PART 3: Weighted automata & transducers

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Definition. A semiring is a structure \((K, +, \cdot, 0, 1)\) where

1. \((K, +, 0)\) is a commutative monoid
2. \((K, \cdot, 1)\) is a monoid
3. The distributivity axioms hold
   \[\forall x, y, z \in K,\]
   \[x \cdot (y + z) = x \cdot y + x \cdot z\]
   \[(y + z) \cdot x = y \cdot x + z \cdot x\]
4. \(\forall x \in K,\)
   \[x \cdot 0 = 0 \cdot x = 0\]
**Definition** A semiring is a structure \((K, +, \cdot, 0, 1)\) where
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- \((K, \cdot, 1)\) is a monoid
- The distributivity axioms hold
  \[\forall x, y, z \in K\]
  \[x \cdot (y + z) = x \cdot y + x \cdot z\]
  \[(y + z) \cdot x = y \cdot x + z \cdot x\]
- \(\forall x \in K\)
  \[x \cdot 0 = 0 \cdot x = 0\]

**Definition** A semiring morphism \(f : (K, +, \cdot, 0, 1) \to (L, \ast, \cdot, 0, 1)\)
is a function \(f : K \to L\) such that
\[\forall x, y \in K\]
\[f(x + y) = f(x) + f(y)\]
\[f(0_K) = 0_L\]
\[\forall x, y \in K\]
\[f(x \cdot y) = f(x) \cdot f(y)\]
\[f(1_K) = 1_L\]
Examples

- Numerical semirings
  \((\mathbb{N}, +, \cdot, 0, 1)\)
  \((\mathbb{Z}, +, \cdot, 0, 1)\)
  \((\mathbb{Q}, +, \cdot, 0, 1)\)
  \((\mathbb{R}, +, \cdot, 0, 1)\)

- Boolean semiring
  \(\mathbb{B} = (\{0, 1\}, \lor, \land, 0_B, 1_B)\)
Examples

- **Numerical semirings**
  
  \((\mathbb{N}, +, \cdot, 0, 1)\)
  \((\mathbb{Z}, +, \cdot, 0, 1)\)
  \((\mathbb{Q}, +, \cdot, 0, 1)\)
  \((\mathbb{R}, +, \cdot, 0, 1)\)

- **Boolean semiring**
  
  \(B = (\{0_B, 1_B\}, \lor, \land, 0_B, 1_B)\)

- **Tropical semirings**
  
  \(\mathbb{N}_{min} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)\)

  \(\# \min (n, \infty) = n \quad \forall n \in \mathbb{N} \cup \{\infty\}\)
  
  \(\# x + \min(y, z) = \min (x + y, x + z)\)
Examples

- numerical semirings
  \((\mathbb{N}, +, \cdot, 0, 1)\)
  \((\mathbb{Z}, +, \cdot, 0, 1)\)
  \((\mathbb{Q}, +, \cdot, 0, 1)\)
  \((\mathbb{R}, +, \cdot, 0, 1)\)

- boolean semiring
  \(B = (\{0_B, 1_B\}, \lor, \land, 0_B, 1_B)\)

- tropical semirings
  \(N_{\min} = (\mathbb{N} \cup \{-\infty\}, \min, +, \infty, 0)\)
  \(\# \min (n, \infty) = n \quad \forall n \in \mathbb{N} \cup \{-\infty\}\)
  \(\# x + \min (y, z) = \min (x + y, x + z)\)
  \(N_{\max} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)\)
  \(Z_{\min} = (\mathbb{Z} \cup \{-\infty\}, \min, +, \infty, 0)\)
  \(Z_{\max} = (\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)\)
If $(M, \cdot)$ is a monoid, then $(\mathcal{P}(M), \cup, \cdot, \emptyset, \{1_M\})$

is a semiring, where, for $X, Y \subseteq M$ we define

$$X \cdot Y = \{ x \cdot y \mid x \in X, y \in Y \}.$$
Examples

If \((M, \cdot)\) is a monoid, then \((\mathcal{P}(M), U, \cdot, \emptyset, \{1_M\})\) is a semiring, where, for \(X, Y \subseteq M\) we define \(X \cdot Y = \{x \cdot y \mid x \in X, y \in Y\}\).

