}$ is defined by

$$\sigma_{\mathbb{R}}(\alpha) = x \quad \longleftrightarrow \quad \begin{cases} \left(q_{\alpha(k)}\right)_{k\in\omega} \text{ is rapidly Cauchy} \\ \text{and } \lim_{k\to\infty} q_{\alpha(k)} = x. \end{cases}$$

2. The enumeration representation of the Scott domain $\mathcal{P}(\omega)$ is the total function $\rho_{\mathsf{En}} : \omega^\omega \to \mathcal{P}(\omega)$ defined by

$$\rho_{\mathsf{En}}(x) = \{n \mid \exists k \; x_k = n + 1\}.$$

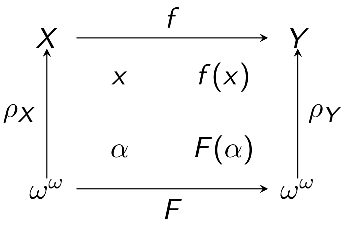
3. If $(V_n)_{n\in\omega}$ is a basis for X, then one can take $\rho :\subseteq \omega^\omega \to X$:

$$\rho(\alpha) = x \quad \longleftrightarrow \quad \{\alpha(k) : k \in \omega\} = \{n : x \in V_n\}.$$

Relatively continuous functions

Let X, Y be 2nd countable T_0 spaces.
A map $f : X \to Y$ is *relatively continuous* if
for some (hence any) admissible representations ρ_X, ρ_Y
there exists a continuous $F : \mathrm{dom}\,\rho_X \to \mathrm{dom}\,\rho_Y$ such that

$$\forall \alpha \in \mathrm{dom}\,\rho_X \quad f \circ \rho_X(\alpha) = \rho_Y \circ F(\alpha)$$



Proposition

Let X, Y be 2nd countable T_0. A map $f : X \to Y$ is relatively continuous iff it is continuous.

Admissible representability and dimension

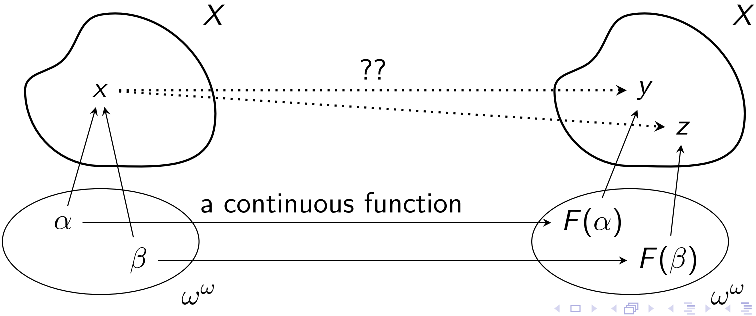
A space is 0-*dimensional* if it admits a basis of clopen sets.

Theorem

Let X be a 2^{nd} countable T_0 space. The following are equivalent.

1. *X is 0-dimensional.*
2. *X admits an injective admissible representation.*

So in case X is not 0-dimensional, there is no injective admissible representation of X.



Relatively continuous relations

Definition (Brattka, Hertling, Weihrauch)

X, Y 2nd countable T_0 spaces.
A total relation $R : X \rightrightarrows Y$ is *relatively continuous* if
for some (hence any) admissible representations ρ_X, ρ_Y
there exists a continuous $F : \mathrm{dom}\,\rho_X \to \mathrm{dom}\,\rho_Y$ such that

$$\forall \alpha \in \mathrm{dom}\,\rho_X \quad R(\rho_X(\alpha), \rho_Y(F(\alpha)))$$

Basic Properties

1. *graphs of continuous functions are relatively continuous.*
2. *relatively continuous relations compose.*
3. *If $R, S : X \rightrightarrows Y$, R relatively continuous and $R \subseteq S$, then S is also relatively continuous.*

In a 0-dimensional space X, for $R : X \rightrightarrows X$:

R is relatively continuous $\quad\longleftrightarrow\quad$ R admits a continuous uniformising function.

