INVESTIGATIONS INTO ALGEBRA AND TOPOLOGY OVER NOMINAL SETS

Thesis submitted for the degree of
Doctor of Philosophy
at the University of Leicester

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June 2011

Abstract

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The last decade has seen a surge of interest in nominal sets and their applications to formal methods for programming languages. This thesis studies two subjects: algebra and duality in the nominal setting.

In the first part, we study universal algebra over nominal sets. At the heart of our approach lies the existence of an adjunction of descent type between nominal sets and a category of many-sorted sets. Hence nominal sets are a full reflective subcategory of a many-sorted variety. This is presented in Chapter 2.

Chapter 3 introduces functors over many-sorted varieties that can be presented by operations and equations. These are precisely the functors that preserve sifted colimits.

They play a central role in Chapter 4, which shows how one can systematically transfer results of universal algebra from a many-sorted variety to nominal sets. However, the equational logic obtained is more expressive than the nominal equational logic of Clouston and Pitts, respectively, the nominal algebra of Gabbay and Mathijssen. A uniform fragment of our logic with the same expressivity as nominal algebra is given.

In the second part, we give an account of duality theory in the nominal setting. Chapter 5 shows that Stone's representation theorem cannot be internalized in nominal sets. This is due to the fact that the adjunction between nominal Boolean algebras and nominal sets is no longer of descent type. We prove a duality theorem for nominal distributive lattices with a restriction operation in terms of nominal bitopological spaces. This duality restricts to duality between nominal Boolean algebras and a category of nominal topological spaces. Our notion of compactness allows for generalisation of Manes' theorem to the nominal setting.

Acknowledgements

You would not be reading this thesis but for the guidance of Alexander Kurz. I am very grateful to him for being a wonderful supervisor, for his contagious enthusiasm for discussing mathematics, for his conceptual approach to everything—the list can go on and on... *Vielen Dank für Deine ständige Ermutigung! Es bedeutet mir sehr viel!*

I would like to thank Andrew Pitts and Andrzej Murawski for agreeing to examine this thesis.

Very special thanks are due to all my other collaborators.

- To Jiří Velebil for his beautiful insights, for teaching me about adjunctions of descent type and for his constant interest in my work. I have benefited a lot from my two visits to Prague and I am very grateful to both him and Marta Bílková for their hospitality. *Děkuji moc!*
- I owe a big thank you to Jamie Gabbay for writing a thesis on something иew, for his enthusiasm and for the long hours on the phone discussing drafts of the paper, proofs, ideas, tips and tricks in nominal techniques.
- I would like to thank Tadeusz Litak for his long and intricate lectures on logic and for all the meaningful feedback.
- I would like to thank Drew Moshier: our joint work is not featured in this thesis, but I thoroughly enjoyed our collaboration. I benefited greatly from all the conversations we had in Leicester, Birmingham, Amsterdam or Tbilisi.

I profited from our exciting Birmingham-Leicester seminars. I am grateful to Achim Jung and to his team for visiting us so often and sharing their ideas. It would be an impossible task to list all the people I have enjoyed discussing or listening to at different conferences and workshops.

I have to thank all the members of our department for providing a vibrant research environment. I am grateful to José Fiadeiro for offering me the chance to stay a bit longer in Leicester and to Roy Crole for encouragement and advice.

I am grateful to all those dear friends who made my stay in Leicester a wonderful experience. Miles away, my parents, Luminiţa and Marcel, are the best in the world. To them and to Maia I say ‘Mulţumesc din inimă!’.

I wish to thank Daniel for crossing the Channel so often to visit me, for his patience and *infinite* support. *Merci de tout mon coeur!*

Contents

Chapter 1

Introduction

The deep connection between categorical algebra and formal semantics for programming languages was made explicit in the ADJ series of papers—see for example [GTWW77]—which coined the term *abstract syntax*. Simple data types are given by many-sorted signatures in the sense of universal algebra. The initial algebra for such a signature is then the *abstract syntax*. Any other algebra is a semantic domain and the unique morphism from the initial algebra assigns meaning to each construct of the language. This model of syntax provides clear principles of structural induction and recursion. But this approach proves inadequate for syntax containing binding constructs. Since binders are ubiquitous, the last decades have witnessed an intensification of research into formal frameworks for reasoning about binding.

In particular, 1999 was a good year for abstract syntax with variable binding, with two papers appearing in the proceedings of LICS: one by Gabbay and Pitts, the other by Fiore, Plotkin and Turi. Both of them provided solutions that would allow the initial algebra semantics for abstract syntax with binding. Their idea was to move away from the category Set and consider abstract syntax as an initial algebra for functors on different base categories. In [FPT99] presheaf models were used, namely $\mathsf{Set}^{\mathbb{F}}$, where $\mathbb{F}$ is the category of finite ordinals and functions between them. On the other hand, [GP99] and the subsequent journal version [GP02] proposed Fraenkel-Mostowski sets, or FM-sets. This model stemmed from the permutation model of set theory with atoms introduced by Fraenkel starting with [Fra22], and later refined by Mostowski [Mos38]. Permutation models were designed to prove the independence of the Axiom of Choice from the axioms of ZFA—Zermelo-Fraenkel set theory with atoms.

Later FM-sets have been replaced by nominal sets, which can be defined in the realm of the classical set theory [Pit03]. The core idea in nominal techniques is that names play a crucial role. Nominal sets can be thought of as sets with an additional structure: a group action of a symmetric group on names satisfying a certain 'finiteness' property. Very roughly, we can think of the elements of a nominal set as having a finite set of 'free names.' The action of a permutation on such an element is then given by renaming the 'free names'.

The nominal sets approach became successful, perhaps because it strikes the right balance between rigorous formalism and informal reasoning. This is nicely illustrated in [Pit06] where principles of structural recursion and induction are explained in nominal sets. Nominal techniques attracted interest in many areas of computer science, for example game semantics, [AGM$^+$04, Tze09], domain theory [Tur09], automata theory [KST11, BKL11], coalgebra [Kli07], functional and logic programming languages [SPG03], theorem provers [Urb08].

For this reason we believe that developing algebra and topology over nominal sets is a meaningful endeavour. Let us recall some of the work that has already been done in this direction.

Equational logic for nominal sets has been introduced in [CP07] and [GM09] and an account of Lawvere theories adapted to nominal setting was given in [Clo11]. Gabbay [Gab08] made the first step towards universal algebra over nominal sets by proving an HSP(A) theorem. But his proof requires ingenuity and ad hoc constructions, since it is based on the analogy between nominal sets and sets and even fundamental notions of universal algebra such as variables and free algebras have to be revisited.

By contrast, this thesis takes its cue from the idea that the category of nominal sets is closely related to a many-sorted variety. In a first part, our aim is to systematically obtain universal algebraic results for algebras over nominal sets.

The second part of this thesis is dedicated to a study of Stone type dualities in nominal setting. Stone type dualities have played an important role in theoretical computer science, starting with Abramsky's Domain Theory in Logical Form [Abr91] and continuing with the developments in coalgebraic modal logic [BK05]. Since coalgebras over nominal sets have already been proposed as models for name-passing calculi [FS06] and the first step towards nominal domain theory has already been taken [TW09] a natural question arises on whether Stone duality can be generalised to nominal sets.

Overview and origins of the chapters

Chapter 2 brings together different characterisations of nominal sets. The category of nominal sets is equivalent to a category of pullback preserving functors from $\mathbb{I}$ to Set, where $\mathbb{I}$ is the category whose objects are finite sets of names and arrows are injective maps between them. Another equivalent category is that of named sets considered by Montanari and Pistore for applications in the study of history dependent automata. From a category theoretic perspective, nominal sets have an extremely rich structure: they form a Grothendieck topos and have a different symmetric monoidal closed structure which allows a nice interpretation of abstraction.

However, the idea that pervades this thesis is that nominal sets are not too far removed from a presheaf category. It is precisely this connection that will allow us to systematically transfer universal algebraic results to nominal sets.

Perhaps the most important element introduced is a functor U from nominal sets to a category of many-sorted sets $\mathsf{Set}^{|\mathbb{I}|}$, where $|\mathbb{I}|$ is the discrete underlying category of $\mathbb{I}$. The idea behind this construction is that the elements of a nominal set are 'stratified' by their support. This functor, although not monadic, has a left adjoint $F : \mathsf{Set}^{|\mathbb{I}|} \to \mathsf{Nom}$. This adjunction gives rise to a monad $\mathbb{T}$ on $\mathsf{Set}^{|\mathbb{I}|}$ whose category of Eilenberg-Moore algebras is isomorphic to $\mathsf{Set}^{\mathbb{I}}$. The adjunction $F \dashv U$ is not monadic, but of descent type. As a result the comparison functor $I : \mathsf{Nom} \to \mathsf{Set}^{\mathbb{I}}$ is full and faithful. Moreover I has a left adjoint which preserves finite limits. This adjunction provides the infrastructure for transferring results from many-sorted universal algebra to nominal sets.

Chapter 3 describes functors on many-sorted varieties having finitary presentations by operations and equations. The notion of finitary presentations for functors on one-sorted varieties was introduced by [BK06]. It generalises from the category of sets the fact that any finitary functor $L : \mathsf{Set} \to \mathsf{Set}$ is a quotient

$$\coprod_{n \in \mathbb{N}} L\underline{n} \times X^n \twoheadrightarrow LX \tag{1.1}$$

of a polynomial functor where $\underline{n}$ is the set $\{1, \ldots, n\}$, and a pair (σ, f) is mapped to $Lf(\sigma)$ (f can be thought of as a map from $\underline{n}$ to X). This is a quotient because L, as a filtered colimit preserving functor, is determined by its values on finite sets. The elements of $L\underline{n}$ can be regarded as the n-ary operations presenting L, satisfying the equations corresponding to the kernel of the above map (for a full account see Adámek and Trnková [AT90, III.4.9]). To summarise, (1.1) provides a presentation (Σ_L, E_L) by operations Σ_L and equations E_L and, therefore, an *equational logic* for L-algebras: the category of L-algebras is isomorphic to the category of algebras for the signature Σ_L and equations E_L.

To generalise (1.1) from Set to an arbitrary variety, one can replace finite sets by finitely generated free algebras. But then L should be determined by its values on finitely generated free algebras, that is, L should preserve a certain class of colimits, called sifted colimits [ARV10]. As shown in [KR06], it is indeed the case that a functor L on a variety $\mathscr{A}$ has a presentation by operations and equations if and only if L preserves sifted colimits.

In this chapter we generalise the results on functors on varieties from the one-sorted to the many-sorted case. We show that, if $\mathscr{A}$ and $L : \mathscr{A} \to \mathscr{A}$ have presentation $(\Sigma_{\mathscr{A}}, E_{\mathscr{A}})$ and $(\Sigma_L, \mathscr{A}_L)$, respectively, then the category of L-algebras has presentation $(\Sigma_{\mathscr{A}} + \Sigma_L, E_{\mathscr{A}} + E_L)$. Moreover, L has a presentation iff it preserves sifted colimits. This generalisation of results from [BK06, KR06] is not difficult, but proves relevant for the developments in the next chapters. As an example we give an equational presentation for $\mathsf{Set}^{\mathbb{I}}$ and a shift functor δ that corresponds to the abstraction functor on nominal sets. A second example looks at Halmos' polyadic algebras—introduced as an algebraic semantics for first-order logic. They are characterised as algebras for a functor on the category $\mathsf{BA}^{\mathbb{F}}$ of Boolean algebra valued presheaves over the category $\mathbb{F}$ of finite ordinals.

This chapter contains results presented in the joint papers with Alexander Kurz [KP10a] and [KP08].

Chapter 4 generalises the HSP theorem to full reflective subcategories of many-sorted varieties. We then apply this result to categories of algebras given by functors on nominal sets. The central idea is to use the flexibility of the notion of functor presented by operations and equations. Given such a functor L on $\mathsf{Set}^{\mathbb{I}}$ we can apply the theory developed in Chapter 3 and conclude that the category of algebras $\mathsf{Alg}(L)$ is monadic over $\mathsf{Set}^{|\mathbb{I}|}$. If $\tilde{L}$ is a functor on nominal sets that, intuitively speaking, behaves similarly to L, the adjunction between nominal sets and $\mathsf{Set}^{\mathbb{I}}$ can be lifted to an adjunction between the categories of algebras $\mathsf{Alg}(\tilde{L})$ and $\mathsf{Alg}(L)$.

However the equational logic obtained in this fashion is more expressive than what is needed in nominal setting. We therefore introduce a fragment of our equational logic, called the uniform fragment. We prove an HSPA theorem in the style of [Gab08]: a class of $\tilde{L}$-algebras is definable by uniform equations if and only if it is closed under quotients, subalgebras, products and abstraction. This shows that the uniform fragment of equational logic has the same expressiveness as the nominal algebra of [Gab08].

We also show how to translate theories of nominal algebra [GM09] and nominal equational logic [CP07] into uniform theories and how to translate uniform theories into synthetic nominal equational logic [FH08]. We prove that these translations are semantically invariant and we obtain an HSPA theorem for nominal equational logic and a new proof of the HSPA theorem of nominal algebra [Gab08].

Chapter 4 is based on the joint paper with Alexander Kurz [KP10b].

Chapter 5 discusses Stone type dualities in nominal setting. In this chapter, adjunctions of descent type again play an important role. We recall the celebrated adjunction between BA^{op} and Set, given by the dualising object 2. This adjunction is of descent type. It gives a monad on Set, whose category of Eilenberg-Moore algebras is isomorphic to the category CHaus of compact Hausdorff topological spaces—a result due to Manes. Thus the comparison functor from BA^{op} to CHaus is full and faithful. Its essential image consists of the zero-dimensional compact Hausdorff spaces, the so called Stone spaces.

We show that the adjunction between BA^{op} and Set can be internalised in nominal sets, but is no longer of descent type. This boils down to the fact that the ultrafilter theorem does not hold in nominal sets. In fact we construct an example of a nominal Boolean algebra and a finitely-supported filter which cannot be extended to a finitely-supported ultrafilter. This shows that Stone duality fails in nominal sets.

We conclude that the Boolean algebra structure of the power object of a nominal set is not enough to generalise Stone duality in nominal sets. Gabbay pointed out a restriction operation on the power object of a nominal set, similar in spirit to the $\mathsf{И}$ quantifier. This led us to consider the category of nominal Boolean algebras with a restriction operation $\mathsf{nBA}_{\mathsf{И}}$.

The power object functor $\mathscr{P} : \mathsf{Nom} \to \mathsf{nBA}_{\mathsf{И}}^{op}$ has a right adjoint and, in this case, the adjunction is of descent type. We can generalise Manes' theorem, using the notion of compactness for nominal topological spaces we already identified in [GLP11]. We prove that the Eilenberg-Moore algebras for the induced monad on Nom are nominal topological spaces that are Hausdorff and n-compact. Thus the comparison functor from $\mathsf{nBA}_{\mathsf{И}}^{op}$ to Nom is full and faithful. Its essential image consists of the zero-dimensional n-compact Hausdorff nominal topological spaces.

Furthermore, we have generalised this result to a duality between nominal distributive lattices and a category of nominal bitopological spaces, following [BBGK10].

The starting point of this chapter is the duality for nominal Boolean algebra with restriction given in the joint work with Jamie Gabbay and Tadeusz Litak [GLP11]. The chapter generalises this work, offering a category-theoretic perspective. We are now able to explain why the notions of n-filters and n-compactness arise naturally. Furthermore, the example of Section 5.2, the generalisation of Manes' theorem and the duality for nominal distributive lattices with restriction are new.

Chapter 2

Nominal Sets and presheaf categories

Nominal sets entered the scene of theoretical computer science with the seminal paper [GP02], as a model for abstract syntax with binding that strikes the right balance between a rigorous formalism and pen-to-paper informal reasoning. This chapter takes stock of the rich structure of the category of nominal sets and presents some new observations relevant for future developments.

In Section 2.1 we recall basic definitions and results scattered across the literature. In Section 2.2 we discuss categorical constructions for nominal sets and we recall some equivalent categories. Each one of these alternative descriptions sheds light on interesting features of nominal sets. Having established that Nom is a topos, we consider, in Section 2.3 and Section 2.4, geometric morphisms relating Nom with Set and a presheaf category. The latter connection is of particular importance in this thesis.

2.1 Nominal sets - basic definitions

We fix an infinite countable set of names $\mathbb{A}$. Elements of $\mathbb{A}$ will be denoted by $a, b \ldots$. Bijective functions on $\mathbb{A}$ are called *permutations* on $\mathbb{A}$. Examples of permutations include *transpositions*, that is, functions of the form $(a\ b)$ which swap a and b, and fix the other elements of $\mathbb{A}$. The permutations on $\mathbb{A}$ form a group, where multiplication is the usual composition of functions and the neutral element is the identity on $\mathbb{A}$. Let $\mathfrak{S}(\mathbb{A})$ denote the subgroup of permutations on $\mathbb{A}$ generated by transpositions. A permutation π on $\mathbb{A}$ can be expressed as a finite product of transpositions if and only if the set

$$\{a \in \mathbb{A} \mid \pi(a) \neq a\} \tag{2.1}$$

is finite. These permutations are called *finitely supported*.

Definition 2.1.1. A $\mathfrak{S}(\mathbb{A})$-*action* is a pair $(\mathbb{X}, \cdot)$ consisting of a set $\mathbb{X}$ and a map $\cdot : \mathfrak{S}(\mathbb{A}) \times \mathbb{X} \to \mathbb{X}$ such that for all $x \in \mathbb{X}$ and for all $\pi, \pi' \in \mathfrak{S}(\mathbb{A})$

$$\begin{aligned} &\mathrm{id} \cdot x = x \\ &(\pi' \circ \pi) \cdot x = \pi' \cdot (\pi \cdot x) \end{aligned} \tag{2.2}$$

Example 2.1.2. The set of names can be equipped with the 'evaluation' action defined by

$$\pi \cdot a = \pi(a). \tag{2.3}$$

$(\mathbb{A}, \cdot)$ is a $\mathfrak{S}(\mathbb{A})$-action.

Example 2.1.3. The powerset of $\mathbb{A}$ equipped with the pointwise action, defined by

$$\pi \cdot X = \{\pi(x) \mid x \in X\} \tag{2.4}$$

for all $X \subseteq \mathbb{A}$, is a $\mathfrak{S}(\mathbb{A})$-action.

Example 2.1.4. $(\mathfrak{S}(\mathbb{A}), *)$ is a $\mathfrak{S}(\mathbb{A})$-action, where $* : \mathfrak{S}(\mathbb{A}) \times \mathfrak{S}(\mathbb{A}) \to \mathfrak{S}(\mathbb{A})$ is the conjugation action defined by

$$\tau * \pi = \tau \pi \tau^{-1}. \tag{2.5}$$

Definition 2.1.5. Consider a $\mathfrak{S}(\mathbb{A})$-action $(\mathbb{X}, \cdot)$ and an element $x \in \mathbb{X}$. We say that a set of names $A \subseteq \mathbb{A}$ *supports* x if for all $a, b \in \mathbb{A} \setminus A$ we have $(a\ b) \cdot x = x$.

We say that x is *finitely supported* if there exists finite $A \subseteq \mathbb{A}$ that supports x.