Consider the free monoid \((A^*, \cdot)\). Then \((\mathcal{P}(A^*), U, \cdot, \emptyset, \{\varepsilon\})\) is a semiring.
If \((M, \cdot)\) is a monoid, then \((\mathcal{P}(M), \cup, \cdot, \emptyset, \{e\}_M)\) is a semiring, where, for \(X, Y \subseteq M\) we define \(X \cdot Y = \{x \cdot y \mid x \in X, y \in Y\}\).

Consider the free monoid \((A^\ast, \cdot)\). Then \((\mathcal{P}(A^\ast), \cup, \cdot, \emptyset, \{\varepsilon\})\) is a semiring.

The set \(\text{Rat}(A^\ast)\) of rational languages carries a semiring structure \((\text{Rat}(A^\ast), \cup, \cdot, \emptyset, \{\varepsilon\})\). That is, the inclusion \(i : \text{Rat}(A^\ast) \rightarrow \mathcal{P}(A^\ast)\) is a semiring morphism.
Examples

If \((K, +, \cdot, 0, 1)\) is a semiring, then the set of matrices of dimension \(a\), denoted by \(K^a\), carries a semiring structure \((K^a, +, \cdot, 0, 1)\).  

Examples

If \((K,+,0,1)\) is a semiring, then the set of matrices of dimension \(Q\), denoted by \(K^{Q \times Q}\), carries a semiring structure \((K^{Q \times Q}, +, \cdot, 0, I)\).

- Addition of matrices
- Multiplication of matrices
- All entries are \(0_k\)
- The identity matrix

\[
\begin{pmatrix}
1_k & 1_k & 0 \\
0 & 1_k
\end{pmatrix}
\]
Weighted automata over a semiring $K$ or $K$-automata

Boolean automata
(most deterministic automata)

$A = (\Sigma, \delta, I, F)$

- $\Sigma$ is a finite alphabet
- $Q$ is a finite set of states
- $I \subseteq Q$ initial states
- $\delta \subseteq Q \times \Sigma \times Q$
  transition relation
- $F \subseteq Q$ accepting states
Weighted automata over a semiring \( R \) or \( K \)-automata

**Boolean automata**

(mon. deterministic automata)

\[ A = (A, Q, I, \delta, F) \]

- \( A \) is a finite alphabet
- \( Q \) is a finite set of states
- \( I \subseteq Q \) initial state
- \( \delta : Q \times A \times Q \to \{0_b, 1_b\} \) transition relation
- \( F \subseteq Q \) accepting states
- \( \chi_F : A \to \{0_b, 1_b\} \)
Weighted automata over a semiring $K$, or $K$-automata

Boolean automata (non-deterministic automata)

$A = (A, Q, I, S, F)$

- $A$ is a finite alphabet
- $Q$ is a finite set of states
- $I \in Q$ initial state
  \[ X_I : Q \rightarrow \{ 0, 1 \} \]
- $S \in Q \times A \times Q$
  transition relation
  \[ S : Q \times A \times Q \rightarrow \{ 0, 1 \} \]
- $F \subseteq Q$ accepting states
  \[ X_F : Q \rightarrow \{ 0, 1 \} \]

$K$-automata

$A = (A, Q, I, S, F)$

- $A$ is a finite alphabet
- $Q$ is a finite set of states
- $I : Q \rightarrow K$ is the initial weight function
- $S : Q \times A \times Q \rightarrow K$ is the transition function
- $F : Q \rightarrow K$ is the final weight function
Weighted automata over a semiring $K$, or $K$-automata

Boolean automata

(non-deterministic automata)

$A = (A, Q, I, S, F)$

- $A$ is a finite alphabet
- $Q$ is a finite set of states
- $I \subseteq Q$: initial state
  $\chi_I: Q \rightarrow \{0, 1\}$
- $S \subseteq Q \times A \times Q$: transition relation
  $\delta: Q \times A \times Q \rightarrow \{0, 1\}$
- $F \subseteq Q$: accepting states
  $\chi_F: Q \rightarrow \{0, 1\}$

$K$-automata

$A = (A, Q, I, S, F)$

- $A$ is a finite alphabet
- $Q$ is a finite set of states
- $I: Q \rightarrow K$ is the initial weight function
- $S: Q \times A \times Q \rightarrow K$ is the transition function
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$A = (A, Q, I, S, F)$