Reduction by relatively continuous relations

Definition

Let X be 2^{nd} countable T_0, $A, B \subseteq X$.
A *is reducible* to B, $A \preccurlyeq B$, if there exists a relatively continuous relation $R : X \rightrightarrows X$ that reduces A to B.

Basic Properties

1. $\preccurlyeq$ *is a quasiorder on subsets of* X.
2. *If* $A \leq_W B$*, then* $A \preccurlyeq B$.
3. *For any admissible representation* ρ *of* X, $A \preccurlyeq B$ *iff there exists* $F : \operatorname{dom} \rho \to \operatorname{dom} \rho$ *continuous such that*

$$\forall \alpha \in \operatorname{dom} \rho \quad \big(\rho(\alpha) \in A \quad \longleftrightarrow \quad \rho(F(\alpha)) \in B\big).$$

If X is 0-dimensional: $A \leq_W B \leftrightarrow A \preccurlyeq B$, that is

Wadge reducibility $\quad = \quad$ reducibility by relativ. cont. relations.

Baire classes again

A result due to Saint Raymond gives

Theorem

Let X be a 2^{nd} countable T_0 space, $\rho :\subseteq \omega^\omega \to X$ an admissible representation of X. For any countable $\alpha > 0$ and $A \subseteq X$

$$A \in \mathbf{\Sigma}^0_\alpha(X) \quad \longleftrightarrow \quad \rho^{-1}(A) \in \mathbf{\Sigma}^0_\alpha(\operatorname{dom}\rho).$$

de Brecht showed that this also holds for Hausdorff-Kuratowski difference classes.

Corollary

Let X be 2^{nd} countable T_0 and $\mathbf{\Gamma}$ be $\mathbf{\Sigma}^0_\alpha(X)$ or $\mathbf{\Pi}^0_\alpha(X)$. Then $B \in \mathbf{\Gamma}$ and $A \preccurlyeq B$ imply $A \in \mathbf{\Gamma}$.

The Baire classes are not only initial segments for $\leq_W$ but for $\preccurlyeq$ too.

A game for the reduction

Let X be a 2^{nd} countable T_0, $\rho :\subseteq \omega^\omega \to X$ an admissible representation of X, and $A, B \subseteq X$.

We define a perfect information two players game $G^\rho(A, B)$ as follows

$$\begin{array}{lccccccc} \mathrm{I}: & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & & \alpha \in \omega^\omega \\ \mathrm{II}: & \beta_0 & \beta_1 & \beta_2 & \beta_3 & \cdots & & \beta \in \omega^\omega \end{array}$$

Player II wins if either $\alpha \notin \operatorname{dom} \rho$, or if $\alpha \in \operatorname{dom} \rho$, $\beta \in \operatorname{dom} \rho$ and

$$\rho(\alpha) \in A \quad \longleftrightarrow \quad \rho(\beta) \in B.$$

Proposition

- *If Player II has a winning strategy, then* $A \preccurlyeq B$.
- *If Player I has a winning strategy, then* $B \preccurlyeq A^{\complement}$.

Borel representable spaces

Definition

A 2^{nd} countable T_0 space X is called *Borel representable space* if there exists an admissible representation ρ of X whose domain is Borel (in ω^ω).

Borel representable spaces include

- every Borel subspace of any Polish space,
  i.e. every Borel subspace of $[0,1]^\omega$.
- every Borel subspace of any quasi-Polish space,
  i.e. every Borel subspace of $\mathcal{P}(\omega)$ with the Scott topology.

Most (all?) properties of Wadge reducibility on 0-dim Polish spaces extends to arbitrary Borel representable spaces via the reducibility by relatively continuous relations.

The nice picture regained

Using Determinacy of Borel games (Martin) and the exact same proof as in the case of the Wadge reducibility on the Baire space, we obtain:

Theorem

Let X be Borel representable.

1. *For every Borel sets $A, B \subseteq X$, either $A \preccurlyeq B$ or $B \preccurlyeq A^{\complement}$ (so antichains have size at most 2).*
2. *$\preccurlyeq$ is well founded on Borel sets.*

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