Remark 2.1.6. Given a set of names A, let *fix*(A) denote the set of permutations in $\mathfrak{S}(\mathbb{A})$ that fix the elements of A. Given an element x of a $\mathfrak{S}(\mathbb{A})$-action, the *stabilizer* of x, denoted by $\mathsf{Stab}(x)$, is the set $\{\pi \in \mathfrak{S}(\mathbb{A}) \mid \pi \cdot x = x\}$. Both *fix*$(A)$ and $\mathsf{Stab}(x)$ are subgroups of $\mathfrak{S}(\mathbb{A})$. Notice that x is supported by a set A when

$$\forall \pi.\ \pi \in \mathit{fix}(A) \quad \Rightarrow \quad \pi \cdot x = x \tag{2.6}$$

This follows from the fact that a permutation is in *fix*(A) if and only if it can be written as a product of transpositions $\prod_{i=1}^{n}(a_i\ b_i)$ such that $a_i, b_i \notin A$.

Definition 2.1.7. A *nominal set* is a $\mathfrak{S}(\mathbb{A})$-action $(\mathbb{X}, \cdot)$ such that each element of $\mathbb{X}$ is finitely supported.

Example 2.1.8. $(\mathbb{A}, \cdot)$ as defined in Example 2.1.2 is a nominal set because each name a is supported by $\{a\}$.

Example 2.1.9. The $\mathfrak{S}(\mathbb{A})$-action on the powerset of names, described in Example 2.1.3, is not a nominal set. A subset of $\mathbb{A}$ is finitely supported if and only if it is finite or cofinite. But the set $\mathscr{P}_{fin}(\mathbb{A})$ of finite subsets of $\mathbb{A}$ and the set $\mathscr{P}(\mathbb{A})$ of finite and cofinite subsets of $\mathbb{A}$ are nominal sets when equipped with the pointwise action. More on the power object construction in Section 5.3.

Example 2.1.10. $(\mathfrak{S}(\mathbb{A}), *)$ defined in Example 2.1.4 is a nominal set, as each permutation $\pi \in \mathfrak{S}(\mathbb{A})$ is supported by the finite set $\{a \mid \pi(a) \neq a\}$.

Definition 2.1.11. A morphism between two nominal sets $f : (\mathbb{X}, \cdot) \to (\mathbb{Y}, \cdot)$ is a map $f : \mathbb{X} \to \mathbb{Y}$ that is *equivariant*, that is,

$$\forall x \in \mathbb{X}.\ \forall \pi \in \mathfrak{S}(\mathbb{A}) \quad f(\pi \cdot x) = \pi \cdot f(x) \tag{2.7}$$

Next we recall some fundamental properties of the notion of support.

Proposition 2.1.12. Let $\mathbb{X}$ be a nominal set and let $x \in \mathbb{X}$. Then the following hold:

1. If $S, S' \in \mathscr{P}_{fin}(\mathbb{A})$ support x then $S \cap S'$ supports x.
2. There exists a least *finite* set $\mathsf{supp}(x) \in \mathscr{P}_{fin}(\mathbb{A})$ that supports x.
3. $\mathsf{supp}(\pi \cdot x) = \pi \cdot \mathsf{supp}(x)$ for all $\pi \in \mathfrak{S}(\mathbb{A})$.
4. If $\pi, \tau \in \mathfrak{S}(\mathbb{A})$ satisfy $\pi_{|\mathsf{supp}(x)} = \tau_{|\mathsf{supp}(x)}$ then $\pi \cdot x = \tau \cdot x$.
5. If S supports x and $S \subseteq S'$ then S' supports x.
6. For all equivariant maps $f : (\mathbb{X}, \cdot) \to (\mathbb{Y}, \cdot)$ we have $\mathsf{supp}(f(x)) \subseteq \mathsf{supp}(x)$.

Complete proofs can be found in [GP02]. In order to prove 1 assume $a \notin S$ and $b \notin S'$. Consider $c \notin S \cup S'$. Such c exists because both S and S' are finite. Observe that $(a\ b) \cdot x = (a\ c)(b\ c)(a\ c) \cdot x$. But $(a\ c)$ and $(b\ c)$ fix x since $a, c \notin S$, respectively $b, c \notin S'$. Thus $(a\ b) \cdot x = x$ and $S \cap S'$ supports x.

Point 2 follows immediately from 1. Notice the importance of the word 'finite' in 2. In general, an element of a nominal set does not have a least supporting set of elements. For example, the element $\{a\}$ of $\mathscr{P}_{fin}(\mathbb{A})$ has as least finite support the set $\{a\}$, but it is also supported by $\mathbb{A} \setminus \{a\}$.

To prove 3, consider $a, b \notin \pi \cdot \mathsf{supp}(x)$. Then $\pi^{-1}(a\ b)\pi \in \mathit{fix}(\mathsf{supp}(x))$, hence $(a\ b)\pi \cdot x = \pi \cdot x$. This shows that $\mathsf{supp}(\pi \cdot x) \subseteq \pi \cdot \mathsf{supp}(x)$. Using this for $\pi^{-1} \cdot x$, we can prove that the reverse inclusion also holds.

Point 4 follows from Remark 2.1.6 applied to $\pi\tau^{-1}$ while 5 and 6 are immediate consequences of Definition 2.1.5 and Definition 2.1.11.

Consider an element x of a nominal set. From Remark 2.1.6 it follows that $\mathit{fix}(\mathsf{supp}(x))$ is a subgroup of $\mathsf{Stab}(x)$. The next lemma shows that $\mathit{fix}(\mathsf{supp}(x))$ is in fact a normal subgroup of $\mathsf{Stab}(x)$.

Lemma 2.1.13. Let $\mathbb{X}$ be a nominal set and let $x \in \mathbb{X}$. Let $\mathfrak{S}(\mathsf{supp}(x))$ denote the group of permutations of $\mathsf{supp}(x)$. Define $\phi : \mathsf{Stab}(x) \to \mathfrak{S}(\mathsf{supp}(x))$ by

$$\pi \longmapsto \pi_{|\mathsf{supp}(x)}.$$

Then ϕ is a well defined group homomorphism and the kernel of ϕ is $\mathit{fix}(\mathsf{supp}(x))$. Consequently, the quotient group $\mathsf{Stab}(x)/\mathit{fix}(\mathsf{supp}(x))$ is isomorphic to a subgroup of $\mathfrak{S}(\mathsf{supp}(x))$.

Proof. Given $\pi \in \mathsf{Stab}(x)$ we have that $\mathsf{supp}(\pi \cdot x) = \mathsf{supp}(x)$. By point 3 of Proposition 2.1.12 it follows that $\pi \cdot \mathsf{supp}(x) = \mathsf{supp}(x)$. Therefore ϕ is well defined. It is easy to check that ϕ is a group homomorphism and that the kernel of ϕ is $\mathit{fix}(\mathsf{supp}(x))$. Hence $\mathit{fix}(\mathsf{supp}(x))$ is a normal subgroup of $\mathsf{Stab}(x)$. The last part of the lemma follows from the First Isomorphism Theorem for groups. □

For any element x of a nominal set we have an infinite, actually cofinite, supply of names outside its support, which we can think of as not interfering with x, or being *fresh* for x:

Definition 2.1.14. Given a nominal set $(\mathbb{X}, \cdot)$, $x \in \mathbb{X}$ and $a \in \mathbb{A}$, we say that a is fresh for x and write $a\#x$ when $a \notin \mathsf{supp}(x)$.

Freshness is an important concept in nominal techniques. In [GP01] Gabbay and Pitts introduced the freshness quantifier, that can express concisely the ability of finding at least one or, as we will see, cofinitely many fresh names satisfying certain properties:

Definition 2.1.15. Given a higher-order logic formula $\phi(a, x_1, \ldots, x_n)$, describing properties of elements of a nominal set X and of names in $\mathbb{A}$, we say

$$\text{И} a.\phi(a, x_1, \ldots, x_n)$$

when $\{a \in \mathbb{A} \mid \phi(a, x_1, \ldots, x_n)\}$ is cofinite, or, equivalently, when $\phi(a, x_1, \ldots, x_n)$ holds for cofinitely many names a.

In nominal proofs it is enough to consider a fresh name and prove that a certain property holds for this particular choice. The next theorem, that goes back to [GP01], says that the property will hold automatically for any other fresh name considered.

Theorem 2.1.16. The following are equivalent:

$\exists a \in \mathbb{A}. a\#x_1,\dots,x_n \wedge \phi(a,x_1,\dots,x_n)$.

$\forall a \in \mathbb{A}. a\#x_1,\dots,x_n \Rightarrow \phi(a,x_1,\dots,x_n)$.

$\reflectbox{N} a.\phi(a,x_1,\dots,x_n)$.

Remark 2.1.17. The $\reflectbox{N}$ quantifier allows for a concise way of expressing freshness. Given an element x of a nominal set and $a \in \mathbb{A}$ we have

$$a\#x \iff \reflectbox{N} b.(a\ b)\cdot x = x$$

Another important concept introduced in [GP01] is that of atom-abstractions. Given a nominal set $\mathbb{X}$ we can define its abstraction $[\mathbb{A}]\mathbb{X}$ as follows. We define an equivalence relation $(a,x)\sim(b,y)$ on $\mathbb{A}\times\mathbb{X}$ by $\reflectbox{N} c.(a\ c)\cdot x = (b\ c)\cdot y$. It can be shown that the equivalence class of (a,x) w.r.t. $\sim$, denoted by $[a]x$, is the set

$$\{(a,x)\} \cup \{(b,(a\ b)\cdot x) \mid b\#x\}.$$

Define $[\mathbb{A}]\mathbb{X}$ to be the set of equivalence classes of $\sim$ with the permutation action given by $\pi\cdot[a]x = [\pi(a)](\pi\cdot x)$. This is indeed a nominal set, and one can prove further that

$$\mathsf{supp}([a]x) = \mathsf{supp}(x)\setminus\{a\}.$$

2.2 The category of nominal sets

Nominal sets and equivariant maps form a category, denoted by Nom. In this section we will analyse the rich categorical structure of Nom. We set about describing finite limits.

Lemma 2.2.1. Nom is a cartesian category.

Proof. Nom has a terminal object 1, that is, a singleton equipped with the trivial action. Nom has pullbacks of pairs of morphisms. Given $f:\mathbb{X}\to\mathbb{Z}$ and $g:\mathbb{Y}\to\mathbb{Z}$, the pullback of (f,g) is computed as follows. Consider

$$\mathbb{T} = \{(x,y)\in\mathbb{X}\times\mathbb{Y} \mid f(x) = g(y)\} \tag{2.8}$$

with the $\mathfrak{S}(\mathbb{A})$-action defined by $\pi\cdot(x,y) = (\pi\cdot x, \pi\cdot y)$. Each pair (x,y) is supported by the finite set $\mathsf{supp}(x)\cup\mathsf{supp}(y)$, thus $\mathbb{T}$ is a nominal set. The projections $p_{\mathbb{X}}:\mathbb{T}\to\mathbb{X}$ and $p_{\mathbb{Y}}:\mathbb{T}\to\mathbb{Y}$, given by $(x,y)\mapsto x$, respectively $(x,y)\mapsto y$, are equivariant and $fp_{\mathbb{X}} = gp_{\mathbb{Y}}$. Furthermore, for all equivariant maps $h:\mathbb{U}\to\mathbb{X}$ and $j:\mathbb{U}\to\mathbb{Y}$ such that $fh = gj$ there exists a unique equivariant map $k:\mathbb{U}\to\mathbb{T}$, defined by $k(u) = (h(u), j(u))$, such that $h = p_{\mathbb{X}}k$ and $j = p_{\mathbb{Y}}k$.

Therefore Nom has all finite limits. □

We will see that Nom is actually complete. While the product of two nominal sets $(\mathbb{X},\cdot)$ and $(\mathbb{Y},\cdot)$ is the nominal set $(\mathbb{X}\times\mathbb{Y},\cdot)$ where $\pi\cdot(x,y)=(\pi\cdot x,\pi\cdot y)$, arbitrary products are more complicated. Consider a family of nominal sets $(\mathbb{X}_i)_{i\in I}$. We can equip the set of all tuples $\{(x_i)_{i\in I} \mid x_i\in\mathbb{X}_i\}$ with the pointwise action given by

$$\pi\cdot(x_i)_{i\in I}=(\pi\cdot x_i)_{i\in I}. \tag{2.9}$$

This is a $\mathfrak{S}(\mathbb{A})$-action, but some tuples may not be finitely supported, as the following example shows:

Example 2.2.2. Consider as indexing set I the set of natural numbers $\mathbb{N}$ and for all $i\in\mathbb{N}$ let $\mathbb{X}_i$ be the nominal set of names $\mathbb{A}$. Consider an ordering of the names in $\mathbb{A}$: $a_1,a_2,\ldots$ Then the tuple $(a_{2i})_{i\in\mathbb{N}}$ is not finitely supported.

Lemma 2.2.3. The product of a family of nominal sets $(\mathbb{X}_i,\cdot_i)_{i\in I}$ is the nominal set $\prod\limits_{i\in I}(\mathbb{X}_i,\cdot_i)$ of tuples of the form $(x_i)_{i\in I}$ that are finitely supported w.r.t. the action of (2.9).

Proof. By construction $\prod\limits_{i\in I}(\mathbb{X}_i,\cdot_i)$ is a nominal set and the projections

$$p_i:\prod_{i\in I}(\mathbb{X}_i,\cdot_i)\to(\mathbb{X}_i,\cdot_i)$$

mapping a tuple to its i component are equivariant. We need to check that the usual universal property is satisfied. Assume $(\mathbb{T},\cdot)$ is a nominal set and $f_i:\mathbb{T}\to\mathbb{X}_i$ are equivariant maps. We define $k:\mathbb{T}\to\prod\limits_{i\in I}(\mathbb{X}_i,\cdot_i)$ by $k(t)=(f_i(t))_{i\in I}$. This is well defined because, by point 6 of Proposition 2.1.12, each $f_i(t)$ is supported by $\mathsf{supp}(t)$, hence $(f_i(t))_{i\in I}$ is supported by $\mathsf{supp}(t)$. □

Lemma 2.2.4. Nom is a cartesian closed category.

Proof. We have to show that for all nominal sets $(\mathbb{X},\cdot_{\mathbb{X}})$ the functor $-\times\mathbb{X}:\mathsf{Nom}\to\mathsf{Nom}$ has a right adjoint $[\mathbb{X},-]:\mathsf{Nom}\to\mathsf{Nom}$. Given a nominal set $(\mathbb{Y},\cdot_{\mathbb{Y}})$ define a $\mathfrak{S}(\mathbb{A})$-action on the set of all Set-functions from $\mathbb{X}$ to $\mathbb{Y}$:

$$\forall x\in\mathbb{X}\quad(\pi\cdot f)(x)=\pi\cdot_{\mathbb{Y}}f(\pi^{-1}\cdot_{\mathbb{X}}x) \tag{2.10}$$

Define $[\mathbb{X},\mathbb{Y}]$ to be the nominal set of all functions $f:\mathbb{X}\to\mathbb{Y}$ such that f is finitely supported w.r.t. the $\mathfrak{S}(\mathbb{A})$-action in (2.10). It is not difficult to check that this construction extends to a right adjoint for $-\times\mathbb{X}$. Indeed, the evaluation map $ev:[\mathbb{X},\mathbb{Y}]\times\mathbb{X}\to\mathbb{Y}$ given by

$$(f,x)\mapsto f(x)$$

is equivariant, natural in $\mathbb{Y}$ and universal from $- \times \mathbb{X}$ to $\mathbb{Y}$. The latter means that for all equivariant maps $h : \mathbb{Z} \times \mathbb{X} \to \mathbb{Y}$ there exists a unique map $\bar{h} : \mathbb{Z} \to [\mathbb{X}, \mathbb{Y}]$ such that $h = ev \circ (\bar{h} \times id_{\mathbb{X}})$. For each $z \in \mathbb{Z}$, $\bar{h}(z) \in [\mathbb{X}, \mathbb{Y}]$ is defined by

$$\bar{h}(z)(x) = h(z, x). \tag{2.11}$$

It is straightforward to check that $\bar{h}(z)$ is supported by $\mathsf{supp}(z)$. □

Lemma 2.2.5. Nom has a subobject classifier $1 \to \Omega$, where Ω is the two-element nominal set with trivial action.

From the lemmas above it follows that Nom is a topos. In fact Nom is equivalent to a Grothendieck topos, the so called Schanuel topos. We give more details on this equivalence in Section 2.4.

Being a topos, Nom has a very rich structure:

1. Nom is complete and cocomplete.
2. Nom has a epi-mono factorisation system.
3. As with any topos, we have a power object construction defined by $\mathscr{P}(\mathbb{X}) = [\mathbb{X}, \Omega]$. Spelling out the definition of the internal hom functor, we obtain a description of the power object in elementary terms: $Y \in \mathscr{P}(\mathbb{X})$ if and only if $Y \subseteq \mathbb{X}$ is finitely supported w.r.t. the action on the powerset of $\mathbb{X}$ given by
$$\pi \cdot Y = \{\pi \cdot y \mid y \in Y\} \tag{2.12}$$
In Section 5.3 we will have a closer look at the power object construction in nominal sets. It is worth mentioning Paré's result, [Par74], that states that the functor $\mathscr{P} : \mathsf{Nom}^{op} \to \mathsf{Nom}$ is monadic. This means that $\mathscr{P}$ has a left adjoint and Nom^{op} is equivalent to the category of Eilenberg-Moore algebras for the resulting monad. See [MLM94, p. 178] for a definition of monadic functors and [MLM94, Theorem 1, p.180] for a proof of the fact that $\mathscr{P}^{op}$ is the left adjoint of $\mathscr{P}$.

2.3 Functors between nominal sets and sets

In the next chapters we will use the connection between nominal sets and many-sorted sets to systematically obtain universal algebraic results. One may ask why we cannot use for this purpose the relation between nominal sets and sets. In this section we will see that the functors that we may consider from Nom to Set do not have the satisfactory properties necessary to transfer universal algebraic concepts.

The first such functor that comes to mind is probably the forgetful functor $V : \mathsf{Nom} \to \mathsf{Set}$ that maps a nominal set $(\mathbb{X}, \cdot)$ to its underlying carrier set $\mathbb{X}$. However a free construction is missing:

Remark 2.3.1. From Example 2.2.2 and Lemma 2.2.3 it follows that V does not preserve arbitrary limits. Hence V does not have a left adjoint.

But we will see next that V is the inverse image of a geometric morphism from Set to Nom. Recall that geometric morphisms are maps between toposes, see [MLM94, Definition 1, p. 348] for a definition. We can construct a right adjoint for V as follows. For each set X we consider the set $\prod\limits_{\sigma \in \mathfrak{S}(\mathbb{A})} X$ equipped with a $\mathfrak{S}(\mathbb{A})$-action given by $\tau * (x_\sigma)_\sigma = (x_{\sigma\tau})_\sigma$. The finitely supported elements of $\prod\limits_{\sigma \in \mathfrak{S}(\mathbb{A})} X$ w.r.t. this action form a nominal set, denoted by RX. Denote by $p_\sigma : RX \to X$ the natural projections. Notice that for all $u \in RX$ we have

$$p_\sigma(\tau * u) = p_{\sigma\tau}(u). \tag{2.13}$$

Given a function $f : X \to Y$ define $Rf : RX \to RY$ by

$$Rf((x_\sigma)_\sigma) = (f(x_\sigma))_\sigma. \tag{2.14}$$

The next lemma shows that we can extend R to a functor from Set to Nom.

Lemma 2.3.2. $V \dashv R : \mathsf{Set} \to \mathsf{Nom}$ is a geometric morphism.