- $A$ is a finite alphabet
- $Q$ is a finite set of states
- $I : Q \rightarrow \mathbb{K}$ is the initial weight function
- $S : Q \times A \times Q \rightarrow \mathbb{K}$ is the transition function
- $F : Q \rightarrow \mathbb{K}$ is the final weight function

Remark: We could also use a transition relation $d \subseteq Q \times A \times (1, 0, k^3) \times Q$

Example

$N_{\text{max}}$ - automaton:
A = (A, Q, I, δ, F)

- A is a finite alphabet
- Q is a finite set of states
- I: Q → K is the initial weight function
- δ: Q × A × Q → K is the transition function
- F: Q → K is the final weight function

Remark: We could also use a transition relation
δ ∈ Q × A × (K ∪ 10K) × Q

Example

N_{max} - automaton:

\[ p \xrightarrow{a_1} \, q \, \xrightarrow{a_0} \, p \]

\[ q \, \xrightarrow{b_1} \, p \, \xrightarrow{b_0} \, q \]

Notation: We write \( p \xrightarrow{a/k} q \) or \( p \xrightarrow{k} q \) for \( δ(p, a, q) = k \).
**Notation**  We write $p \xrightarrow{k} q$ or $p \xrightarrow{k_0} q$ for $d(p,a,q) = k$.

**Convection**  For the numerical reminders, we omit the unit 1 of the multiplication.
**Notation** We write \( p \overset{a}{\rightarrow} q \) or \( p \xrightarrow{k_a} q \) for \( s(p,a,q) = k \).

**Convention** For the numerical reminders we omit the unit 1 of the multiplication.

**Example of an \( N \)-automaton**

\[ \begin{array}{c}
I(p) = 1 \quad I(q) = 0 \\
F(p) = 0 \quad F(q) = 1 \\
s(p,a,p) = 1; \\
s(p,a,q) = 1; \quad \text{etc...}
\end{array} \]
Definition

A path in a weighted automaton is a sequence of the form \( p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \xrightarrow{...} p_n \)

where \( \forall i \in \{0, ..., n-1\} \)

\( p_i \xrightarrow{k_{i+1}} p_{i+1} \) is a transition, i.e.

\( \delta(p_i, a_{i+1}, p_{i+1}) = k_{i+1} \)
Definition  A path in a weighted automaton is a sequence of the form \( p_0 \xrightarrow{k_1 \cdot a_1} p_1 \xrightarrow{k_2 \cdot a_2} p_2 \xrightarrow{} \ldots \xrightarrow{} p_m \)

where \( \forall i \in \{0, \ldots, n-1\} \)

\( p_i \xrightarrow{k_i \cdot a_i} p_{i+1} \) is a transition, i.e.

\[ d(p_i, a_i, p_{i+1}) = k_i \]

Definition  Given a path \( d : p_0 \xrightarrow{k_1 \cdot a_1} p_1 \xrightarrow{k_2 \cdot a_2} \ldots \xrightarrow{} p_m \)

- the label of \( d \) is defined as \( l(d) = a_1 \ldots a_m \)
- the weight of \( d \) is defined as \( w(d) = k_1 \ldots k_m \)
- the weighted label of \( d \) is defined as \( \omega_l(d) = (k_1, \ldots, k_m) a_1 \ldots a_m \)
Definition: A computation in a weighted automaton is a path together with the input and final values of the source, respectively the target of the path:

\[ I(p_0) \xrightarrow{h \cdot a} p_0 \xrightarrow{p_1} \ldots \xrightarrow{k \cdot a_n} F(p_m) \]
Definition: A computation in a weighted automaton is a path together with the input and final values of the source, respectively the target of the path:

\[ C: \quad I(p_0) \xrightarrow{k_1} p_1 \xrightarrow{k_2} p_2 \rightarrow \ldots \rightarrow k_m \xrightarrow{F(p_m)} \]

The weight of \( C \) is defined as \( w(C) = I(p_0) \cdot k_1 \cdot \ldots \cdot k_m \cdot F(p_m) \).

The label of \( C \) is the label of the path, that is, \( l(C) = a_1 \ldots a_m \).
**Definition** A computation in a weighted automaton is a path together with the input and final values of the source, respectively the target of the path:

\[
C : I(p_0) \xrightarrow{k_1 a_1} p_1 \xrightarrow{k_2 a_2} \cdots \xrightarrow{k_n a_n} F(p_n)
\]

The weight of \( C \) is defined as

\[
\text{weight}(C) = I(p_0) \cdot k_1 \cdot a_1 \cdots k_n \cdot F(p_n)
\]

The label of \( C \) is the label of the path, that is,

\[
\text{label}(C) = a_1 \cdots a_n
\]

**Definition** The behaviour of a \( \mathbb{K} \)-automaton \( A \) is a function \( \mathcal{L} : \mathbb{A}^* \rightarrow \mathbb{K} \) defined by

\[
\forall w \in \mathbb{A}^*. \quad \mathcal{L}(w) = \sum_{\text{computations } C \text{ in } A} \text{weight}(C)
\]

\[
\text{label}(C) = w
\]
Example

The behavior of the N-automaton

Diagram:
Example

The behavior of the $N$-automaton

\[ A (aabb) = ? \]

Some example of computation:

\[ 1 \xrightarrow{a} p \xrightarrow{a} p \xrightarrow{b} p \xrightarrow{a} \]

\[ \rightarrow \text{weight } 0! \]
Example

The behavior of the N-automaton

\[ (a^2 b) = ? \]

Some example of computation

\[ 1 \xrightarrow{p} a \xrightarrow{p} o \xrightarrow{p} o \xrightarrow{o} \quad \rightarrow \text{weight 0!} \]

\[ 1 \xrightarrow{p} a \xrightarrow{p} q \xrightarrow{b} q_1 \xrightarrow{1} \quad \rightarrow \text{weight 1} \]
Example

The behavior of the \( N \)-automaton

\[
\begin{align*}
1 \text{A1} (\text{aab}) &= \omega (\rightarrow p \xrightarrow{a} q \xrightarrow{a} q \xrightarrow{b} q \xrightarrow{} ) + \omega (\rightarrow p \xrightarrow{a} p \xrightarrow{a} q \xrightarrow{b} q \xrightarrow{} ) \\
\text{Some example of computation} &
\end{align*}
\]

\[
\begin{align*}
1 \rightarrow p \xrightarrow{a} p \xrightarrow{a} p \xrightarrow{b} p \xrightarrow{0} \\
\uparrow \rightarrow p \xrightarrow{a} p \xrightarrow{a} q \xrightarrow{b} q \xrightarrow{1}
\end{align*}
\]

\[\implies \text{weight 0!} \]

\[\implies \text{weight 1} \]

In general, \( 1 \text{A1} (w) = |w|_{a} \). (the number of a's in \( w \))
Example

Recall the $N_{\text{max}}$ automaton

\[ A : \]

\[ a/1 \quad \rightarrow \quad \rightarrow \quad 0 \quad 1 \]

\[ b/0 \]

for the semiring $\mathbb{N}_{\text{max}} = (\mathbb{N} \cup \{-\infty\}, \text{max}, +, -\infty, 0)$.

What is the behaviour of $A$ on a word $w = \{a,b\}^*$?
Example

Recall the $N_{\text{max}}$-automaton

For the semiring $N_{\text{max}} = (\mathbb{N} \cup \{ -\infty \}, \max, +, -\infty, 0)$.

What is the behaviour of $A$ on a word we $\{a, b, z\}^*$?
Exercise: Find a $\mathbb{Z}$-automaton $A$ such that

$$|A| \cdot |w| = |w|_{a} - |w|_{b}.$$
**K-series**

**Definition** A K-series over $A^*$ is a function $\delta : A^* \rightarrow K$. The set of K-series over $A^*$ is denoted by $K \langle A^* \rangle$ and can be equipped with the following operations:
Definition: A K-series over $A^*$ is a function $s: A^* \rightarrow K$. The set of K-series over $A^*$ is denoted by $\mathcal{K}(A^*)$ and can be equipped with the following operations:

- **Pointwise Addition**

For $s, t \in \mathcal{K}(A^*)$ define $s + t \in \mathcal{K}(A^*)$ by

$$(s + t)(w) = s(w) + t(w).$$
**K-series**

**Definition** A K-series over $A^*$ is a function $s: A^* \rightarrow K$.

The set of K-series over $A^*$ is denoted by $K \langle A^* \rangle$ and can be equipped with the following operations:

- **Pointwise addition**
  For $s, t, r \in K \langle A^* \rangle$ define $s + t \in K \langle A^* \rangle$ by
  $$(s + t)(w) = s(w) + \sum_{v \in A^*} t(w).$$