Proof. The functor V preserves finite limits, as we can easily see from the proof of Lemma 2.2.1. It remains to check that we have indeed an adjunction $V \dashv R$. For each nominal set $\mathbb{X}$, consider the map $\eta_{\mathbb{X}} : \mathbb{X} \to RV\mathbb{X}$ defined by

$$x \mapsto (\sigma \cdot x)_\sigma \tag{2.15}$$

We can easily check that $\eta_{\mathbb{X}}$ is natural in $\mathbb{X}$ and that for all $x \in \mathbb{X}$ and $\tau \in \mathfrak{S}(\mathbb{A})$ we have $\eta(\tau \cdot x) = \tau * \eta(x) = (\sigma\tau \cdot x)_\sigma$. Further, $\eta_{\mathbb{X}}$ is universal from $\mathbb{X}$ to R, that is, for any set Y and any equivariant map $h : \mathbb{X} \to RY$, we can find $\bar{h} : V\mathbb{X} \to Y$ such that

$$h = R\bar{h} \circ \eta_{\mathbb{X}}. \tag{2.16}$$

Indeed, define $\bar{h}$ by $\bar{h}(x) = p_{\mathrm{id}}(h(x))$. The commutativity of (2.16) amounts to the fact that $p_{\mathrm{id}}(h(\sigma \cdot x)) = p_\sigma(h(x))$. This follows from the equivariance of h and (2.13). $\square$

As a left adjoint, V preserves all colimits. Moreover, V creates colimits, thus:

Corollary 2.3.3. Colimits in Nom are computed as in Set.

Next we will look at another functor from Nom to Set which, although has a left adjoint, fails to be faithful.

Since Nom is equivalent to a Grothendieck topos, there exists a geometric morphism from Nom to Set that corresponds to the global sections functor and its left adjoint, to the constant sheaf functor. The global sections functor is just the hom-functor out of the terminal object $\mathsf{Nom}(1,-) : \mathsf{Nom} \to \mathsf{Set}$. Its left adjoint $\Delta : \mathsf{Set} \to \mathsf{Nom}$ is given by

$$\Delta X = (X, \cdot) \tag{2.17}$$

where $\cdot$ is the trivial action on X.

Notice that, given a nominal set $(\mathbb{X}, \cdot)$, we have that $\mathsf{Nom}(1, \mathbb{X})$ is isomorphic to the set of elements of $\mathbb{X}$ that are supported by the empty set. We will call such elements the *equivariant* elements of $\mathbb{X}$.

As with any cartesian closed category, this functor relates the exponential and the hom functors. More precisely, we have $\mathsf{Nom}(1, [\mathbb{X}, \mathbb{Y}]) = \mathsf{Nom}(\mathbb{X}, \mathbb{Y})$.

The functor $\mathsf{Nom}(1,-)$ is not faithful as the next example shows.

Example 2.3.4. Consider the power object of the nominal set of names $\mathscr{P}(\mathbb{A})$ and $f : \mathscr{P}(\mathbb{A}) \to \mathscr{P}(\mathbb{A})$ that maps proper subsets of $\mathbb{A}$ to their complement, but maps $\emptyset$ to $\emptyset$ and $\mathbb{A}$ to $\mathbb{A}$. f is an equivariant map obviously different than $\mathrm{id}_{\mathscr{P}(\mathbb{A})}$. However $\mathsf{Nom}(1, f) = \mathsf{Nom}(1, \mathrm{id}_{\mathscr{P}(\mathbb{A})})$.

Further, Δ has a left adjoint $\Pi_0 : \mathsf{Nom} \to \mathsf{Set}$ defined as follows.

Definition 2.3.5. Given a nominal set $(\mathbb{X}, \cdot)$ and $x \in \mathbb{X}$, the orbit of x is defined as

$$O_x = \{\pi \cdot x \mid \pi \in \mathfrak{S}(\mathbb{A})\} \tag{2.18}$$

Two elements x, y have the same orbit provided that there exists $\pi \in \mathfrak{S}(\mathbb{A})$ such that $x = \pi \cdot y$. The functor Π_0 maps a nominal set to the set of its orbits. Given an equivariant map $f : \mathbb{X} \to \mathbb{Y}$, we put $\Pi_0 f(O_x) = O_{f(x)}$. This is well defined since $O_x = O_y$ implies that $O_{f(x)} = O_{f(y)}$.

Lemma 2.3.6. We have two adjunctions $\Pi_0 \dashv \Delta \dashv \mathsf{Nom}(1,-)$:[1]

$$\begin{tikzcd} \mathsf{Nom} \arrow[r, bend left, "\Pi_0"] \arrow[r, bend right, "\mathsf{Nom}(1,-)"'] & \mathsf{Set} \arrow[l, "\Delta" description] \end{tikzcd} \tag{2.19}$$

---

[1] The existence of the left adjoint for Δ shows that Nom is a locally connected topos.

Proof. We will give the unit and counit for both adjunctions.

$\eta^1_{\mathbb{X}} : \mathbb{X} \to \Delta\Pi_0\mathbb{X}$ is given by

$$x \mapsto O_x$$

while $\varepsilon^1_X : \Pi_0\Delta X \to X$ is given by

$$O_x \mapsto x.$$

It is easy to check that η^1 and ε^1 are well defined and satisfy the usual triangular identities.

$\eta^2_X : X \to \mathsf{Nom}(1, \Delta X)$ is given by $x \mapsto x$. This is well defined since each $x \in \Delta X$ is equivariant. The counit $\varepsilon^2_{\mathbb{X}} : \Delta\mathsf{Nom}(1, \mathbb{X}) \to \mathbb{X}$ is given by $x \mapsto x$. It is easy to check that $\varepsilon^2_{\mathbb{X}}$ is equivariant and that the usual triangular diagrams commute. □

Proposition 2.3.7. $(\mathbb{X}, \cdot)$ is a finitely presentable object in Nom if and only if $(\mathbb{X}, \cdot)$ has finitely many orbits.

Proof. Let $(\mathbb{X}, \cdot)$ be a finitely presentable object in Nom, J a filtered category and $F : J \to \mathsf{Set}$ a diagram with colimit $\mathrm{colim}_{j \in J} F(j)$. Then we have

$$\begin{array}{rcll} \mathsf{Set}(\Pi_0\mathbb{X}, \mathrm{colim}_{j\in J} F(j)) & \simeq & \mathsf{Nom}(\mathbb{X}, \Delta\mathrm{colim}_{j\in J} F(j)) & (\text{ by Lemma 2.3.6}) \\ & \simeq & \mathsf{Nom}(\mathbb{X}, \mathrm{colim}_{j\in J} \Delta F(j)) & (\Delta \text{ is left adjoint}) \\ & \simeq & \mathrm{colim}_{j\in J} \mathsf{Nom}(\mathbb{X}, \Delta F(j)) & (\mathbb{X} \text{ is f.p.}) \\ & \simeq & \mathrm{colim}_{j\in J} \mathsf{Set}(\Pi_0\mathbb{X}, F(j)) & (\text{ by Lemma 2.3.6}) \end{array}$$

This shows that $\Pi_0\mathbb{X}$ is finitely presentable in Set, hence $\mathbb{X}$ has finitely many orbits.

Conversely, we first prove that each nominal set $\mathbb{X}$ with exactly one orbit is finitely presentable. Consider a filtered diagram $F : J \to \mathsf{Nom}$ and let ι_j denote the injections from $F(j)$ to $\mathrm{colim}_{j\in J} F(j)$. We show that each equivariant $f : \mathbb{X} \to \mathrm{colim}_{j\in J} F(j)$ factors through ι_k for some $k \in J$.

Using Corollary 2.3.3 we have that $\mathrm{colim}_{j\in J} F(j)$ is computed as a quotient

$$\coprod_{j\in J} F(j)/\sim$$

where for $u_1 \in F(j_1)$ and $u_2 \in F(j_2)$ we have $u_1 \sim u_2$ if and only if there exist morphisms $l_1 : j_1 \to j$ and $l_2 : j_2 \to j$ such that $F(l_1)(u_1) = F(l_2)(u_2)$. The equivalence class of $u \in F(j)$ is denoted by $\widehat{u}$. The permutation action on $\mathrm{colim}_{j\in J} F(j)$ is given by $\pi \cdot \widehat{u} = \widehat{\pi \cdot u}$.

Consider $x \in \mathbb{X}$ and let $f(x) = \widehat{u} \in \mathrm{colim}_{j\in J} F(j)$. First we will show that there exists $j \in J$ and $v \in F(j)$ such that $v \sim u$ and $\mathsf{supp}(v) = \mathsf{supp}(\widehat{u})$. Since ι_j are equivariant maps, point 6 of Proposition 2.1.12 implies that for all v with $v \in \widehat{u}$ we have $\mathsf{supp}(\widehat{u}) \subseteq \mathsf{supp}(v)$.

Assume by contradiction that for all $v \in \widehat{u}$ we have that $\mathsf{supp}(v) \neq \mathsf{supp}(\widehat{u})$ and consider $v \in F(j)$ for some j such that $\widehat{v} = \widehat{u}$ and $\mathsf{supp}(v) \setminus \mathsf{supp}(\widehat{u})$ has the minimum number of elements. Let $a \in \mathsf{supp}(v) \setminus \mathsf{supp}(\widehat{u})$. Then

$$\begin{array}{lll} a\#\widehat{v} & \iff \text{И}b.(a\ b)\cdot \widehat{v} = \widehat{v} & \text{(by Remark 2.1.17)} \\ & \iff \text{И}b.(a\ b)\cdot v \sim v & \\ & \iff \exists b\#a, v.\exists l \in J(j,j').F(l)((a\ b)\cdot v) = F(l)(v) & \text{(by Thorem 2.1.16)} \\ & \iff \exists l \in J(j,j').\text{И}b.(a\ b)\cdot F(l)(v) = F(l)(v) & \\ & \iff \exists l \in J(j,j').a\#F(l)(v) & \text{(by Remark 2.1.17)} \end{array}$$

Since $a \in \mathsf{supp}(v)$ we have that $\mathsf{supp}(F(l)(v)) \subsetneq \mathsf{supp}(v)$. Since $F(l)(v) \sim v$ and $\mathsf{supp}(F(l)(v)) \setminus \mathsf{supp}(\widehat{u}) \subsetneq \mathsf{supp}(v) \setminus \mathsf{supp}(\widehat{u})$ we obtain a contradiction.

Hence there exists $v \in F(j)$ such that $v \sim u$ and $\mathsf{supp}(v) = \mathsf{supp}(\widehat{u})$.

By Lemma 2.1.13 we have that each $\pi \in \mathsf{Stab}(x)$ can be written as the composition of a permutation in $\mathit{fix}(\mathsf{supp}(x))$ and a permutation in a finite group $\{\sigma_1, \ldots, \sigma_n\}$ of permutations of $\mathsf{supp}(x)$. Since $f(x) = \widehat{v}$, it follows that $\sigma_i \cdot v \sim v$ for all $1 \leq i \leq n$. Using the fact that J is a filtered category there exists a morphism $l : j \to k$ in J such that $F(l)(v) = F(l)(\sigma_i \cdot v)$ for all $1 \leq i \leq n$. Put $w = F(l)(v) \in F(k)$. Then $\sigma_i \cdot w = w$ for all $1 \leq i \leq n$. Moreover we have that $w \sim v$ and $\mathsf{supp}(w) = \mathsf{supp}(\widehat{v}) \subseteq \mathsf{supp}(x)$, hence $\mathit{fix}(\mathsf{supp}(x)) \subseteq \mathsf{Stab}(w)$. Thus

$$\mathsf{Stab}(x) \subseteq \mathsf{Stab}(w). \tag{2.20}$$

We define $\bar{f} : \mathbb{X} \to F(k)$ by $f(\pi \cdot x) = \pi \cdot w$. The fact that $\bar{f}$ is well defined follows by (2.20). It is easy to check that $\bar{f}$ is equivariant and that $f = \iota_k \circ \bar{f}$. This shows that for a one-orbit nominal set $\mathbb{X}$ we have:

$$\mathsf{Nom}(\mathbb{X}, \mathrm{colim}_{j\in J} F(j)) \simeq \mathrm{colim}_{j\in J} \mathsf{Nom}(\mathbb{X}, F(j)).$$

Next, if $\mathbb{X}$ has finitely many orbits, we have $\mathbb{X} \simeq \coprod_{i=1}^{n} \mathbb{X}_i$ with $\mathbb{X}_i$ being nominal sets with one orbit. The conclusion follows since by [ARV10, Lemma 5.11] finitely presentable objects are closed under finite colimits. □

Orbits also play an important role in understanding the connection between nominal sets and named sets, [GMM06, FS06]. Named sets were introduced in [FMP02] and can be regarded as more efficient implementations of nominal sets, useful for history-dependent automata [MP05]: finitely-presentable nominal sets correspond to finite named sets.

2.4 Nominal sets and presheaf categories

In this section we emphasise the connection between nominal sets and a category of presheaves $\mathsf{Set}^{\mathbb{I}}$. In a certain sense, $\mathsf{Set}^{\mathbb{I}}$ is the closest many-sorted variety to nominal sets. We start by giving a forgetful functor from Nom to a category of many-sorted sets. Although this functor is not monadic, it is at the heart of our approach to understanding algebra over nominal sets.

Definition 2.4.1. Let $\mathbb{I}$ be the category whose objects are finite subsets of $\mathbb{A}$ and morphisms are injective maps between them.

Let $\mathbb{B}$ be the subcategory of $\mathbb{I}$ whose objects are finite subsets of $\mathbb{A}$ and morphisms are bijective functions.

Let $|\mathbb{I}|$ denote the underlying discrete subcategory of $\mathbb{I}$.

The notion of support (Definition 2.1.5) allows us to 'stratify' the elements of a nominal set. We have a forgetful functor $U : \mathsf{Nom} \to \mathsf{Set}^{|\mathbb{I}|}$ defined by

$$U\mathbb{X}(S) = \{x \in \mathbb{X} \mid S \text{ supports } x\} \tag{2.21}$$

for all $S \in |\mathbb{I}|$. Given equivariant $f : \mathbb{X} \to \mathbb{Y}$ define $Uf : U\mathbb{X} \to U\mathbb{Y}$ by

$$U\mathbb{X}(S) \ni x \mapsto f(x) \in U\mathbb{Y}(S) \tag{2.22}$$

This is well defined by point 6 of Proposition 2.1.12.

We now show that the functor U has a left adjoint $F : \mathsf{Set}^{|\mathbb{I}|} \to \mathsf{Nom}$. Given $X \in \mathsf{Set}^{|\mathbb{I}|}$, the carrier set of FX is

$$FX = \coprod_{S,T \in |\mathbb{I}|} \mathbb{B}(S,T) \times X(S).$$

For $\sigma \in \mathbb{B}(S,T)$ and $\pi \in \mathfrak{S}(\mathbb{A})$ we define $\pi \bullet \sigma : S \to (\pi \cdot T)$ by

$$s \mapsto \pi(\sigma(s)).$$

Above, $\pi \cdot T$ is computed in the nominal set $\mathscr{P}_{fin}(\mathbb{A})$, as in Example 2.1.9.

We define a $\mathfrak{S}(\mathbb{A})$-action on FX by

$$\pi \cdot (\sigma, x) = (\pi \bullet \sigma, x)$$

for all $\pi \in \mathfrak{S}(\mathbb{A})$ and $(\sigma, x) \in FX$. It is easy to check that this is a well defined $\mathfrak{S}(\mathbb{A})$-action. Moreover, if $\sigma : S \to T$ is a bijection of finite sets, then the element $(\sigma, x) \in FX$ is supported by T. Hence FX is a nominal set.

For a morphism $f : X \to Y$ in $\mathsf{Set}^{|\mathbb{I}|}$ define $Ff : FX \to FY$ by $Ff(\sigma, x) = (\sigma, f(x))$, which is clearly equivariant. Thus we have a functor $F : \mathsf{Set}^{|\mathbb{I}|} \to \mathsf{Nom}$.

Before showing that F is left adjoint to U, let us observe that the nominal sets of the form FX are precisely the strong nominal sets in the sense of [Tze09]. Recall from [Tze09, Definition 2.9] that a finite set S is a strong support for an element x of a nominal set $\mathbb{X}$ when $\mathit{fix}(S)$ is exactly the stabilizer subgroup of x

$$\forall \pi.\ \pi \in \mathit{fix}(S) \quad \Longleftrightarrow \quad \pi \cdot x = x \tag{2.23}$$

and $\mathbb{X}$ is called a strong nominal set when all its elements have a strong support.

Lemma 2.4.2. A nominal set $\mathbb{X}$ is a strong nominal set if and only if it is of the form FX for some $X \in \mathsf{Set}^{|\mathbb{I}|}$.

Proof. Consider $X \in \mathsf{Set}^{|\mathbb{I}|}$ and $(\sigma, x) \in FX$, with $\sigma \in \mathbb{B}(S,T)$ and $x \in X(S)$. Recall that $\pi \cdot (\sigma, x) = (\pi \bullet \sigma, x)$. If $\pi \in \mathit{fix}(T)$ then $\pi \cdot (\sigma, x) = (\sigma, x)$. Conversely, if $\pi \cdot (\sigma, x) = (\sigma, x)$ then $\pi \bullet \sigma = \sigma$, or equivalently, $\pi(\sigma(s)) = \sigma(s)$ for all $s \in S$. Since σ is bijective we have that $\pi \in \mathit{fix}(T)$.

Now let $\mathbb{X}$ be a strong nominal set. Using the Axiom of Choice, for each orbit $O \in \Pi_0\mathbb{X}$ pick $x_O \in O$ and consider the set

$$X(S) = \{x_O \mid O \in \Pi_0\mathbb{X} \text{ and } S \text{ is a strong support for } x_O\}.$$

We will show that FX and $\mathbb{X}$ are isomorphic nominal sets. Define $\phi : FX \to \mathbb{X}$ by

$$\phi(\sigma, x_O) = \overline{\sigma} \cdot x_O \tag{2.24}$$

where $\sigma \in \mathbb{B}(S,T)$, $x_O \in X(S)$ and $\overline{\sigma}$ is any permutation in $\mathfrak{S}(\mathbb{A})$ that is equal to σ when restricted to S. Clearly such permutations exist and (2.24) does not depend on the choice we made since x_O is supported by S. It is not difficult to check that ϕ is equivariant and bijective. □

Lemma 2.4.3. We have an adjunction $F \dashv U : \mathsf{Nom} \to \mathsf{Set}^{|\mathbb{I}|}$.

Proof. Define the unit $\eta_X : X \to UFX$ by

$$X(T) \ni x \mapsto (\mathrm{id}_T, x) \in UFX(T). \tag{2.25}$$

This is well defined because $(\mathrm{id}_T, x) \in FX$ is supported by T. Define the counit $\varepsilon_{\mathbb{X}} : FU\mathbb{X} \to \mathbb{X}$ by

$$(b, x) \mapsto \overline{b} \cdot x. \tag{2.26}$$

Above $b : S \to T$ is a bijection, x is an element of $\mathbb{X}$ supported by S and $\overline{b}$ is an arbitrary permutation such that $\overline{b}_{|S} = b$. Such permutations exist and the choice made is not important since x is supported by S. It can be easily checked that $\varepsilon_{\mathbb{X}}$ is equivariant.