- **Cauchy product**
  $$(s \cdot t)(w) = \sum_{v \in A^*} s(v) \cdot t(v) \quad \text{if } w = uv$$

- **Exterior multiplications**
  For $k \in K$ and $s \in K \langle A^* \rangle$ define $k \cdot s \in K \langle A^* \rangle$ by
  $$(k \cdot s)(w) = s(w)$$
  $$(s \cdot k)(w) = s(w) \cdot k$$

**Axioms:**

\begin{align*}
\forall s, t, r \in K \langle A^* \rangle \\
(s + t) \cdot r &= s \cdot r + t \cdot r \\
r \cdot (s + t) &= r \cdot s + r \cdot t
\end{align*}
**K-series**

**Definition** A **K-series** over \( A^* \) is a function \( s : A^* \rightarrow K \).

The set of K-series over \( A^* \) is denoted by \( K \langle A^* \rangle \) and can be equipped with the following operations:

* **Pointwise addition**

For \( s, t \in K \langle A^* \rangle \) define \( s + t \in K \langle A^* \rangle \) by

\[
(s + t)(w) = s(w) + \chi_k t(w)
\]

* **Cauchy product**

\[
(s \cdot t)(w) = \sum_{u, v \in A^*} s(u) \cdot k \cdot t(v)
\]

where \( w = uv \)

* **Exterior multiplications**

For \( k \in K \) and \( s \in K \langle A^* \rangle \)

\[
(k \cdot s)(w) = k \cdot s(w)
\]

\[
(s \cdot k)(w) = s(w) \cdot \chi_k k
\]

**Axioms:**

\[
\forall s, t, r \in K \langle A^* \rangle
\]

\[
(s + t) \cdot r = s \cdot r + t \cdot r
\]

\[
r \cdot (s + t) = r \cdot s + r \cdot t
\]

\[
\forall k \in K, \forall s, t \in K \langle A^* \rangle
\]

\[
k \cdot (s + t) = k \cdot s + k \cdot t
\]

\[
(s + t) \cdot k = s \cdot k + t \cdot k
\]

\[
k \cdot (s \cdot t) = (k \cdot s) \cdot t
\]

\[
(s \cdot t) \cdot k = s \cdot (t \cdot k)
\]
**K-series**

**Definition.** A **K-series** over $A^*$ is a function $s : A^* \rightarrow K$. The set of K-series over $A^*$ is denoted by $K\langle A^* \rangle$ and can be equipped with the following operations:

- **Pointwise addition**
  For $s, t, r \in K\langle A^* \rangle$ define $s + t$ by $s + t : (s + t)(w) = s(w) + t(w)$.

- **Cauchy product**
  $s \cdot t : (s \cdot t)(w) = \sum_{u, v \in A^*} s(u) \cdot t(v)$ where $w = uv$.

- **Exterior multiplication**
  For $k \in K$ and $s \in K\langle A^* \rangle$ define $k \cdot s$ by $(k \cdot s)(w) = k \cdot s(w)$ and $(s \cdot k)(w) = s(w) \cdot k$.

**Axioms:**

- $\forall s, t, r \in K\langle A^* \rangle$
  $s + t + r = s + (t + r)$
  $(s + t) \cdot r = s \cdot r + t \cdot r$
  $r \cdot (s + t) = r \cdot s + r \cdot t$

- $\forall k \in K$, $\forall s, t \in K\langle A^* \rangle$
  $k \cdot (s + t) = k \cdot s + k \cdot t$
  $(s + t) \cdot k = s \cdot k + t \cdot k$
  $k \cdot (s \cdot t) = (k \cdot s) \cdot t$
  $(s \cdot t) \cdot k = s \cdot (t \cdot k)$

$K\langle A^* \rangle$ is a **K-algebra**!
Definition: The support of a $K$-series $v \in K \ll A^*$ is defined by $\text{Supp}(v) = \{ w \in A^* \mid v(w) \neq 0, k \}$.
Definition: The support of a $1K$-series $s \in 1K\langle A^* \rangle$ is defined by
\[ \text{supp}(s) = \{ w \in A^* | s(w) \neq 0_K \} \]

Given a $1K$-automaton $A$ we obtain a Boolean automaton $\text{supp}A$ by replacing every non-zero weight in $K$ with $1_B$.

Exercise:
1) Prove that $|\text{supp}A| \leq |\text{support}|$.
2) Find a sufficient condition so that equality holds.
Definition: The support of a $1K$-series $s \in 1K \langle A^* \rangle$ is defined by $\text{supp}(s) = \{ w \in A^* | s(w) \neq 0K \}$.

Given a $1K$-automaton $A$, we obtain a Boolean automaton $\text{supp } A$ by replacing every non-zero weight in $1K$ with $1B$.

Exercise: a) Prove that $\text{supp } |A| \leq |\text{support } A|$.
   b) Find a sufficient condition so that equality holds.

Notation: Given $s \in 1K \langle A^* \rangle$, we write $s = \sum_{w \in A^*} s(w) \cdot w$.

If $s$ has a finite support, it is called a polynomial.

Examples: $2ab + 3aabb \in 1K \langle A^* \rangle$. 
The matrix representation of a $1K$-automaton

Let $A = (Q, \Sigma: Q \to 1K, S: Q \times A \times Q \to 1K, \bar{F}: Q \to 1K)$ be a $1K$-automaton.

\[
\begin{align*}
\delta: & Q \times A \times Q \to 1K \\
\Delta: & Q \times Q \to 1K^A
\end{align*}
\]
The matrix representation of a 1K-automaton.

Let $A = (Q, \Sigma: Q \to \mathbb{K}, \delta: Q \times A \times Q \to \mathbb{K}, F: Q \to \mathbb{K})$ be a 1K-automaton.

\[ Q \times A \times Q \to \mathbb{K} \]

But $K^A \subseteq K\langle A^* \rangle$, so $\delta$ yields a map $\Delta: Q \times Q \to K\langle A^* \rangle$, i.e. a $1K\langle A^* \rangle$-matrix of dimension $Q$. 

The matrix representation of a $1K$-automaton

Let $A = (Q, I:Q \to 1K, \Delta:Q \times A \times Q \to 1K, F:Q \to 1K)$ be a $1K$-automaton.

\[
\begin{array}{c}
Q \times A \times Q \\ \rightarrow 1K \\
\hline
Q \times Q \\ \rightarrow 1K^A
\end{array}
\]

But $1K^A \leq 1K^{\langle A^* \rangle}$, so $\Delta$ yields a map $\Delta:Q \times Q \to 1K^{\langle A^* \rangle}$, i.e. a $1K^{\langle A^* \rangle}$-matrix of dimension $Q$.

We think of $I:Q \to 1K$ as a row vector and of $F:Q \to 1K$ as a column vector.
The matrix representation of a $1K$-automaton

Let $A = (Q, I: Q \to 1K, \delta: Q \times A \times Q \to 1K, F: Q \to 1K)$ be a $1K$-automaton.

$$Q \times A \times Q \to 1K$$
$$Q \times Q \to 1K^A$$

But $1K^A \subseteq 1K\langle A^* \rangle$, so $\delta$ yields a map $\Delta: Q \times Q \to 1K\langle A^* \rangle$, i.e. a $1K\langle A^* \rangle$-matrix of dimension $Q$.

We think of $I: Q \to 1K$ as a row vector and of $F: Q \to 1K$ as a column vector.

Example Consider the $\mathbb{N}$-automaton

```
\begin{center}
\begin{tikzpicture}
  \node (p) at (0,0) {$p$};
  \node (q) at (1,0) {$q$};
  \draw[->] (p) -- node[above] {$a$} (q);
  \draw[->] (q) -- node[above] {$b$} (p);
\end{tikzpicture}
\end{center}
```

Its matrix representation is

\begin{align*}
(1, 0),
(a+b, a),
(0, 2a+b),
(1)
\end{align*}
Lemma Consider the matrix representation \((I, \Delta F)\) of a 1k-automaton \(A\) and let \(w \in A^*\) of length \(n\). Then
\[
\Delta^n(w) = (I \cdot \Delta X \cdot F)(w)
\]

Example Consider the \(N\)-automaton

![Diagram](image)

Its matrix representation is

\[
(1, 0), \begin{pmatrix} a + b & a \\ 0 & 2a + b \end{pmatrix}, (0)
\]
**Lemma** Consider the matrix representation \((I, \Delta F)\) of a \(k\)-automaton \(A\) and let \(w \in \mathbb{A}^*\) of length \(n\). Then

\[
\mathcal{L}(w) = (I \cdot \Delta^n \cdot F)(w)
\]