By a simple verification, the usual triangular diagrams commute:

$$\begin{array}{rcl} \varepsilon_{FX}(F\eta_X(b,x)) & = & \varepsilon_{FX}(b,(\mathrm{id}_T,x)) \\ & = & \overline{b}\cdot(\mathrm{id}_T,x) \\ & = & (\overline{b}\bullet\mathrm{id}_T,x) \\ & = & (b,x) \end{array} \tag{2.27}$$

for all $(b,x)\in FX$ with $b\in\mathbb{B}(S,T)$ and $x\in X(S)$. Also:

$$\begin{array}{rcl} U\varepsilon_{\mathbb{X}}(\eta_{U\mathbb{X}}(x)) & = & U\varepsilon_{\mathbb{X}}(\mathrm{id}_S,x)) \\ & = & \overline{\mathrm{id}_S}\cdot x \\ & = & x \end{array} \tag{2.28}$$

for all $x\in U\mathbb{X}(S)$. □

Lemma 2.4.4. The adjunction $F\dashv U:\mathsf{Nom}\to\mathsf{Set}^{|\mathbb{I}|}$ yields a monad $(\mathbb{T},\eta,\mu)$ on $\mathsf{Set}^{|\mathbb{I}|}$ given by

$$\mathbb{T}X(T)=\coprod_{S\in|\mathbb{I}|}\mathbb{I}(S,T)\times X(S)$$

with the unit $\eta_X:X\to\mathbb{T}X$ defined by

$$X(S)\ni x\mapsto(\mathrm{id}_S,x)\in\mathbb{T}X(S)$$

and multiplication $\mu_X:\mathbb{T}\mathbb{T}X\to\mathbb{T}X$ by

$$\mathbb{T}\mathbb{T}X(S)\ni(f,(g,x))\mapsto(f\circ g,x)\in\mathbb{T}X(S).$$

Proof. Notice that

$$UFX(T)=\{(b,x)\mid b\in\mathbb{B}(S',S), x \text{ is supported by } S', S\subseteq T\}.$$

Since each injective map $f:S\to T$ can be uniquely written as $f=ib$ with i an inclusion and b a bijection, we conclude that $UFX=\coprod_{S\in|\mathbb{I}|}\mathbb{I}(S,-)\times X(S)$. The rest of the proof is an easy verification. □

It is well known that the category of Eilenberg-Moore algebras for the monad $(\mathbb{T},\eta,\mu)$ on $\mathsf{Set}^{|\mathbb{I}|}$ given in Lemma 2.4.4 is exactly the functor category $\mathsf{Set}^{\mathbb{I}}$. This is a particular instance of [ARV10, Remark 1.20].

Thus, by [ML71, Theorem 1,p. 138], there exists a unique, so-called *comparison* functor $I:\mathsf{Nom}\to\mathsf{Set}^{\mathbb{I}}$:

$$\begin{tikzcd}
\mathsf{Nom} \arrow[rr, bend right=10, "I"'] \arrow[dr, bend left=15, "U"] & & \mathsf{Set}^{\mathbb{I}} \arrow[dl, bend left=15, "U^{\mathbb{I}}"] \\
& \mathsf{Set}^{|\mathbb{I}|} \arrow[ul, bend left=15, "F"] \arrow[ur, bend left=15, "F^{\mathbb{I}}"] \arrow[loop right, "\mathbb{T}"] &
\end{tikzcd} \tag{2.29}$$

We can easily check using (2.21) and (2.26) that the functor I works as follows:

$$I\mathbb{X}(S) = \{x \in \mathbb{X} \mid S \text{ supports } x\} \tag{2.30}$$

and for $j : S \to T$ the map $I\mathbb{X}(j) : I\mathbb{X}(S) \to I\mathbb{X}(T)$ is given by

$$I\mathbb{X}(j)(x) = \overline{j} \cdot x \tag{2.31}$$

where $\overline{j}$ is any finitely supported permutation such that $\overline{j}_{|S} = j$. For equivariant $f : \mathbb{X} \to \mathbb{Y}$ the map $If : I\mathbb{X} \to I\mathbb{Y}$ is given by

$$I\mathbb{X}(S) \ni x \mapsto f(x) \in I\mathbb{Y}(S) \tag{2.32}$$

for all objects S in $\mathbb{I}$.

We can easily check that I is full and faithful. Notice that for any nominal set $\mathbb{X}$ the presheaf $I\mathbb{X}$ preserves all pullbacks of the form

$$\begin{tikzcd} S_1 \cap S_2 \arrow[r, hook] \arrow[d, hook] & S_1 \arrow[d, hook] \\ S_2 \arrow[r, hook] & T \end{tikzcd} \tag{2.33}$$

This is by point 1 of Proposition 2.1.12. It follows easily that $I\mathbb{X}$ preserves all pullbacks. In fact the essential image of the functor I consists precisely of the pullback preserving presheaves. To prove this we need the following definition.

Definition 2.4.5. Given a presheaf $P \in \mathsf{Set}^{\mathbb{I}}$, $T \in |\mathbb{I}|$ and $x \in P(T)$, we say that $S \subseteq T$ is a P-support for x when for all $i, j : T \to T'$ such that $i_{|S} = j_{|S}$ we have $P(i)(x) = P(j)(x)$.

Lemma 2.4.6. Let $P \in \mathsf{Set}^{\mathbb{I}}$ be a pullback preserving functor. Then the following hold:

1. If $x \in P(T)$ and S is a P-support for x then there exists a unique $y \in P(S)$ such that $P(S \hookrightarrow T)(y) = x$.

2. For all $x \in P(T)$ there exists a least $S \subseteq T$ that is a P-support for x and there exists a unique $s(x) \in P(S)$ such that $P(S \hookrightarrow T)(s(x)) = x$.

Proof. For all $S \subseteq T$ there exist a finite set of names T' and $i, j : T \to T'$ such that the following square is a pullback:

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d, hook] & T \arrow[d, "i"] \\ T \arrow[r, "j"'] & T' \end{tikzcd} \tag{2.34}$$

Since P preserves this pullback and $x \in P(T)$ is P-supported by S, the first part of the lemma follows. The second part follows from the first part and the fact that P preserves in particular pullbacks of the form (2.33). □

Consider the set $\mathbb{X} = \{s(x) \mid T \in |\mathbb{I}|, x \in P(T)\}$, where $s(x)$ is as in part 2 of Lemma 2.4.6. The set $\mathbb{X}$ can be equipped with a $\mathfrak{S}(\mathbb{A})$-action given by

$$\pi \cdot s(x) = s(P(\overline{\pi})(x))$$

where $\overline{\pi}$ is any injective map such that $\overline{\pi}_{|S} = \pi_{|S}$. We can show that this is well defined and, moreover, if $s(x) \in P(S)$ then $s(x)$ is supported by S. Thus $\mathbb{X}$ is a nominal set. We can see that $I\mathbb{X} \simeq P$.

It is well-known, see [Joh02, Example A2.1.11(h)], that a functor in $\mathsf{Set}^{\mathbb{I}}$ preserves pullbacks if and only if it is a sheaf w.r.t. the atomic topology on $\mathbb{I}^{op}$. These sheaves form a Grothendieck topos $\mathrm{Sh}(\mathbb{I}^{op})$, called the Schanuel topos.

We have recovered the following well-known fact:

Proposition 2.4.7. The category of nominal sets is equivalent to $\mathrm{Sh}(\mathbb{I}^{op})$.

In Chapter 4 we often make use of the above equivalence. In what follows, we will denote by I_* the inclusion functor $\mathrm{Sh}(\mathbb{I}^{op}) \hookrightarrow \mathsf{Set}^{\mathbb{I}}$.

Proposition 2.4.8. The functor $I_* : \mathrm{Sh}(\mathbb{I}^{op}) \to \mathsf{Set}^{\mathbb{I}}$ has a left adjoint I^* which preserves finite limits.

This is a particular instance of [MLM94, Theorem 1, pp.128].

Remark 2.4.9. The functors I and I_* preserve filtered colimits and coproducts. This is because colimits are computed pointwise in $\mathsf{Set}^{\mathbb{I}}$ and both filtered colimits and coproducts commute with pullbacks in Set.

Also I_* preserves all limits and I^* preserves all colimits. This follows from the fact that I_* is right adjoint to I^*.

We proceed to describe the left adjoint I^*, following the proof of [MLM94, Theorem 1, Chapter III.5]. We will need some technical details regarding the characterisation of $\mathrm{Sh}(\mathbb{I}^{op})$ as a Grothendieck topos of sheaves on $\mathbb{I}^{op}$ for the atomic topology. We refer the reader to [MLM94, Chapter III] for the general definitions of notions such as Grothendieck topology, sieve, matching family, amalgamation, sheaf or atomic topology. However we shall make these definitions explicit in the case of $\mathbb{I}^{op}$. We have to underscore that unlike in [MLM94, Chapter III] we work here only with covariant functors, so a presheaf on $\mathbb{I}^{op}$ in the sense of [MLM94, Chapter III] is a Set valued covariant functor on $\mathbb{I}$.

Definition 2.4.10. A *sieve* $\mathscr{S}$ on an object S in $\mathbb{I}$ is a family of morphisms in $\mathbb{I}$ with domain S, and such that $f \in \mathscr{S}$ implies $gf \in \mathscr{S}$ whenever the composition is possible.

Given $j \in \mathbb{I}(S,T)$ and $\mathscr{S}$ a sieve on S the set

$$j^*(\mathscr{S}) = \{g \mid \exists T'.\ g \in \mathbb{I}(T,T') \text{ and } gj \in \mathscr{S}\}$$

is a sieve on T. The atomic topology on $\mathbb{I}^{op}$ is obtained by considering as covers for an object S all the non-empty sieves on S, see [MLM94, Chapter III].

Given a nonempty sieve $\mathscr{S}$ on S, there exists a least natural number n and an arrow $f : S \to T$ in $\mathscr{S}$ with $|T| - |S| = n$, where $|T|$ denotes the cardinal of T. Furthermore, we can assume that the sieve $\mathscr{S}$ is generated by an inclusion $i : S \hookrightarrow T$, that is,

$$\mathscr{S} = \{f : S \to T' \mid \exists g : T \to T' \text{ such that } f = gi\}.$$

Definition 2.4.11. A *matching family* for a sieve $\mathscr{S}$ of elements of $P : \mathbb{I} \to \mathsf{Set}$ is a family of elements $(x_f)_{f \in \mathscr{S}}$ indexed by the elements of $\mathscr{S}$ such that

1. If f in $\mathscr{S}$ has codomain T then $x_f \in P(T)$;
2. $P(g)(x_f) = x_{gf}$ for all $f : S \to T$ in $\mathscr{S}$ and g arbitrary morphisms in $\mathbb{I}$ that can be composed with f.

Lemma 2.4.12. If the sieve $\mathscr{S}$ is generated by $i : S \hookrightarrow T$ then matching families for $\mathscr{S}$ of elements of P are in one-to-one correspondence with elements of $P(T)$ that have S as a P-support.

Proof. Indeed, given a matching family for $\mathscr{S}$ we can consider the element $x_i \in P(T)$. To see that x_i is P-supported by S, consider $j, k : T \to T'$ that coincide on S. Then $ji = ki : S \to T'$ is an element of $\mathscr{S}$. We have that $P(j)(x_i) = x_{ji}$ and $P(k)(x_i) = x_{ki}$. Since $ji = ki$ we obtain that $P(j)(x_i) = P(k)(x_i)$. Conversely, let $x \in P(T)$ be an element P-supported by S. If $f : S \to T'$ in $\mathscr{S}$ is of the form gi for some $g : T \to T'$, put $x_f = P(g)(x)$. This does not depend on the chosen g because x is P-supported by S. It is easy to check that $(x_f)_{f \in \mathscr{S}}$ is a matching family. $\square$

Definition 2.4.13. An *amalgamation* for a matching family $(x_f)_{f \in \mathscr{S}}$ for a sieve $\mathscr{S}$ on S of elements of $P : \mathbb{I} \to \mathsf{Set}$ is an element $x \in P(S)$ such that $P(f)(x) = x_f$ for all $f \in \mathscr{S}$.

A functor $P : \mathbb{I} \to \mathsf{Set}$ is *separated* when each matching family has at most one amalgamation and is *a sheaf* when each matching family has a unique amalgamation. Thus Lemma 2.4.6 shows that pullback-preserving functors are indeed sheaves. Given $P : \mathbb{I} \to \mathsf{Set}$, we first obtain a separated functor P^+. If P is separated then P^+ is a sheaf. Thus the sheaf $I^*(P)$ is obtained as $(P^+)^+$. We spell out the $(-)^+$ functor in our case.

Following the proof of [MLM94, Theorem 1, pp.128] we have that $P^+(S)$ is defined as a colimit of matching families taken over covering sieves of S ordered by reverse inclusion. To get a simpler definition for $P^+(S)$ we use the correspondence of Lemma 2.4.12. We define an equivalence relation on the elements of P that have S as a P-support.

Definition 2.4.14. Consider $x \in P(T)$ and $y \in P(T')$ having S as a P-support. We say that x and y are equivalent when there exists $i : T \to T''$ and $j : T' \to T''$ such that $i_{|S} = j_{|S}$ and $P(i)(x) = P(j)(y)$. Write $\overline{x}$ for the equivalence class of x.

We then have

Lemma 2.4.15. $P^+(S)$ is the set of equivalence classes of elements of P that have S as a P-support.

Given $j \in \mathbb{I}(S,S')$ the arrow $P^+(j) : P^+(S) \to P^+(S')$ is defined as follows. Let $x \in P(T)$ have S as P-support. There exists T' and $\overline{j} \in \mathbb{I}(T,T')$ such that the square commutes:

$$\begin{tikzcd}
S \arrow[r, hook] \arrow[d, "j"'] & T \arrow[d, "\overline{j}"] \\
S' \arrow[r, hook] & T'
\end{tikzcd} \tag{2.35}$$

Then $P^+(j)(\overline{x}) = \overline{P(\overline{j})(x)}$. This is well defined.

We have an arrow $\eta^+_P : P \to P^+$ natural in P, given by

$$P(S) \ni x \mapsto \overline{x} \in P^+(S).$$

If P is a sheaf we have that η^+_P is an isomorphism.

The unit of the adjunction $I^* \dashv I_*$ is then obtained as the composition

$$P \xrightarrow{\eta^+_P} P^+ \xrightarrow{\eta^+_{P^+}} (P^+)^+ \tag{2.36}$$

In Chapter 4 we will also use quotients in the Schanuel topos:

Proposition 2.4.16. A morphism between two presheaves is an epimorphism if and only if it is so pointwise. A sheaf morphism $f : A \to B$ is an epimorphism in $\mathrm{Sh}(\mathbb{I}^{op})$ if and only if for all finite sets of names S and all $y \in B(S)$ there exists an inclusion $l : S \to T$ in $\mathbb{I}$ and $x \in A(T)$ such that $f_T(x) = B(l)(y)$.

Proof. This follows from [MLM94, Corollary III.7.5.]. □

We conclude this section with a very brief account of a different symmetric monoidal closed structure on Nom. On the presheaf category $\mathsf{Set}^{\mathbb{I}}$ one can consider Day's convolution, see [Day70]. This yields a separating tensor on Nom given by

$$\mathbb{X} \otimes \mathbb{Y} = \{(x, y) \in \mathbb{X} \times \mathbb{Y} \mid \mathsf{supp}(x) \cap \mathsf{supp}(y) = \emptyset\} \tag{2.37}$$

Then $(\mathsf{Nom}, \otimes, 1)$ is a symmetric monoidal category, which is moreover closed. Let $\multimap$ denote the internal hom w.r.t this structure. It was shown in [Men03] that $\mathbb{A} \multimap \mathbb{X}$ is isomorphic to $[\mathbb{A}]\mathbb{X}$. In general, $\mathbb{X} \multimap \mathbb{Y}$ was described in [Sch06]. This monoidal structure plays an important role in [FH07].

Chapter 3

Universal algebraic functors

In this chapter we discuss an important notion in the category theoretic toolkit needed in the development of this thesis, namely, that of presentations by operations and equations for functors. This concept was pioneered in [BK06] and further studied in [KR06].

In Section 3.1 we recall briefly the category theoretic account of universal algebra and we discuss sifted colimits. In Section 3.2 we introduce functors on many-sorted varieties that can be presented by operations and equations and we generalise to this setting a characterisation theorem and a compositionality theorem. The former states that, as in the one-sorted case, the functors on many-sorted varieties that admit presentations by operations and equations are exactly those preserving sifted colimits. The latter shows that the category of algebras for a sifted colimit preserving functor L on a many sorted variety $\mathscr{A}$ has a presentation by operations and equations obtained by merging a presentation of L and a presentation of $\mathscr{A}$.

Although the move from the one-sorted to the many-sorted case is technically straightforward, we can now apply these results to an array of examples, as seen in Sections 3.4 and 3.3. Furthermore, in [KP10a] we give further applications to compositionality for coalgebraic modal logic.

3.1 Preliminaries

In this section we fix notation and we recall the connection between equational categories, finitary monads and algebraic theories. A very nice overview of this deep connection is given in [HP07]. The importance of sifted colimits is emphasised in the monograph [ARV10].

3.1.1 Equational categories.

Let S be a set (of sorts). A *signature* Σ is a set of operation symbols together with an arity map $a : \Sigma \to S^* \times S$ which assigns to each element $\sigma \in \Sigma$ a pair $(s_1 \dots s_n; s)$ consisting of a finite word in the alphabet S indicating the sort of the arguments of σ and an element of S indicating the sort of the result of σ. To each signature we can associate an endofunctor on Set^S, which will be denoted for simplicity with the same symbol Σ:

$$(\Sigma X)_s = (\coprod_{k \in S^*} \Sigma_{k,s} \times X^k)_s$$

If $k = s_1 \dots s_n \in S^*$ then $\Sigma_{k,s}$ is the set of operations of arity $(s_1 ... s_n; s)$ and X^k is isomorphic in Set with the finite product $X_{s_1} \times \cdots \times X_{s_n}$. If we regard S as a discrete category, a word $k = s_1 \dots s_n \in S^*$ corresponds to a finite coproduct $\coprod_{i=1}^n S(s_i, -) \in \mathsf{Set}^S$, so we can identify X^k with $\mathsf{Set}^S(k, X)$. Conversely, to each polynomial endofunctor on Set^S given as above corresponds a signature $\coprod\limits_{k \in S^*} \Sigma_{k,s}$. In this chapter we will make no notational difference between the signature and the corresponding functor, and it will be clear from the context when we refer to the set of operation symbols or to a Set^S endofunctor. The algebras for a signature Σ are precisely the algebras for the corresponding endofunctor, and form the category denoted by $\mathsf{Alg}(\Sigma)$. The terms over an S-sorted set of variables X are defined in the standard manner and form an S-sorted set denoted by $\mathrm{Term}_\Sigma(X)$, in fact this is the underlying set of the free Σ-algebra generated by X. An equation

$$\Gamma \vdash \tau_1 = \tau_2$$

consists of a context $\Gamma = \{x_1 : s_1, \dots, x_n : s_n\}$ of variables x_i of sort s_i, and a pair of terms in $\mathrm{Term}_\Sigma(\Gamma)$ of the same sort. A Σ-algebra A satisfies this equation if and only if, for any interpretation of the variables of Γ, we obtain equality in A. A full subcategory $\mathscr{A}$ of $\mathsf{Alg}(\Sigma)$ is called a *variety* or an *equational class* if there exists a set of equations E such that an algebra lies in $\mathscr{A}$ if and only if it satisfies all the equations of E. In this case, the variety $\mathscr{A}$ will be denoted by $\mathsf{Alg}(\Sigma, E)$.

An important example of a (finitary) variety of algebras is the functor category $\mathsf{Set}^{\mathscr{C}}$ for any small category $\mathscr{C}$. The sorts are the objects of $\mathscr{C}$, the operations symbols are the morphisms of $\mathscr{C}$ (all of them with arity 1), and the equations are given by the commutative diagrams in $\mathscr{C}$.

Endofunctors may appear via composition of functors between different varieties. Therefore, it is useful to consider a slight generalisation of the notion of signature. If S_1 and S_2 are sets of sorts we will consider operations with arguments of sorts in S_1 and returning a result of a sort in S_2, encompassed in the signature functor $\Sigma : \mathsf{Set}^{S_1} \to \mathsf{Set}^{S_2}$

$$\Sigma X = (\coprod_{k \in S_1^*} \Sigma_{k,s} \times X^k)_{s \in S_2} \tag{3.1}$$

3.1.2 The monadic approach to universal algebra.

The forgetful functor

$$U : \mathsf{Alg}(\Sigma, E) \to \mathsf{Set}^S$$

preserves filtered colimits and has a left adjoint F. The variety $\mathsf{Alg}(\Sigma, E)$ is isomorphic to the Eilenberg-Moore category $(\mathsf{Set}^S)^T$ for the finitary monad $T = UF$ (see [AR94, Theorem 3.18]).

In fact, the forgetful functor U preserves a wider class of colimits, namely *sifted colimits*, see Definition 3.1.1.

Conversely, for each finitary monad on Set^S, the category of Eilenberg-Moore algebras is a variety. However its equational presentation is not unique, and for practical purposes finding a good presentation is important.

3.1.3 Algebraic theories and sifted colimits.

Lawvere introduced algebraic theories which give a category-theoretic treatment of the notion of clone from universal algebra. More generally, one can define an algebraic theory simply to be a small category $\mathscr{T}$ with finite products. The category $\mathsf{Alg}(\mathscr{T})$ of algebras for $\mathscr{T}$ is defined to be the full subcategory of finite product preserving functors in $\mathsf{Set}^{\mathscr{T}}$.

In Lawvere's words, (see [ARV10, Foreword]), 'the bedrock ingredient for all of general algebra's aspects is the use of finite cartesian products'. For this reason, the crucial role in understanding algebraic categories is also played by those colimits that commute in Set with finite products:

Definition 3.1.1. A small category $\mathscr{D}$ is called *sifted* when colimits over $\mathscr{D}$ commute with finite products in Set. *Sifted colimits* are colimits of sifted diagrams.

Then the category of algebras for a theory $\mathscr{T}$ is exactly the free completion under sifted colimits of $\mathscr{T}^{op}$, see [ARV10, Theorem 4.13]. Sifted colimits play a similar role for algebraic categories as filtered colimits play for locally finitely presentable categories. A result similar to Gabriel-Ulmer duality, namely a 2-categorical duality for the 2-category of algebraic categories, functors preserving limits and sifted colimits and natural transformations, was given in [ALR03].

The most important examples of sifted colimits are filtered colimits and reflexive coequalizers.

An object in a category is called *strongly finitely presentable* if its hom-functor preserves sifted colimits. It is shown in [ARV10] that any object in a variety is a sifted colimit of strongly finitely presentable algebras, which in a variety are the retracts of finitely generated free algebras. An important observation is that sifted colimit preserving functors on varieties are determined by their action on free algebras.

3.2 Presentations for functors

3.2.1 Presenting Algebras and Functors

The notion of a finitary presentation by operations and equations for a functor was introduced in [BK06]. It generalises the notion of a presentation for an algebra, in the usual sense of universal algebra. An algebra A in a variety $\mathscr{A}$ is presented by a set of generators G and a set of equations E, if A is isomorphic to the free algebra on G, quotiented by the equations E. In a similar fashion, an endofunctor L on $\mathscr{A}$ is presented by operations Σ and equations E, if for each object A of $\mathscr{A}$, LA is isomorphic to the free algebra over ΣUA quotiented by the equations E. Below we extend this notion to the case of functors between possibly different many-sorted varieties.

A presentation for a (many-sorted) algebra in a variety $\mathscr{A}$ can be regarded as a coequalizer, as in the next definition. This category-theoretical perspective will allow us to generalise this notion to functors.

Definition 3.2.1. Let A be a many-sorted algebra in a variety $\mathscr{A}$. We say that (G,E) is a presentation for A if G is an S-sorted set of generators and $E=(E_s)_{s\in S}$ with $E_s \subseteq (UFG)_s \times (UFG)_s$ is an S-sorted set of equations such that q_A is the coequalizer of the following diagram:

$$\begin{tikzcd} FE \arrow[r, shift left, "\pi_1^\sharp"] \arrow[r, shift right, "\pi_2^\sharp"'] & FG \arrow[r, "q_A"] & A \end{tikzcd} \tag{3.2}$$

The maps $\pi_1^\sharp, \pi_2^\sharp$ are induced, via the adjunction, by the projections π_1, π_2 of E on UFG.

Next we want to define a presentation for a functor $L : \mathscr{A}_1 \to \mathscr{A}_2$ between many-sorted varieties. For $i \in \{1,2\}$, denote by S_i the set of sorts for $\mathscr{A}_i$ respectively, by $U_i : \mathscr{A}_i \to \mathsf{Set}^{S^i}$ the corresponding forgetful functor, and by F_i its left adjoint. We will do this in the same fashion as in [KR06] and [BK06], keeping in mind that we need to extend (3.2) uniformly: this means that the generators and equations for each LA will depend functorially on A. Suppose A is a many-sorted algebra in $\mathscr{A}_1$. The generators $\Sigma U_1 A$ for the algebra LA will be given by a signature functor $\Sigma : \mathsf{Set}^{S_1} \to \mathsf{Set}^{S_2}$ as in (3.1). The equations that we will consider are of **rank 1**, meaning that in the terms involved every variable is under the scope of precisely one operation symbol in Σ, and are given by an S_2-sorted set E. In detail, for each sort $s \in S_2$ and each S_1-sorted set of variables V with the property that $\bigcup\limits_{t \in S_1} V_t$ is finite, we consider a set $E_{V,s}$ of equations over the set V, of terms of sort s, which is defined as a subset of $(U_2 F_2 \Sigma U_1 F_1 V)^2_s$. Now take $E_V = (E_{V,s})_{s \in S_2}$ and $E = \bigcup E_V$, where the union is taken over finite many-sorted sets of variables V.

Definition 3.2.2. Let S_1, S_2 be sets of sorts, $\mathscr{A}_1$ be an S_1-sorted variety and $\mathscr{A}_2$ be an S_2-sorted variety. A presentation for a functor $L : \mathscr{A}_1 \to \mathscr{A}_2$ is a pair (Σ, E) defined as above. A functor $L : \mathscr{A}_1 \to \mathscr{A}_2$ is presented by (Σ, E), if

(i) for every algebra $A \in \mathscr{A}_1$, there exists a morphism $q_A : F_2 \Sigma U_1 A \to LA$ that is the joint coequalizer of the next diagram

$$\begin{tikzcd} F_2 E_V \arrow[r, shift left, "\pi_1^\sharp"] \arrow[r, shift right, "\pi_2^\sharp"'] & F_2 \Sigma U_1 F_1 V \arrow[r, "F_2 \Sigma U_1 v^\sharp"] & F_2 \Sigma U_1 A \arrow[r, "q_A"] & LA \end{tikzcd} \tag{3.3}$$

taken over all finite sets of S_1-sorted variables V and all valuations $v : V \to U_1 A$. Here $v^\sharp$ denotes the adjoint transpose of a valuation v.

(ii) for all morphisms $f : A \to B$ the diagram commutes:

$$\begin{tikzcd} F_2 \Sigma U_1 A \arrow[r, "q_A"] \arrow[d, "F_2 \Sigma U_1 f"] & LA \arrow[d, "Lf"] \\ F_2 \Sigma U_1 B \arrow[r, "q_B"] & LB \end{tikzcd} \tag{3.4}$$

Example 3.2.3. Let $\mathscr{C}$ be a small category. We have seen that $\mathsf{Set}^{\mathscr{C}}$ is a many-sorted variety over $\mathsf{Set}^{|\mathscr{C}|}$, where $|\mathscr{C}|$ denotes the discrete subcategory of objects of $\mathscr{C}$. Let U denote the forgetful functor and F its left adjoint. Then for all $X \in \mathsf{Set}^{|\mathscr{C}|}$ and $C \in |\mathscr{C}|$ the set UFX_C consists of pairs (f, x) with $f \in \mathscr{C}(D, C)$ and $x \in X_D$ for some $D \in |\mathscr{C}|$.

We give a presentation for the functor $L : \mathsf{Set}^{\mathscr{C}} \to \mathsf{Set}^{\mathscr{C}}$ given by $LX = X \times X$. For all $C \in |\mathscr{C}|$ we consider a binary operation symbol with arity $op_C : C \times C \to C$.

The corresponding signature functor $\Sigma : \mathsf{Set}^{|\mathscr{C}|} \to \mathsf{Set}^{|\mathscr{C}|}$ is given by

$$(\Sigma X)_C = \{op_C\} \times X_C \times X_C.$$

For a finite many-sorted set of equations V and $C \in |\mathscr{C}|$ the set

$$E_{V,C} \subseteq (UF\Sigma UFV)_C \times (UF\Sigma UFV)_C$$

consists of pairs of terms of the form

$$(f, (op_D, ((\mathrm{id}_D, x), (\mathrm{id}_D, y)))) = (\mathrm{id}_C, (op_C((f,x),(f,y)))$$

with $f \in \mathscr{C}(D,C)$ and $x, y \in V_D$ for some $D \in |\mathscr{C}|$. Of course, we prefer to write such an equation in an abbreviated form

$$\{x : D, y : D\} \vdash f\, op_D(x,y) = op_C(fx, fy).$$

Then (Σ, E) is a presentation for L. In this case the equations express that the interpretation of the operations op_C is natural in C.

3.2.2 The Equational Logic Induced by a Presentation of L

If $\mathscr{A} = \mathsf{Alg}(\Sigma_{\mathscr{A}}, E_{\mathscr{A}})$ is an S-sorted variety and the endofunctor $L : \mathscr{A} \to \mathscr{A}$ has a finitary presentation (Σ_L, E_L), we can obtain an equational calculus for $\mathsf{Alg}(L)$, regarding the equations $E_{\mathscr{A}}$ and E_L as equations containing terms in $\mathrm{Term}_{\Sigma_{\mathscr{A}}+\Sigma_L}$. First remark that formally, for an arbitrary set of variables V, $E_{L,V}$ is a subset of the S-sorted set $(UF\Sigma_L UFV)^2$. But for each set X, UFX is a quotient of $\mathrm{Term}_{\Sigma_{\mathscr{A}}} X$ modulo the equations. Thus, if we choose a representative for each equivalence class in $UF\Sigma_L UFV$, we can obtain a set of equations in $\mathrm{Term}_{\Sigma_{\mathscr{A}}} \Sigma_L \mathrm{Term}_{\Sigma_{\mathscr{A}}} V$. We can use the natural map from $\mathrm{Term}_{\Sigma_{\mathscr{A}}} \Sigma_L \mathrm{Term}_{\Sigma_{\mathscr{A}}} V$ to $\mathrm{Term}_{\Sigma_{\mathscr{A}}+\Sigma_L} V$ to obtain a set of equations on terms $\mathrm{Term}_{\Sigma_{\mathscr{A}}+\Sigma_L} V$. By abuse of notation we will denote this set with E_L as well.

Theorem 3.2.4. Let $\mathscr{A} = \mathsf{Alg}(\Sigma_{\mathscr{A}}, E_{\mathscr{A}})$ be an S-sorted variety and let $L : \mathscr{A} \to \mathscr{A}$ be a functor presented by operations Σ_L and equations E_L. Then $\mathsf{Alg}(L) \cong \mathsf{Alg}(\Sigma_{\mathscr{A}} + \Sigma_L, E_{\mathscr{A}} + E_L)$.

Proof. We define a functor $H : \mathsf{Alg}(L) \to \mathsf{Alg}(\Sigma_{\mathscr{A}} + \Sigma_L, E_{\mathscr{A}} + E_L)$. Suppose $\alpha : LA \to A$ is an L-algebra. Then the underlying set of HA is defined to be UA. HA inherits the algebraic structure of A: the interpretation of the operation symbols of $\Sigma_{\mathscr{A}}$ is the same as in the algebra A and it satisfies the equations $E_{\mathscr{A}}$. As far as the operation symbols of Σ_L are concerned, their interpretation is given by the composition:

$$F\Sigma_L UA \xrightarrow{q_A} LA \xrightarrow{\alpha} A \tag{3.5}$$

Explicitly, the interpretation of an operation symbol σ of arity $(s_1 \ldots s_n; s)$ is the morphism $\sigma^A : A_{s_1} \times \cdots \times A_{s_n} \to A_s$ defined by

$$\sigma^A(x_1, \ldots, x_n) = \alpha(q_A((\sigma, x_1, \ldots, x_n)))$$

Now it is clear that the equations E_L are satisfied in HA, because q_A is a coequalizer as in (3.3). If f is a morphism of L-algebras, we define $Hf = f$ and we only have to check that $f(\sigma(a_1, \ldots, a_k)) = \sigma(f(a_1), \ldots, f(a_k))$ for all $\sigma \in \Sigma_L$. But this follows from the definition of the interpretation of the operations, the commutativity of diagram (3.4) and the fact that f is an L-algebra morphism.

Conversely, we define a functor $J : \mathsf{Alg}(\Sigma_{\mathscr{A}} + \Sigma_L, E_{\mathscr{A}} + E_L) \to \mathsf{Alg}(L)$. Suppose A is an algebra in $\mathsf{Alg}(\Sigma_{\mathscr{A}} + \Sigma_L, E_{\mathscr{A}} + E_L)$. The map $\rho_A : \Sigma_L UA \to UA$ defined by:

$$(\sigma_{(s_1 \ldots s_n; s)}, x_{i_1}, \ldots, x_{i_n}) \mapsto \sigma_{(s_1 \ldots s_n; s)}(x_{i_1}, \ldots, x_{i_n})$$

induces a map $\rho_A^\sharp : F\Sigma_L UA \to A$. The fact that equations E_L are satisfied implies that $\rho_A^\sharp \circ F\Sigma_L Uv^\sharp \circ \pi_1^\sharp = \rho_A^\sharp \circ F\Sigma_L Uv^\sharp \circ \pi_2^\sharp$ as depicted in (3.6). But q_A is a coequalizer in $\mathsf{Alg}(\Sigma_{\mathscr{A}}, E_{\mathscr{A}})$, therefore there exists a morphism $\alpha_A : LA \to A$ such that $\alpha_A \circ q_A = \rho_A^\sharp$. We define JA to be the L-algebra α_A. For any morphism $f : A \to B$ in $\mathsf{Alg}(\Sigma_{\mathscr{A}} + \Sigma_L, E_{\mathscr{A}} + E_L)$ we define $Jf = U_0 f$, where $U_0 : \mathsf{Alg}(\Sigma_{\mathscr{A}} + \Sigma_L, E_{\mathscr{A}} + E_L) \to \mathsf{Alg}(\Sigma_{\mathscr{A}}, E_{\mathscr{A}})$ is the forgetful functor. This is well defined and we can check this easily by proving that the rightmost square of diagram (3.6) is commutative:

$$\begin{tikzcd}
 & & F\Sigma_L UA \arrow[r, "q_A"] \arrow[d, "F\Sigma Uf"] \arrow[rr, bend left, "\rho_A^\sharp"] & LA \arrow[r, dashed, "\alpha_A"] \arrow[d, "Lf"] & A \arrow[d, "f"] \\
FE_L \arrow[r, shift left, "\pi_1^\sharp"] \arrow[r, shift right, "\pi_2^\sharp"'] & F\Sigma_L UFV \arrow[ur, "F\Sigma Uv_1^\sharp"] \arrow[dr, "F\Sigma Uv_2^\sharp"] & & & \\
 & & F\Sigma_L UB \arrow[r, "q_B"] \arrow[rr, bend right, "\rho_B^\sharp"'] & LB \arrow[r, dashed, "\alpha_B"] & B
\end{tikzcd} \tag{3.6}$$

Now it is straightforward to check that $J \circ H$ and $H \circ J$ are the identities. □

3.2.3 The Characterisation Theorem

The characterisation theorem of endofunctors having finitary presentation was given in [KR06] for monadic categories over Set and it can be easily extended if we replace Set with the presheaf category Set^S. The result holds even if we work with functors between different varieties.

Theorem 3.2.5. Let S_1, S_2 be sets of sorts, $\mathscr{A}_1$ be an S_1-sorted variety and $\mathscr{A}_2$ be an S_2-sorted variety. For a functor $L : \mathscr{A}_1 \to \mathscr{A}_2$ the following conditions are equivalent:

(i) L has a finitary presentation by operations and equations;
(ii) L preserves sifted colimits.

Proof. $(i) \Rightarrow (ii)$. Assume L has a finitary presentation (Σ, E). Let D be a sifted category and $a_i : A_i \to A$ be a sifted colimit in $\mathscr{A}_1$. Let $d_i : LA_i \to B$ be an arbitrary cocone. As we have seen in the preliminaries, the corresponding forgetful functors and their left adjoints U_1, U_2, F_1, F_2 preserve sifted colimits. Σ shares the same property because sifted colimits are computed point-wise and commute with finite products. Therefore we obtain that $F_2\Sigma U_1 a_i : F_2\Sigma U_1 A_i \to F_2\Sigma U_1 A$ is a colimiting cocone in $\mathscr{A}_2$, hence there exists a map $d : F_2\Sigma U_1 A \to B$ such that $d \circ F_2\Sigma U_1 a_i = d_i \circ q_{A_i}$ for all i in D.