\(\in \mathbb{A}^*\)

**Example** Consider the \(N\)-automaton

\[
\begin{array}{c}
\circlearrowleft \\
\circlearrowleft \\
\end{array}
\]

Its matrix representation is

\[
(1, 0), \begin{pmatrix} a + b & a \\ 0 & 2a + b \end{pmatrix}, (0)
\]

\[
\Delta^2 = \begin{pmatrix} & & \\ & & \\ \end{pmatrix}
\]

\[
\mathcal{L}(a^2) = ?
\]
**Lemma** Consider the matrix representation \((I, \Delta F)\) of a \(W\)-automaton \(A\) and let \(w \in A^*\) of length \(n\). Then
\[
|A|(w) = (I \cdot \Delta^m \cdot F)(w)
\]

\(\in \mathbb{K} \subseteq A^*\)

**Proof Idea** Show by induction that \(\Delta^m\) is the sum of weighted labels of paths of length \(m\) from \(p\) to \(q\).

**Example** Consider the \(W\)-automaton

![Automaton Diagram]

Its matrix representation is

\[
(1, 0), \begin{pmatrix} a + b & a \\ 0 & 2a + b \end{pmatrix}, (0)
\]

\(\Delta^2 = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}\)

\(|A|(a^2) = ?\)
Corollary: Consider the matrix representation $(I, \Delta, F)$ of a \( \mathbb{K} \)-automaton \( A \). Then
\[
(A) = \sum \limits_{n \geq 0} I \Delta^n F = I \sum \limits_{n \geq 0} \Delta^n F = I \Delta^\infty F
\]

Remark: The infinite sum above is well defined in the semiring \( \mathbb{K} \ll \mathbb{A}^\infty \mathbb{Q} \times \mathbb{Q} \).
Corollary. Consider the matrix representation \((I, \Delta, F)\) of an \(HK\)-automaton \(A\). Then

\[
\det(I - A) = \sum_{n=0}^{\infty} I \cdot \Delta^n \cdot F = I \cdot \sum_{n=0}^{\infty} \Delta^n \cdot F = I \cdot \Delta^* \cdot F
\]

Remark. The infinite sum above is well defined in the semiring \(HK \ll \mathbb{T} \gg \mathbb{Q} \times \mathbb{Q}\).

More generally, given an arbitrary semiring \(HK\), we would like to define an operation \((\cdot)^*\) by

\[
\mathbb{T}^* = \sum_{n=0}^{\infty} \mathbb{T}^n
\]

This is only a partial operation. We will see sufficient conditions for \(\mathbb{T}^*\) to be defined.
Topological seminings

A topological semiring is a semiring and a topological space and the operations $+_{K}$ and $\cdot_{K}$ are continuous.

In practice we will consider topologies induced by some distances. For example, on $\mathbb{Q}$ or $\mathbb{R}$ we consider the Euclidean distance.

Then $\lim_{n \to \infty} k_n = \lim_{n \to \infty} k + k_n$ and $\lim_{n \to \infty} k_n = \lim_{n \to \infty} k \cdot k_n$. 
Topological seminings

A topological semiring is if it is a semiring and a topological space and the operations $+K$ and $\cdot K$ are continuous.

In practice we will consider topologies induced by some distance. For example, on $\mathbb{Q}$ a $K$ we consider the Euclidean distance. Then $k + \lim_{n \to \infty} k_n = \lim_{n \to \infty} k + k_n$ and $k \cdot \lim_{n \to \infty} k_n = \lim_{n \to \infty} k \cdot k_n$.

Given a distance $d$ on $K$, we obtain a distance $\tilde{d}$ on $K \langle A^+ \rangle$ defined by

$$
\tilde{d}(s, t) = \frac{1}{2} \sum \frac{1}{2^k} \max_{1 \leq l \leq n} d(s(w), t(w))
$$

$(s_n)_n$ converges to $s \in K \langle A^+ \rangle$ if for all $A^+$ $(s_n(w))$ converges to $s(w)$. 
Summable families. A family \((k_i)_{i \in I}\) of elements of \(K\) is called \(\text{summable}\) when

\[\exists k \in K, \forall \varepsilon > 0, \exists I_{\varepsilon} \subseteq I \text{ finite}, \forall j \in I_{\varepsilon} \text{ finite}.\]

\[\varepsilon \in J \rightarrow d(\sum_{i \in J} k_i, k) < \varepsilon\]