Choose an arbitrary S_1-sorted set of variables $V = (V_s)_{s\in S_1}$ such that $\bigcup\limits_{s\in S_1} V_s$ is finite and a morphism $v : V \to U_1 A$. Since V is strongly finitely presentable in the category Set^{S_1}, and U_1 preserves sifted colimits, we have that $\mathsf{Set}^{S_1}(V, U_1 A)$ is the sifted colimit of $\mathsf{Set}^{S_1}(V, U_1 A_i)$. In particular there exists an index i and a morphism $v_i : V \to U_1 A_i$ such that $v = U_1 a_i \circ v_i$. From the fact that q_{A_i} is a joint coequalizer, it follows that d makes the bottom line of diagram (3.7) commutative.

$$\begin{tikzcd}
 & & F_2\Sigma U_1 A_i \arrow[d, "F_2\Sigma U_1 a_i"] \arrow[dr, "d_i \circ q_{A_i}"] & \\
F_2 E_V \arrow[r, shift left, "\pi_1^\sharp"] \arrow[r, shift right, "\pi_2^\sharp"'] & F_2\Sigma U_1 F_1 V \arrow[ur, "F_2\Sigma U_1 v_i^\sharp"] \arrow[r, "F_2\Sigma U_1 v^\sharp"] & F_2\Sigma U_1 A \arrow[r, "d"] & B
\end{tikzcd} \tag{3.7}$$

Using that q_A is a joint coequalizer, we obtain $b : LA \to B$ such that $b \circ q_A = d$. Now it is immediate to check that diagram (3.8) is commutative, and this shows that the cocone $La_i : LA_i \to LA$ is universal.

$$\begin{tikzcd}
F_2\Sigma U_1 A_i \arrow[r, "q_{A_i}"] \arrow[d, "F_2\Sigma U_1 a_i"] & LA_i \arrow[d, "La_i"] \arrow[ddr, "d_i"] & \\
F_2\Sigma U_1 A \arrow[r, "q_A"] \arrow[drr, "d"] & LA \arrow[dr, "b"] & \\
 & & B
\end{tikzcd} \tag{3.8}$$

$(ii) \Rightarrow (i)$ Being a sifted colimit preserving functor, L is determined by its values on finitely generated free algebras. Given k a finite many-sorted set in Set^{S_1} and given $s \in S_2$ we can view the elements of the set $(U_2 L F_1 k)_s$ as operations symbols which take $|k_t|$ arguments of sort t for all $t \in S_1$ and return a result of sort s. Explicitly, we consider for all algebras A the map r_A given componentwise by:

$$\coprod_{k \in S^*} (U_2 L F_1 k)_s \times (U_1 A)^k \xrightarrow{r_{A,s}} (U_2 L A)_s \tag{3.9}$$

$$(\sigma, x) \mapsto (U_2 L \varepsilon_A \circ U_2 L F_1 x)_s(\sigma)$$

where $\varepsilon_A : F_1 U_1 A \to A$ is the counit of the adjunction. In the definition of the map $r_{A,s}$ we have interpreted x as a morphism in $\mathsf{Set}^{S_1}(k, U_1 A)$. Now the operations that we will consider are encompassed in the functor $\Sigma : \mathsf{Set}^{S_1} \to \mathsf{Set}^{S_2}$ defined by

$$\Sigma X = (\coprod_{k \in S_1^*} (U_2 L F_1 k)_s \times X^k)_{s \in S_2} \tag{3.10}$$

Note that r is a natural transformation from ΣU_1 to $U_2 L$.

For an arbitrary S_1-sorted set of variables V, the equations are induced by the map $r_{F_1 V} : \Sigma U_1 F_1 V \to U_2 L F_1 V$ as in (3.9), more precisely E_V is defined to be the kernel pair of the map $U r^{\sharp}_{F_1 V} : U_2 F_2 \Sigma U_1 F_1 V \to U_2 L F_1 V$. We will prove that L is presented by (Σ, E). For all $k \in {S_1}^*$ the following diagram is a split coequalizer because E_k is a kernel pair.

$$\begin{tikzcd}
E_k \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & U_2 F_2 \Sigma U_1 F_1 k \arrow[r, "U_2 r^{\sharp}_{F_1 k}"] \arrow[l, bend left, "t"] & U_2 L F_1 k \arrow[l, bend left, "s"]
\end{tikzcd} \tag{3.11}$$

One can check that it follows that

$$\begin{tikzcd}
U_2 F_2 E_k \arrow[r, shift left, "U_2\pi_1^\sharp"] \arrow[r, shift right, "U_2\pi_2^\sharp"'] & U_2 F_2 \Sigma U_1 F_1 k \arrow[r, "U_2 r^\sharp_{F_1 k}"] \arrow[l, bend left, "U_2 F_2 t \circ \eta_{U_2 F_2 \Sigma U_1 F_1 k}"] & U_2 L F_1 k \arrow[l, bend left, "s"]
\end{tikzcd} \tag{3.12}$$

is again a split coequalizer. U_2 is a monadic functor, hence it creates split coequalizers, and we obtain that

$$\begin{tikzcd}
F_2 E_k \arrow[r, shift left, "\pi_1^\sharp"] \arrow[r, shift right, "\pi_2^\sharp"'] & F_2 \Sigma U_1 F_1 k \arrow[r, "r^\sharp_{F_1 k}"] & L F_1 k
\end{tikzcd} \tag{3.13}$$

is a coequalizer. Now it is straightforward to show that

$$\begin{tikzcd}
F_2 E_V \arrow[r, shift left, "\pi_1^\sharp"] \arrow[r, shift right, "\pi_2^\sharp"'] & F_2 \Sigma U_1 F_1 V \arrow[r, "F_2 \Sigma U_1 v^\sharp"] & F_2 \Sigma U_1 F_1 k \arrow[r, "r^\sharp_{F_1 k}"] & L F_1 k
\end{tikzcd} \tag{3.14}$$

is a joint coequalizer. This proves that L coincides on finitely generated algebras with the functor presented by the finitary presentation (Σ, E), and therefore it is presented by (Σ, E). ☐

3.3 Example: presentations for $\mathsf{Set}^{\mathbb{I}}$ and δ

We have seen that $\mathsf{Set}^{\mathbb{I}}$ is given by a finitary monad on $\mathsf{Set}^{|\mathbb{I}|}$. A very large presentation for $\mathsf{Set}^{\mathbb{I}}$ is obtained by taking a unary operation with arity $S \to S'$ for each injection in $\mathbb{I}(S, S')$. The equations are then commutative diagrams in $\mathbb{I}$. But we can find a smaller presentation. The operation symbols should correspond to morphisms that generate all the arrows in $\mathbb{I}$. One would be tempted to use operation symbols of the form $(a, b)_S$, which correspond to swapping the names a and b of a set S, and $w_{S,a}$, which correspond to inclusions of S into $S \cup \{a\}$. However swappings and inclusions fail to generate all the bijections in $\mathbb{I}$. For example, if $a \neq b$ and $a, b \notin S$, they cannot generate a bijection from $S \cup \{a\}$ to $S \cup \{b\}$ which maps a to b and acts as identity on the remaining elements of S.

This example suggests the following set $\Sigma_{\mathbb{I}}$ of operation symbols with specified arity:

$$\begin{array}{ll} (b/a)_S : S \cup \{a\} \to S \cup \{b\} & a \neq b,\ a \notin S,\ b \notin S \\ w_{S,a} : S \to S \cup \{a\} & a \notin S \end{array} \tag{3.15}$$

We will refer to operation symbols of the form $(b/a)_S$ as 'substitutions' and to operations symbols of the form $w_{S,a}$ as 'inclusions'. When the arity can be inferred from the context, or is irrelevant, we will omit S from the subscript.

We consider the set $E_{\mathbb{I}}$ of equations of the form:

$$\begin{array}{lcll} x : S\cup\{a\} & \vdash & (a/b)_S(b/a)_S(x) = x & (\mathrm{E}_1) \\ x : S\cup\{a,c\} & \vdash & (b/a)_{S\cup\{d\}}(d/c)_{S\cup\{a\}}(x) = & \\ & & (d/c)_{S\cup\{b\}}(b/a)_{S\cup\{c\}}(x) & (\mathrm{E}_2) \\ x : S\cup\{a\} & \vdash & (c/b)_S(b/a)_S(x) = (c/a)_S(x) & (\mathrm{E}_3) \\ x : S\cup\{a\} & \vdash & (b/a)_{S\cup\{c\}} w_{S\cup\{a\},c}(x) = w_{S\cup\{b\},c}(b/a)_S(x) & (\mathrm{E}_4) \\ x : S\cup\{a\} & \vdash & (b/a)_S w_{S,a}(x) = w_{S,b}(x) & (\mathrm{E}_5) \\ x : S & \vdash & w_{S\cup\{b\},a} w_{S,b}(x) = w_{S\cup\{a\},b} w_{S,a}(x) & (\mathrm{E}_6) \\ x : S\cup\{a\} & \vdash & w_{S\cup\{b\},a}(b/a)_S(x) = w_{S\cup\{a\},b}(x) & (\mathrm{E}_7) \end{array} \qquad (3.16)$$

Theorem 3.3.1. $(\Sigma_{\mathbb{I}}, E_{\mathbb{I}})$ is a presentation for $\mathsf{Set}^{\mathbb{I}}$.

A proof of this theorem can be found in Section 3.5.

Next we will define a 'shift' functor δ on $\mathsf{Set}^{\mathbb{I}}$. This functor corresponds to the abstraction functor on nominal sets and plays an important role in Chapter 4.

Definition 3.3.2. Assume $P : \mathbb{I} \to \mathsf{Set}$ is a presheaf and $S \subseteq \mathbb{A}$ is a finite set of names. We define an equivalence relation $\equiv$ on $\coprod_{a\notin S} P(S\cup\{a\})$. If $a, b \notin S$, $x \in P(S\cup\{a\})$ and $y \in P(S\cup\{b\})$ we will say that x and y are equivalent if and only if $P((b/a)_S)(x) = y$. We define $(\delta P)(S)$ as the set of equivalence classes of the elements of $\coprod_{a\notin S} P(S\cup\{a\})$. If $x \in P(S\cup\{a\})$ the equivalence class of x is denoted by $\overline{x}^{S,a}$.

If $f : S \to T$ is a morphism in $\mathbb{I}$ and $a \notin S\cup T$, $f + a : S\cup\{a\} \to T\cup\{a\}$ denotes the function which restricted to S is f and which maps a to a. We define $(\delta A)(f)(\overline{x}^{S,a}) = \overline{A(f+a)(x)}^{T,a}$ for some $a \notin S$.

One can easily check that $(\delta A)(f)$ is well defined and that δ is a functor.

Next, we give a presentation for δ. For each finite subset of names $S \subseteq \mathbb{A}$ and for each $a \notin S$ we consider an operation symbol $[a]_S : S\cup\{a\} \to S$, and we will denote by Σ_δ the corresponding functor on $\mathsf{Set}^{|\mathbb{I}|}$. This is given by

$$(\Sigma_\delta X)_S = \coprod_{a\notin S} \{[a]_S\} \times X_{S\cup\{a\}}.$$

We denote by $U : \mathsf{Set}^{\mathbb{I}} \to \mathsf{Set}^{|\mathbb{I}|}$ the forgetful functor and by F its left adjoint. For any functor $P : \mathbb{I} \to \mathsf{Set}$ we can give an interpretation of these operation symbols, captured by a natural transformation

$$\rho_P : \Sigma_\delta UP \to U\delta P$$

defined by:

$$\forall \alpha \in P(S\cup\{a\}) \qquad ([a]_S,\alpha)\mapsto \overline{\alpha}^{S,a} \in (U\delta P)(S) \tag{3.17}$$

The equations should correspond to the kernel pair of the adjoint transpose

$$\rho^{\sharp}_P : F\Sigma_\delta UP \to \delta P, \tag{3.18}$$

as in Definition 3.2.2. We will use the fact that for any $X = (X_S)_{S\in|\mathbb{I}|}$ we have $(FX)_S = \coprod_{T\in|\mathbb{I}|} X_T \cdot \hom(T,S)$, where $\cdot$ is the copower. For $f : T \to S$ and $x \in X_T$ we denote by fx the element of $(FX)_S$ which is the copy of f corresponding to x. The equations E_δ will have the form:

$$\begin{array}{lcl} t : S\cup\{a,b\} & \vdash & (c/b)_S[a]_{S\cup\{b\}}t = [a]_{S\cup\{c\}}(c/b)_{S\cup\{a\}}t \\ t : S\cup\{a\} & \vdash & [a]_S t = [b]_S(b/a)_S t \\ t : S\cup\{a\} & \vdash & w_{S,b}[a]_S t = [a]_{S\cup\{b\}} w_{S\cup\{a\},b}\, t \end{array} \tag{3.19}$$

Theorem 3.3.3. $(\Sigma_\delta, E_\delta)$ is a presentation for δ.

Proof. First we show that the map $\rho^{\sharp}_P$ makes the next diagram commutative for all finite many-sorted sets of variables V and all valuations $v : V \to UP$.

$$\begin{tikzcd} FE_{\delta,V} \arrow[r, shift left, "\pi^{\sharp}_1"] \arrow[r, shift right, "\pi^{\sharp}_2"'] & F\Sigma_\delta UFV \arrow[r, "F\Sigma_\delta Uv^{\sharp}"] & F\Sigma_\delta UP \arrow[r, "\rho^{\sharp}_P"] & \delta P \end{tikzcd} \tag{3.20}$$

Given $S \in |\mathbb{I}|$, consider an element of $FE_{\delta,V}(S)$ of the form $f([a]_T t, [b]_T(b/a)_T t))$ for some $f \in \mathbb{I}(T,S)$, $t \in V(T\cup\{a\})$ and $([a]_T t, [b]_T(b/a)_T t)) \in E_{\delta,V}(T)$. Then

$$\begin{array}{lcl} \pi^{\sharp}_1(f([a]_T t, [b]_T(b/a)_T t)) & = & f[a]_T t \\ \pi^{\sharp}_2(f([a]_T t, [b]_T(b/a)_T t)) & = & f[b]_T(b/a)_T t \end{array} \tag{3.21}$$

Assume $v(t) = p \in P(T\cup\{a\})$. Then

$$\begin{array}{lcl} F\Sigma_\delta UFV(f([a]_T t) & = & (f,[a]_T p) \\ F\Sigma_\delta UFV(f([b]_T(b/a)_T t) & = & (f,[b]_T P((b/a)_T)(p)) \end{array} \tag{3.22}$$

We can check that

$$\begin{array}{lcl} \rho^{\sharp}_P((f,[a]_T p)) & = & \delta P(f)(\overline{p}^{T,a}) \\ \rho^{\sharp}_P((f,[b]_T P((b/a)_T)(p))) & = & \delta P(f)(\overline{P((b/a)_T(p)}^{T,b}) \end{array} \tag{3.23}$$

By Definition 3.3.2 we have that $\overline{p}^{T,a} = \overline{P((b/a)_T(p)}^{T,b}$. Using (3.21), (3.22) and (3.23) we conclude that diagram (3.20) commutes. The proof of the fact that the other two types of equations are satisfied is similar and is left to the reader.

Next we show that $\rho_P^\sharp$ is indeed the joint coequalizer of (3.20) taken over all finite many-sorted sets of variables and all possible valuations. Consider $\alpha : F\Sigma_\delta UP \to X$ a morphism in $\mathsf{Set}^{\mathbb{I}}$ such that

$$\alpha \circ F\Sigma_\delta UFv^\sharp \circ \pi_1^\sharp = \alpha \circ F\Sigma_\delta UFv^\sharp \circ \pi_2^\sharp.$$

We show that there exists a unique $\widehat{\alpha} : \delta P \to X$ such that $\widehat{\alpha} \circ \rho_P^\sharp = \alpha$. We define $\widehat{\alpha}_T : \delta P(T) \to X(T)$ by

$$\widehat{\alpha}(\overline{p}^{T,a}) = \alpha(\mathrm{id}([a]_T, p)) \tag{3.24}$$

The definition of $\widehat{\alpha}$ does not depend on the representative $p \in P(T \cup \{a\})$. This follows from the fact that α satisfies the second equation in (3.19). The fact that $\widehat{\alpha}$ is natural in T follows from the fact that α satisfies the other two types of equations in (3.19). □

Notation 3.3.4. We will denote the element $\overline{\alpha}^{S,a} \in (\delta P)(S)$ by $\{[a]_S \alpha\}_{\delta P}$.

3.4 Example: polyadic algebras as algebras for a functor

The aim of this section is to understand algebraic semantics of first-order logic, more specifically Halmos' polyadic algebras in terms of algebras for a sifted colimit preserving functor on a variety of nominal substitutions. The nominal substitutions considered by Staton in [Sta09] for his study of the open bisimulation of π-calculus correspond to functors in $\mathsf{Set}^{\mathbb{F}_+}$ where $\mathbb{F}_+$ is the category of positive ordinals and maps between them.

We introduce a category of algebras for a functor on $\mathsf{Set}^{\mathbb{F}_+}$, that we call FOL-algebras (Definition 3.4.3), and we prove that polyadic algebras are precisely FOL-algebras. First, we need to give a presentation for $\mathsf{Set}^{\mathbb{F}_+}$. The set of sorts are the positive integers. We will consider a signature consisting only of unary operation symbols, whose arity is specified below:

$$\begin{array}{lll} \sigma_n^{(i)} : n \to n & 1 \leq i < n & n > 1 \\ w_n : n \to n+1 & & n > 0 \\ c_n : n+1 \to n & & n > 0 \end{array} \tag{3.25}$$

The intended interpretation is the following: $\sigma_n^{(i)}$ corresponds to the transposition $\sigma_n^{(i)} = (i, i+1)$ of the set $\underline{n}$, c_n corresponds to the contraction $c_n : \underline{n+1} \to \underline{n}$ defined by $c_n(i) = i$ for $i \leq n$ and $c_n(n+1) = n$, and w_n to the inclusion w_n of $\underline{n}$ into $\underline{n+1}$.

In [KP10a] we gave a presentation for $\mathsf{Set}^{\mathbb{F}}$, where $\mathbb{F}$ is the category of all finite ordinals. We can obtain a presentation for $\mathsf{Set}^{\mathbb{F}_+}$ by dropping the sort $\underline{0}$ and all the operations of arity $\underline{0}$ and all the equations containing such operations. We obtain that the equations needed to present $\mathsf{Set}^{\mathbb{F}_+}$ are as follows:

Firstly, we consider the equations coming from the presentation of the symmetric group:

$$\begin{array}{lll} (\sigma_n^{(i)})^2 = id_n & 1 \leq i < n & \\ \sigma_n^{(i)}\sigma_n^{(j)} = \sigma_n^{(j)}\sigma_n^{(i)} & j \neq i \pm 1; 1 \leq i, j < n & \quad (\mathrm{E}_1) \\ (\sigma_n^{(i)}\sigma_n^{(i+1)})^3 = id_n & 1 \leq i < n-1 & \end{array}$$

Each permutation of the set $\underline{n}$ can be written as a composition of transpositions $\sigma_n^{(i)}$ and we choose for each permutation such a representation. The permutations that will appear in equation (E_9) below, should be regarded as abbreviations of their representation in terms of the corresponding $\sigma_n^{(i)}$.

Secondly, we use the next set of equations:

$$\begin{array}{lll} c_n\sigma_{n+1}^{(n)} = c_n & & (\mathrm{E}_2) \\ c_n w_n = id_n & & (\mathrm{E}_3) \\ \sigma_{n+1}^{(i)} w_n = w_n \sigma_n^{(i)} & 1 \leq i < n & (\mathrm{E}_4) \\ \sigma_{n+2}^{(n+1)} w_{n+1} w_n = w_{n+1} w_n & & (\mathrm{E}_5) \\ \sigma_n^{(i)} c_n = c_n \sigma_{n+1}^{(i)} & i < n-1 & (\mathrm{E}_6) \\ c_n \sigma_{n+1}^{(n-1)} \sigma_{n+1}^{(n)} w_n = \sigma_n^{(n-1)} w_{n-1} c_{n-1} & n \geq 2 & (\mathrm{E}_7) \\ c_n c_{n+1} \sigma_{n+2}^{(n)} = c_n c_{n+1} & & (\mathrm{E}_8) \\ ((2, n-1)(1, n) w_{n-1} c_{n-1})^2 = (w_{n-1} c_{n-1} (2, n-1)(1, n))^2 & n \geq 4 & (\mathrm{E}_9) \end{array}$$

We will consider a category of algebras for a functor on $\mathsf{BA}^{\mathbb{F}_+}$, where BA is the category of Boolean algebras. The basic observation, essentially going back to Lawvere [Law69], is that presheaves taking values in the category BA

$$A : \mathbb{F}_+ \to \mathsf{BA}$$

where the weakenings w_n have left-adjoints $\exists_n$

$$\exists_n a \leq b \iff a \leq w_n b \tag{3.26}$$

are (algebraic) models of first-order logic (we write $w_n b$ for $A(w_n)(b)$).

$\mathsf{BA}^{\mathbb{F}_+}$ is the subvariety of $\mathsf{Set}^{\mathbb{F}_+}$, obtained by adding for each sort the Boolean connectives $\vee_n$ and $\neg_n$, satisfying the usual axioms for Boolean algebras and commuting with the operations from $\mathbb{F}_+$, that is, we have $w_n \vee_n = \vee_{n+1} w_n$ and $c_n \vee_{n+1} = \vee_n c_n$ and $\sigma_n^{(i)}(x \vee_n y) = \sigma_n^{(i)} x \vee_n \sigma_n^{(i)} y$ as well as the analogous equations for $\neg_n$. We are looking for algebras $\mathsf{Q}B \to B$ where the structure at sort

$\underline{n}$, $(\mathsf{Q}B)(\underline{n}) \to B(\underline{n})$ interprets the quantifier $\exists_n$ binding the new name in $\underline{n+1}$. Thus, the quantifier corresponds to a map $B(\underline{n+1}) \to B(\underline{n})$ and, being an existential quantifier, it preserves joins. Since arrows in BA are Boolean homomorphisms, we account for this by letting $(\mathsf{Q}B)(\underline{n})$ be the free BA over the finite-join-semilattice $B(\underline{n+1})$, or, explicitly

Definition 3.4.1. Define $\mathsf{Q} : \mathsf{BA}^{\mathbb{F}_+} \to \mathsf{BA}^{\mathbb{F}_+}$ as the functor mapping $B \in \mathsf{BA}^{\mathbb{F}_+}$ to the presheaf

- generated, at sort $\underline{n}$, by $\exists_n a$, $a \in B(\underline{n+1})$
- modulo equations specifying that $\exists_n$ preserves finite joins, explicitly $\exists_n(0) = 0$ and $\exists_n(a \vee b) = \exists_n a \vee \exists_n b$.

Remark 3.4.2. Boolean algebra homomorphisms $\mathsf{Q}B(\underline{n}) \to B(\underline{n})$ are in bijective correspondence with finite-join preserving maps $B(\underline{n+1}) \to B(\underline{n})$.

Furthermore, using the (co)unit of the adjunction, the two implications (3.26) are easily transformed into equations (recall $a \leq b \iff a = a \wedge b$), leading to

Definition 3.4.3. The category of FOL-algebras is the category of those Q-algebras satisfying the additional equations $\phi \leq w_n \exists_n \phi$ and $\exists_n w_n \psi \leq \psi$, where ϕ is a variable of sort $\underline{n+1}$ and ψ is a variable of sort $\underline{n}$.

Algebraic semantics of first-order logic was first studied by Tarski and collaborators [HMT71] and Halmos [Hal62]. A polyadic algebra [Hal62] on a set of variables V is a Boolean algebra with some additional structure that captures quantifiers and an action of the monoid of functions V^V, subject to several axioms.

Definition 3.4.4. Let B be a Boolean algebra, a map $\exists : B \to B$ is called a *quantifier* if
$\exists 0 = 0$
$\exists p \geq p$ for all $p \in B$
$\exists p \vee \exists q = \exists(p \vee q)$ for all $p, q \in B$
$\exists\exists p = \exists p$ for all $p \in B$
$\exists\neg\exists p = \neg\exists p$ for all $p \in B$

Definition 3.4.5. A polyadic algebra $\mathbb{B}$ over a set of variables V is a tuple $(B, V, \mathscr{S}, \exists)$ such that B is a Boolean algebra, $\mathscr{S} : V^V \to \mathsf{End} B$ and $\exists$ is a map from $\mathscr{P} V$ to the set of quantifiers on B, such that
(P1) $\exists(\emptyset)$ is the identity map on B.
(P2) $\exists(J \cup K) = \exists(J)\exists(K)$ for all $J, K \subseteq V$

(P3) $\mathscr{S}$ maps the identity on V to the identity on B.

(P4) $\mathscr{S}(\sigma\tau) = \mathscr{S}(\sigma)\mathscr{S}(\tau)$ for all $\sigma, \tau \in V^V$

(P5) $\mathscr{S}(\sigma)\exists(J) = \mathscr{S}(\tau)\exists(J)$, if σ and τ coincide on $V \setminus J$.

(P6) $\exists(J)\mathscr{S}(\tau) = \mathscr{S}(\tau)\exists(\tau^{-1}J)$ for all transformations τ which are injective when restricted to $\tau^{-1}J$.

Definition 3.4.6. A polyadic algebra $\mathbb{B} = (B, V, \mathscr{S}, \exists)$ is called locally finite if for each $P \in B$ there exists a finite set $W \subseteq V$ such that $\exists(J)P = P$ for all $J \subseteq V$ such that J and W are disjoint.

Ouellet [Oue82] reformulated Halmos' polyadic algebras using Boolean valued presheaves. He characterised the locally finite polyadic algebras on a set of variables V as Boolean algebra objects in the category of *locally finite V-actions* that admit suprema indexed by V. A V-action on a set X is locally finite if each element $x \in X$ has a finite support (or is finitely supported), that is, there exists a finite subset W of V, such that any function $\xi : V \to V$ that acts as the identity on W has no effect on x, i.e. $\xi x = x$. Note that any locally finite polyadic algebra is equipped with a V-action given by $(P3)$ and $(P4)$, which is locally finite because of $(P5)$. The locally finite V-actions are exactly the nominal substitutions in [Sta09].

Ouellet [Oue82] uses the equivalence [Oue81] between the category of locally finite V-actions and $\mathsf{Set}^{\mathbb{F}_+}$. The proof of the next theorem shows how this equivalence restricts to an equivalence between FOL-algebras and locally finite polyadic algebras.

Theorem 3.4.7. The category of FOL-algebras is equivalent to the category of locally finite polyadic algebras.

Proof. First we construct a functor from FOL-algebras to locally finite polyadic algebras. Let $\alpha : \mathsf{Q}B \to B$ be a FOL-algebra. Let us fix an infinite set of variables V.

We consider the Boolean algebra $B^\flat = \mathsf{Lan}_i B(V)$, where $\mathsf{Lan}_i B$ is the left Kan extension of B along the inclusion $i : \mathbb{F}_+ \to \mathsf{Set}$. Notice that $B^\flat$ is computed as a colimit in the comma category (i, V), more explicitly, it is isomorphic to $\varinjlim_{f:\underline{n}\to V} B(\underline{n})$. So $B^\flat$ is a quotient of the disjoint union of $B(\underline{n})$ taken over all $n \geq 1$ and all maps $f : \underline{n} \to V$. Consider an element P in the copy of $B(\underline{n})$ that corresponds to a function $f : \underline{n} \to V$ mapping $i \in \underline{n}$ to $v_i \in V$. We will denote by $[P(v_1, \ldots, v_n)]$ the equivalence class of P. Two elements of $B^\flat$, $[P(v_1, \ldots, v_n)]$ and $[Q(w_1, \ldots, w_m)]$, are equal iff there exist maps $l : \underline{n} \to \underline{p}$, $k : \underline{m} \to \underline{p}$ and $h : \underline{p} \to V$ such that $h(l(i)) = v_i$ for all $i \in \underline{n}$, $h(k(j)) = w_i$ for all $j \in \underline{m}$ and $B(l)(P) = B(k)(Q)$.

For any map $\xi : V \to V$ we define $\mathscr{S}(\xi)$ to be the Boolean algebra morphism $\mathsf{Lan}_i(B)(\xi)$. So we have a V-action structure on $B^\flat$. Moreover $B^\flat$ is a locally finite V-action, because each element is finitely supported. Indeed, an element of $B^\flat$ of the form $[P(v_1,\dots,v_n)]$ is supported by the finite set with elements $v_1,\dots,v_n$. In fact each $x \in B^\flat$ has a minimal support denoted by $\mathsf{supp}(x)$. Moreover if $\mathsf{supp}(x) = \{v_1,\dots,v_n\}$ for some $n \geq 1$, then there exists $P \in B(\underline{n})$ such that $x = [P(v_1,\dots,v_n)]$. If x has empty support, then for any tuple of variables $(v_1,\dots,v_n)$ there exists $P \in B(\underline{n})$ such that $x = [P(v_1,\dots,v_n)]$.

Next, for each subset $W \subseteq V$ we define an existential quantifier $\exists W$. First we do this for singleton sets. Assume $v \in V$, and $x \in B^\flat$. There exists $n \geq 1$ and $P \in B(\underline{n+1})$ such that $x = [P(v_1,\dots,v_n,v)]$ for some variables $v_1,\dots,v_n$, all different than v. We define $\exists v(x) = [(\exists_n P)(v_1,\dots,v_n)]$. One can check that this definition does not depend on the choice of P or of the variables $v_1,\dots,v_n$. Note that $\exists_n P$ is just an abbreviation for $\alpha_n(\exists_n P)$.

Remark 3.4.8. We have that $\mathsf{supp}(\exists v(x)) = \mathsf{supp}(x) \setminus \{v\}$.

We need to show that $\exists v$ is indeed an existential quantifier on $B^\flat$.

1. $\exists v 0 = 0$ follows from $\exists_n 0_{n+1} = 0_n$.

2. Let us prove that $\exists v(x) \geq x$. With the notations above we have that

$$\begin{aligned} \exists v(x) &= [(\exists_n P)(v_1,\dots,v_n)] \\ &= [(w_n \exists_n P)(v_1,\dots,v_n,v)] \\ &\geq [P(v_1,\dots,v_n,v)] \\ &= x. \end{aligned} \tag{3.27}$$

3. The fact that $\exists v(x \vee y) = \exists v(x) \vee \exists v(y)$ follows from the corresponding equation for $\exists_n$.

4. Let us prove that $\exists v(\exists v(x)) = \exists v(x)$. Using Remark 3.4.8 it is enough to show that $\exists v(x) = x$ for all x whose support does not contain v. Indeed, if x is such that $v \notin \mathsf{supp}(x) \subseteq \{v_1,\dots,v_n\}$, then $x = [P(v_1,\dots,v_n)]$ for some $P \in B(\underline{n})$. Then $\exists v(x) = [(\exists_n w_n P)(v_1,\dots,v_n)] \leq [P(v_1,\dots,v_n)] = x$. On the other hand we know that $\exists v(x) \geq x$.

5. In order to prove that $\exists v(\neg \exists v(x)) = \neg \exists v(x)$ we use the same argument as above, plus the observation that $\mathsf{supp}(\neg x) = \mathsf{supp}(x)$ for all $x \in B^\flat$.

Lemma 3.4.9. For $u, v \in V$ and $x \in B^\flat$ we have $\exists v(\exists u(x)) = \exists u(\exists v(x))$

Proof. There exists $P \in B(\underline{n+2})$ such that $x = [P(v_1, \dots, v_n, u, v)]$, for $n \geq 1$ and for variables $v_1, \dots, v_n$ different form u, v. It remains to show that

$$\exists_n \exists_{n+1} \sigma^{(n+1)}_{n+2}(P) = \exists_n \exists_{n+1}(P). \tag{3.28}$$

From the equations it follows that

$$\begin{array}{rl} & w_{n+1} w_n \exists_n \exists_{n+1}(P) \geq P \\ \Leftrightarrow & \sigma^{(n+1)}_{n+2} w_{n+1} w_n \exists_n \exists_{n+1}(P) \geq \sigma^{(n+1)}_{n+2} P \\ \Leftrightarrow & w_{n+1} w_n \exists_n \exists_{n+1}(P) \geq \sigma^{(n+1)}_{n+2} P \\ \Leftrightarrow & \exists_n \exists_{n+1}(P) \geq \exists_n \exists_{n+1} \sigma^{(n+1)}_{n+2}(P). \end{array} \tag{3.29}$$

Applying the last inequality to $\sigma^{(n+1)}_{n+2}(P)$ instead of P, we get that $\exists_n \exists_{n+1}(P) \leq \exists_n \exists_{n+1} \sigma^{(n+1)}_{n+2}(P)$, so in fact we have equality. □

Now we can define the existential quantifier $\exists W$ for an arbitrary subset $W \subseteq V$. If $x \in B^\flat$ is such that $\mathsf{supp}(x) \cap W = \{v_1, \dots, v_n\}$, then we define $\exists W(x) = \exists v_1 \dots \exists v_n(x)$. The above lemma implies that $\exists W$ is well defined.

We have to show that these existential quantifiers satisfy the equations defining a polyadic algebra. It is straightforward to check (P1)-(P5), so we will only give the proof for (P6). Assume $W \subseteq V$ and $\xi \in V^V$ is injective when restricted to $\xi^{-1}(W)$. We need to show that $\exists W \circ \xi = \xi \circ \exists \xi^{-1}(W)$. This is immediate using the observation that $\mathsf{supp}(\xi(x)) \subseteq \xi(\mathsf{supp}(x))$.

The polyadic algebra obtained in this way is locally finite in the sense of Definition 3.4.6. Indeed, for $x \in B^\flat$ we have $\exists(J)x = x$ for all sets J, such that $J \cap \mathsf{supp}(x) = \emptyset$.

Conversely, given a locally finite polyadic algebra $(\mathbb{B}, V, \mathscr{S}, \exists)$, let us construct a FOL-algebra $\mathbb{B}^\sharp$. The map $\mathscr{S} : V^V \to \mathsf{End}(\mathbb{B})$ determines a V-action structure on $\mathbb{B}$ such that each element is finitely supported. For each $n > 0$ define $\mathbb{B}^\sharp(n)$ to be the set of V-action morphisms from V^n to $\mathbb{B}$, where V^n is endowed with the component-wise evaluation action. If $f : \underline{n} \to \underline{m}$ is a morphism in $\mathbb{F}$, and $P : V^n \to \mathbb{B}$ is an element of $\mathbb{B}^\sharp(\underline{n})$ then

$$\mathbb{B}^\sharp(f)(P)(v_1, \dots, v_m) = P(v_{f(1)}, \dots, v_{f(n)})$$

We have to construct an algebra $\alpha : \mathsf{Q}\mathbb{B}^\sharp \to \mathbb{B}^\sharp$. This will be determined by the maps $\exists_n : \mathbb{B}^\sharp(\underline{n+1}) \to \mathbb{B}^\sharp(\underline{n})$ defined as follows: For $P \in \mathbb{B}^\sharp(\underline{n+1})$ define $(\exists_n P)(v_1, \dots, v_n) = \exists v(P(v_1, \dots, v_n, v))$ for some v distinct from all the v_i-s. From (P6) it follows that this is well-defined. It is trivial to check that $\exists_n$ preserves joins.

We can check that for all $P \in \mathbb{B}^\sharp(\underline{n})$ we have that $\exists_n w_n P = P$. Indeed

$$\exists_n w_n P(v_1, \dots, v_n) = (\exists v)((w_n P)(v_1, \dots, v_n, v)) \tag{3.30}$$

for some v different than $v_1, \dots, v_n$. Therefore

$$\exists_n w_n P(v_1, \dots, v_n) = (\exists v)(P(v_1, \dots, v_n)) = P(v_1, \dots, v_n).$$

The last equality holds because $\mathsf{supp}(P(v_1, \dots, v_n)) \subseteq \{v_1, \dots, v_n\}$ does not contain v.

For $P \in \mathbb{B}^\sharp(\underline{n+1})$ we have that $(w_n \exists_n P)(v_1, \dots, v_n, v_{n+1}) = (\exists_n P)(v_1, \dots, v_n) = (\exists v)(P(v_1, \dots, v_n, v) \geq P(v_1, \dots, v_n, v_{n+1})$ for some $v \notin \{v_1, \dots, v_n\}$. The last inequality follows from (P5) and the fact that $(\exists v)(P(v_1, \dots, v_n, v)) \geq P(v_1, \dots, v_n, v)$.

One can check that the functors $\flat$ and $\sharp$ give an equivalence of categories.

$\square$

3.5 Appendix: Proof of theorem 3.3.1

Proof. (a) First we show that $\mathsf{Set}^{\mathbb{I}}$ is a category of $(\Sigma_{\mathbb{I}}, E_{\mathbb{I}})$-algebras. The intended interpretation of the operation symbols above is the following: If $a, b \notin S$ then $(b/a)_S$ corresponds to the bijective map from $S \cup \{a\}$ to $S \cup \{b\}$ which substitutes b for a. The symbol $w_{S,a}$ corresponds to the inclusion of S into $S \cup \{a\}$. It easy to check that these morphisms satisfy the equations listed above. We have to check that each morphism in $\mathbb{I}$ can be written as a composition of such inclusions and substitutions. First notice that the swapping of elements a, b of a set $S \cup \{a, b\}$, is obtained as the composition

$$\sigma_{a,b} = (b/c)_{S \cup \{a\}} (a/b)_{S \cup \{c\}} (c/a)_{S \cup \{b\}} \tag{3.31}$$

where $c \notin S \cup \{a, b\}$. Therefore all bijections are generated by substitutions. If the cardinality of a subset S of $\mathbb{A}$ is less or equal than the cardinality of a finite subset T of $\mathbb{A}$, then one can construct an injective map $i : S \to T$, by enlarging S with elements of $T \setminus S$ (using the inclusions) until it reaches the cardinality of T, and then by substituting the remaining elements of $T \setminus S$ for those of $S \setminus T$. Now any other map $j : S \to T$ is obtained by composing i with a bijection on T.

(b) Conversely, it is enough to check that different representations of an injective map $\iota : S \to T$ in $\mathbb{I}$ as composition of inclusions and substitutions are equivalent via the equations $E_{\mathbb{I}}$. Using (E_4) and (E_7) one can prove that each representation of ι can be reduced to one of the form $s_1 \dots s_k w_{a_1} \dots w_{a_l}$ where the s_i-s stand for substitutions. Using (E_1) one can reduce the problem to showing that if the equality $s_1 \dots s_k w_{a_1} \dots w_{a_l} = w_{b_1} \dots w_{b_h}$ holds in $\mathbb{I}$ then it can be derived from $E_{\mathbb{I}}$. For cardinality reasons we must have $l = h$. Notice that using (E_5) and (E_6) we can reduce this to the simpler problem in which $\{a_1, \dots, a_l\} = \{b_1, \dots, b_l\}$. Assume that w_{a_l} has arity $S \rightarrow S \cup \{a_l\}$. The arities for the rest of the w's can be now deduced. Because the equality holds in $\mathbb{I}$ we have that $s_1 \dots s_k$ is a permutation on $S \cup \{a_1, \dots, a_l\}$ which is the identity when restricted to S.

We finalize the proof using the well known presentation of the symmetric groups. Firstly, Lemma 3.5.2 asserts that the equations which are enough to give a presentation for the symmetric group are satisfied by $\sigma_{a,b}$-s, where $\sigma_{a,b}$ is the abbreviation introduced in (3.31). Secondly, Lemma 3.5.1 asserts that a sequence of substitutions whose interpretation is a permutation, can be reduced to a sequence of $\sigma_{a,b}$-s. Then the sequence $s_1 \dots s_k$ can be rewritten as a sequence of $\sigma_{a,b}$-s with the a and b only from the set $\{a_1, \dots, a_l\}$. To finish, notice that it is straightforward to derive from $E_{\mathbb{I}}$ that $\sigma_{a_i,a_l} w_{a_i} w_{a_l} = w_{a_i} w_{a_l}$. □

Lemma 3.5.1. Let $s_1, ..., s_k$ be a sequence of substitutions, such that the composition $s_1 \dots s_k$ of their interpretation in $\mathbb{I}$ is possible and moreover it is a permutation. Then we can reduce $s_1 \dots s_k$ to a sequence of σ_{a_i,b_i}'s, where each σ_{a_i,b_i} is a sequence of 3 substitutions as defined in (3.31).