Then \(\sum_{i \in I} k_i = k\).
**Summable families** A family \((k_i)_{i \in I}\) of elements of \(K\) is called *summable* when
\[
\exists k \in K. \forall \varepsilon > 0. \exists I \subseteq I \text{ finite}. \forall i \in I \text{ finite}. \forall i \in I \exists J \subseteq I \text{ finite}. I \subseteq J \implies \delta(\sum_{i \in J} k_i, k) < \varepsilon
\]

Then \(\sum_{i \in I} k_i = k\).

The semirings we are interested in are of the form \(KK \ll A^i\).

**Definition** A family \((M_i)_{i \in I} \subseteq KK \ll A^i\) is *locally finite* when \(\forall w \in A^i. \{ i \in I | S_i(w) \neq 0, k \} \text{ is finite.}\)
**Summable families** A family \((k_i)_{i \in I}\) of elements of \(K\) is called **summable** when

\[ \exists \epsilon \in K, \forall \delta > 0, \exists I_{\epsilon} \subseteq I \text{ finite, } \forall \delta \in I \text{ finite.} \]

\[ I_{\epsilon} \subseteq I \Rightarrow d(\sum_{i \in I_{\epsilon}} k_i, k) < \epsilon \]

Then \( \sum_{i \in I} k_i = k. \)

The semirings we are interested in are of the form \( \mathbb{K} \ll A^* \).

**Definition** A family \((s_i)_{i \in I} \subseteq \mathbb{K} \ll A^*\) is **locally finite** when \( \forall w \in A^*, \{ i \in I | s_i(w) \neq \emptyset \} \text{ is finite.} \)

**Prop** If \((s_i)_{i \in I}\) is locally finite then \((s_i)_{i \in I}\) is summable.
Proper series

Definition: A series \( s \in \mathcal{K} \ll A^* \) is called proper when \( s(\varepsilon) = 0_{1k} \).

Prop: If \( s \in \mathcal{K} \ll A^* \) is proper then the family \((s^n)_{n \geq 0}\) is summable.

Idea: Show that \((s^n)_{n \geq 0}\) is locally finite.
The $*$ operation

**Definition** The $*$ operation on a semiring $K$ in the partial operation defined by

$$k^* = \sum_{n \geq 0} k^n$$

**Lemma** If $h^*$ exists in a topological semiring, then

$$h^* = h \cdot h^* + 1$$

$$h^* = h^* \cdot h + 1$$
Rationally closed subsets of series

**Definition.** A subset of $\mathbb{K} \langle A^* \rangle$ is called rationally closed if it is closed under right and left exterior multiplication.
Rationally closed subsets of series

**Definition.** A subset of $1K < A^*$ is called rationally closed if it is closed under:
  1. right and left exterior multiplication
  2. pointwise addition
  3. Cauchy product
  4. the $*$ operation when defined.
Rationally closed subsets of series

**Definition:** A subset of $1K\llbracket A^* \rrbracket$ is called rationally closed if it is closed under
* right and left exterior multiplication
* pointwise addition
* Cauchy product
* the * operation whenever defined.

An intersection of rationally closed sets is rationally closed. Define the rational closure of $X \subseteq 1K\llbracket A^* \rrbracket$ as the intersection of all rationally closed supersets of $X$. 
Rational $K$-series

**Definition.** The set of rational $K$-series on $A^*$ is defined as the rational closure of the set $K\langle A^* \rangle = \{ s \in K\llangle A^* \rrangle \mid \text{supp}(s) \text{ finite} \}$ of $K$-polynomials over $A^*$.

**Theorem.** If $K$ is a strong semiring then a series in $K\llangle A^* \rrangle$ is rational if and only if it is the behaviour of some finite $K$-automaton.
Arden's lemma  Let $s, t \in \mathbb{K} \ll A^\ast \gg$, with $s$ proper.

The equation $X = sX + t$ has a unique solution $s^\ast t$.

The equation $X = Xs + t$ has a unique solution $t s^\ast$. 
Exercise  

Let $s, t \in K \langle A^* \rangle$ be proper series. Show 

$(s \cdot t)^* = s^* (t^* s^*)^*$.  

that