Proof. The proof is by induction on k. If $k = 0$ we have nothing to prove. Assume the statement of the lemma has been proved for $k - 1$, let us prove it for k. Assume $s_k = (c/a)$. This means that $s_1 \dots s_k$ is a permutation on a set which contains a. In particular a is in the image of $s_1 \dots s_k$. Therefore there exists i such that $1 \leq i < k$ and s_i is of the form $(a/y)_T$ for some atom y and some set T. Consider the i maximal with this property. The idea is to rewrite the sequence using the equations such that the rightmost substitution of the form $(a/y)_T$ can be moved to position $k - 1$. If $i < k - 1$ we know that the substitutions $s_{i+1}, \dots, s_{k-1}$ do not involve a. We have two cases:

1. y does not appear in the substitutions $s_{i+1}, \dots, s_{k-1}$. We can use (E_2) to prove that the sequence $s_1 \dots s_k$ can be rewritten to a sequence $s'_1 \dots s'_{k-1} s_k$ such that $s'_{k-1} = (a/y)$. If $y = c$ we can reduce the sequence to a shorter one, via (E_1) and apply the induction hypothesis. If $y \neq c$, then, by (E_1), we know that $s'_1 \dots s'_{k-1} s_k = s'1 \dots s'_{k-2}(c/y)(y/c)(a/y)(c/a)$. But this is equal to $s'_1 \dots s'_{k-2}(c/y)\sigma_{a,y}$. By the induction hypothesis $s'_1 \dots s'_{k-2}(c/y)$, which is of length $k - 1$ can be reduced to a sequence of transpositions.

2. y does appear in the substitutions $s_{i+1}, \dots, s_{k-1}$. Because of (E_2), we may assume without loss of generality that $s_{i+1} = (y/w)$. But now we can use

(E$_3$) to reduce the sequence $s_i s_{i+1}$ to (a/w). The resulting sequence is shorter and the conclusion follows by the induction hypothesis.

□

Lemma 3.5.2. The following can be derived from the equations $E_{\mathbb{I}}$:

1. For all pairwise distinct names $a, b, x, y \notin S$ we have that

$$(b/x)_{S\cup\{a\}}(a/b)_{S\cup\{x\}}(x/a)_{S\cup\{b\}} = (b/y)_{S\cup\{a\}}(a/b)_{S\cup\{y\}}(y/a)_{S\cup\{b\}}.$$

   In what follows we will abbreviate $(b/x)_{S\cup\{a\}}(a/b)_{S\cup\{x\}}(x/a)_{S\cup\{b\}}$ by $\sigma_{a,b}$.

2. If $a \neq b$ then $\sigma^2_{a,b} = \mathrm{id}$

3. If a, b, c, d are pairwise distinct names then $\sigma_{a,b}\sigma_{c,d} = \sigma_{c,d}\sigma_{a,b}$.

4. If a, b, c are pairwise distinct names then $(\sigma_{a,b}\sigma_{b,c})^3 = \mathrm{id}$.

Proof. 1. From the equations we can derive:

$$\begin{aligned}
&(b/x)_{S\cup\{a\}}(a/b)_{S\cup\{x\}}(x/a)_{S\cup\{b\}} \\
&\quad =^{(E_1)} (b/x)_{S\cup\{a\}}(x/y)_{S\cup\{a\}}(y/x)_{S\cup\{a\}}(a/b)_{S\cup\{x\}}(x/a)_{S\cup\{b\}} \\
&\quad =^{(E_3)} (b/y)_{S\cup\{a\}}(y/x)_{S\cup\{a\}}(a/b)_{S\cup\{x\}}(x/a)_{S\cup\{b\}} \\
&\quad =^{(E_2)} (b/y)_{S\cup\{a\}}(a/b)_{S\cup\{y\}}(y/x)_{S\cup\{b\}}(x/a)_{S\cup\{b\}} \\
&\quad =^{(E_3)} (b/y)_{S\cup\{a\}}(a/b)_{S\cup\{y\}}(y/a)_{S\cup\{b\}}.
\end{aligned} \tag{3.32}$$

2. Choose x, y distinct from a, b. Then

$$\begin{aligned}
&\sigma^2_{a,b} \\
&\quad = (b/x)_{S\cup\{a\}}(a/b)_{S\cup\{x\}}(x/a)_{S\cup\{b\}}(b/y)_{S\cup\{a\}}(a/b)_{S\cup\{y\}}(y/a)_{S\cup\{b\}} \\
&\quad =^{(E_2)} (b/x)_{S\cup\{a\}}(a/b)_{S\cup\{x\}}(b/y)_{S\cup\{x\}}(x/a)_{S\cup\{y\}}(a/b)_{S\cup\{y\}}(y/a)_{S\cup\{b\}} \\
&\quad =^{(E_3)} (b/x)_{S\cup\{a\}}(a/y)_{S\cup\{x\}}(x/b)_{S\cup\{y\}}(y/a)_{S\cup\{b\}} \\
&\quad =^{(E_2)} (b/x)_{S\cup\{a\}}(x/b)_{S\cup\{a\}}(a/y)_{S\cup\{b\}}(y/a)_{S\cup\{b\}} \\
&\quad =^{(E_1)} \mathrm{id}_{S\cup\{a,b\}}.
\end{aligned} \tag{3.33}$$

3. This follows easily from point 1. above and (E$_2$).

4. This can be proved in the same spirit as point 2. above. The key is to show that $\sigma_{a,b}\sigma_{b,c} = (c/y)(a/c)(b/a)(y/b)$ for some $y \notin \{a, b, c\}$.

□

Chapter 4

HSP like theorems in nominal sets

In this chapter we will prove an HSP-theorem for algebras over the topos $\mathrm{Sh}(\mathbb{I}^{op})$ in a systematic way.

In the first section we prove general results using categorical techniques. To set the scene we outline a categorical proof of Birkhoff's HSP theorem. Then, in Theorem 4.1.5, we show how to obtain an HSP-theorem for a full reflective subcategory $\mathscr{A}$ of a category of algebras $\mathscr{C}$, if some additional conditions are met. Recall that $\mathscr{A}$ is a reflective subcategory of $\mathscr{C}$ when the inclusion functor has a left adjoint, see [ARV10]. Essentially, this is achieved by 'pushing' the proof of the general HSP-theorem through the adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, bend right] \end{tikzcd} .$$

This result is interesting because $\mathscr{A}$ might not be a variety. We also prove a general result, Proposition 4.1.6, concerning a lifting property of an adjunction to categories of algebras for certain functors.

Secondly, in Section 4.2, we apply these results to the nominal setting. Consider the following diagram

$$\begin{tikzcd}
\mathsf{Alg}(\tilde{L}) \arrow[d, "\tilde{U}"'] \arrow[r, bend right, "\overline{I_*}"'] \arrow[r, phantom, "\perp"] & \mathsf{Alg}(L) \arrow[l, bend right, "\overline{I^*}"'] \arrow[d, "U"] \\
\mathsf{Sh}(\mathbb{I}^{op}) \arrow[loop left, "\tilde{L}"] \arrow[r, bend right, "I_*"'] \arrow[r, phantom, "\perp"] & \mathsf{Set}^{\mathbb{I}} \arrow[l, bend right, "I^*"'] \arrow[loop right, "L"]
\end{tikzcd} \tag{4.1}$$

where L is an endofunctor on $\mathsf{Set}^{\mathbb{I}}$ that preserves sifted colimits, and $\tilde{L}$ is an endofunctor on $\mathsf{Sh}(\mathbb{I}^{op})$, such that $\tilde{L}I^* \simeq I^*L$. Using Proposition 4.1.6 we prove that the adjunction between $\mathsf{Sh}(\mathbb{I}^{op})$ and $\mathsf{Set}^{\mathbb{I}}$ can be lifted to an adjunction between $\mathsf{Alg}(\tilde{L})$ and $\mathsf{Alg}(L)$. On the right hand side of this diagram we have categories monadic over $\mathsf{Set}^{|\mathbb{I}|}$, for which the classical HSP-theorem holds. We derive an HSP theorem for $\mathsf{Alg}(\tilde{L})$, by applying Theorem 4.1.5.

4.1 HSP theorems for full reflective subcategories

Birkhoff's HSP Theorem. Given a category of algebras $\mathscr{C}$, a full subcategory $\mathscr{B} \subseteq \mathscr{C}$ is closed under quotients (H for homomorphic images), subalgebras (S), and products (P) iff $\mathscr{B}$ is definable by equations.

This theorem can be proved at different levels of generality. We assume here that $\mathscr{C}$ is monadic over Set^{κ}, for some cardinal κ. We denote by $U : \mathscr{C} \to \mathsf{Set}^{\kappa}$ the forgetful functor and by F its left adjoint. Then we can identify a class of equations Φ in variables X with quotients $FX \twoheadrightarrow Q$. Indeed, given Φ we let Q be the quotient FX/Φ and, conversely, given $FX \twoheadrightarrow Q$ we let Φ be the kernel of $FX \twoheadrightarrow Q$. Further, an algebra $A \in \mathscr{C}$ satisfies the equations iff all $FX \to A$ factor through $FX \twoheadrightarrow Q$ as in the diagram

$$\begin{tikzcd}
A \models \Phi & \Leftrightarrow & FX \arrow[rr] \arrow[dr, "\forall"'] & & FX/\Phi \arrow[dl, dotted, "\exists"] \\
& & & A &
\end{tikzcd} \tag{4.2}$$

Proof of 'if'. We want to show that a subcategory $\mathscr{B}$ defined by equations Φ is closed under HSP. Closure under subobjects $A' \rightarrowtail A$ follows since quotients and subobjects form a factorisation system (see e.g. [AHS90, 14.1]). Indeed, according to (4.2), to show $A \models \Phi \Rightarrow A' \models \Phi$ one has to find the dotted arrow in

$$\begin{tikzcd}
FX \arrow[r] \arrow[d] & FX/\Phi \arrow[d] \arrow[dl, dotted] \\
A' \arrow[r] & A
\end{tikzcd}$$

which exists because of the diagonal fill-in property of factorisation systems. A similar argument works for products (because of their universal property) and for quotients (using that free algebras are projective [AHS90, 9.27]).

Proof of 'only if'. Given $\mathscr{B} \subseteq \mathscr{C}$, we first need to find the equations. Since $\mathscr{B}$ is closed under SP, $\mathscr{B}$ is a full reflective subcategory, that is, the inclusion $\mathscr{B} \to \mathscr{C}$ has a left-adjoint H and, moreover, the unit $A \to HA$ is a quotient.[1] We take as equations all $FX \to HFX$. That all $A \in \mathscr{B}$ satisfy these equations, again using (4.2), follows immediately from the universal property of the left-adjoint H. Conversely, suppose that A satisfies all equations. Consider the equations $q : FUA \to HFUA$. Because of (4.2), the counit $e : FUA \to A$ must factor as $e = f \circ q$. Since e and q are quotients, so is f. Hence A is a quotient of $HFUA$, which is in $\mathscr{B}$.

Remark 4.1.1. Notice that in the proof above we allow quotients $FX \to HFX$ for arbitrary κ-sorted sets X. If the set X is infinite, we allow equations involving infinitely many variables. Therefore we no longer reason within finitary logic. If we impose that the equations involve only finitely many variables, then the HSP theorem is *not* true for arbitrary many-sorted varieties. Indeed, in the many-sorted case, closure under homomorphic images, sub-algebras and products is no longer enough to deduce equational definability (see [ARV10, Example 10.14.2]). One needs an additional constraint, namely closure under directed unions, [ARV10, Theorem 10.12]. But in the motivating example of $\mathsf{Set}^{\mathbb{I}}$, we will prove that closure under HSP implies closure under directed unions.

In the following, we show that it is possible to obtain an HSP theorem for certain subcategories of varieties, by pushing the argument above through an adjunction. But first let us say what we mean in this context by **equationally definable** and **closed under HSP**.

We will work in the following setting. Let $\mathscr{C}$ be a category monadic over Set^κ for some cardinal κ, let $U : \mathscr{C} \to \mathsf{Set}^\kappa$ denote the forgetful functor and let F be its

---

[1] To construct HA given A, consider all arrows $f : A \to B_f$ with codomain in $\mathscr{B}$; factor $f = A \xrightarrow{q_f} \bar{B}_f \xrightarrow{i_f} B$; up to isomorphism, there is a only a proper set of different q_f; now factor $A \xrightarrow{\langle q_f \rangle} \prod_f \bar{B}_f$ as $A \to HA \to \prod_f \bar{B}_f$ to obtain the unit $A \to HA$.

left adjoint. Consider $\mathscr{A}$ a full reflective subcategory of $\mathscr{C}$. Let $I : \mathscr{A} \to \mathscr{C}$ denote the inclusion functor and let $R : \mathscr{C} \to \mathscr{A}$ denote its left adjoint. Assume that $\mathscr{A}$ has a factorization system (E, M), such that for all regular epimorphisms e in $\mathscr{C}$ we have $Re \in E$ and for all monomorphisms m in $\mathscr{C}$ we have that $Rm \in M$.

Definition 4.1.2. We say that $\mathscr{B} \hookrightarrow \mathscr{A}$ is **equationally definable** if there exists a set of equations Φ in $\mathscr{C}$, such that an object A of $\mathscr{A}$ lies in $\mathscr{B}$ iff $IA \models \Phi$ (where Φ and $\models$ are as in (4.2)). We say that $\mathscr{B}$ is **closed under HSP** if and only if

1. For all morphisms $e : B \to B'$ such that $e \in E$ and Ie is a quotient, we have that $B \in \mathscr{B}$ implies $B' \in \mathscr{B}$.
2. For all morphisms $m : B \to B'$ such that $m \in M$ we have $B' \in \mathscr{B}$ implies $B \in \mathscr{B}$.
3. If B_i are in $\mathscr{B}$ then their product in $\mathscr{A}$ is an object of $\mathscr{B}$.

Remark 4.1.3. In general, the inclusion functor I does not preserve epimorphisms. We will assume that the arrows in M are monomorphisms. Being a right adjoint, I preserves products and monomorphisms, but we cannot infer from $B' \to IB$ being a monomorphism in $\mathscr{C}$ that B' is (isomorphic to an object) in $\mathscr{A}$.

Remark 4.1.4. The third item of Definition 4.1.2 makes sense only if $\mathscr{A}$ has products. But $\mathscr{A}$ is complete, since $\mathscr{A}$ is a full reflective category of a complete category, see [Bor94, Proposition 3.5.3].

If $\mathscr{C}$ is a category monadic over Set^κ for some cardinal κ and $\mathscr{A}$ is a full reflective subcategory of $\mathscr{C}$, then $\mathscr{A}$ is complete and is well-powered because $\mathscr{C}$ is. Hence we can equip $\mathscr{A}$ with a strong-epi/mono factorisation system [Bor94, 4.4.3]. We can prove:

Theorem 4.1.5. Let $\mathscr{C}$ be a category monadic over Set^κ for some cardinal κ and $\mathscr{A}$ a full reflective subcategory of $\mathscr{C}$, such that the left adjoint of the inclusion functor preserves monomorphisms. Then $\mathscr{B} \subseteq \mathscr{A}$ is closed under HSP in the sense of Definition 4.1.2 if and only if $\mathscr{B}$ is equationally definable.

Proof. We will use U to denote the forgetful functor $\mathscr{C} \to \mathsf{Set}^\kappa$ and F to denote its left adjoint. I denotes the full and faithful functor $\mathscr{A} \to \mathscr{C}$ and R denotes its left adjoint.

Note that in $\mathscr{A}$ (as in $\mathscr{C}$) strong epis coincide with extremal epis [Bor94, 4.3.7] and with regular epis [AHS90, 14.14, 14.22]. The proof of the theorem relies on the following two properties

$$e \text{ regular epi in } \mathscr{C} \;\Rightarrow\; Re \text{ regular epi in } \mathscr{A} \tag{4.3}$$

$$m \text{ mono in } \mathscr{A} \;\Rightarrow\; Im \text{ mono in } \mathscr{C} \tag{4.4}$$

(4.3) holds because R is a left-adjoint and (4.4) because I is a right adjoint. Also note that we have the converse of (4.4), since I is full and faithful.

Let us prove that equational definability implies closure under HSP. Let $\mathscr{B}$ be an equationally definable subcategory of $\mathscr{A}$. That means that there exists an equationally definable subcategory $\mathscr{B}'$ of $\mathscr{C}$ such that B is an object of $\mathscr{B}$ iff IB is an object of $\mathscr{B}'$. The proof of the fact that $\mathscr{B}$ is closed under HSP in the sense of Definition 4.1.2, follows from the HSP theorem applied for $\mathscr{B}'$ and the following observations:

1. The quotients $e \in E$ considered in Definition 4.1.2 are exactly those for which Ie is a regular epimorphism in $\mathscr{C}$.
2. I preserves monomorphisms.
3. I preserves products.

Conversely, let $\mathscr{B}$ be a subcategory of $\mathscr{A}$, closed under HSP, as in the previous definition. We will prove that $\mathscr{B}$ is equationally definable. We proceed in three steps:

Step 1 (construction of the equations that define $\mathscr{B}$): Let C be an arbitrary object of $\mathscr{C}$. We will consider all morphisms $f_i : RC \to B_i$ in $\mathscr{A}$ such that B_i is in $\mathscr{B}$. For each i, the corresponding morphism in $\mathscr{C}$, $f_i^\sharp : C \to IB_i$ factors in $\mathscr{C}$:

$$\begin{tikzcd}
C \arrow[dd, "f_i^\sharp"'] \arrow[dr, "e_i"] & \\
& \overline{B_i} \arrow[dl, "m_i"] \\
IB_i &
\end{tikzcd}$$

We will denote by η and ε the unit, respectively co-unit, of the adjunction $R \dashv I$. One can easily show that the following diagram commutes:

$$\begin{tikzcd}
RC \arrow[dd, "f_i"'] \arrow[dr, "Re_i"] & \\
& R\overline{B_i} \arrow[dl, "\varepsilon_{B_i} \circ Rm_i"] \\
B_i &
\end{tikzcd} \tag{4.5}$$

Since R preserves regular epis and monos, and ε_{B_i} is an isomorphism (I is full and faithful), we have that (4.5) is a factorisation of f_i in $\mathscr{A}$. But $\mathscr{B}$ is closed under subobjects, hence $R\overline{B_i}$ is actually an object of $\mathscr{B}$. Since $\mathscr{C}$ is co-well-powered, there is only a proper set of different e_i up to isomorphism, so we can take the product P of the objects of the form $R\overline{B_i}$, obtained as above. P is again an object of $\mathscr{B}$, and we have a morphism $\alpha : RC \to P$, uniquely determined by the Re_i. We consider a factorisation in $\mathscr{C}$ of the adjoint map $\alpha^\sharp : C \to IP$:

$$\begin{tikzcd}
C \arrow[dd, "\alpha^\sharp"'] \arrow[dr, "e"] & \\
& Q_C \arrow[dl, "m"] \\
IP &
\end{tikzcd} \tag{4.6}$$

Using a similar argument to the above we deduce that RQ_C is an object of $\mathscr{B}$ and the following diagram commutes:

$$\begin{tikzcd}
RC \arrow[rr, "Re"] \arrow[d, "f_i"] & & RQ_C \arrow[d, "\varepsilon_P \circ Rm"] \\
B_i & R\overline{B_i} \arrow[l, "\varepsilon_B \circ Rm_i"] & P. \arrow[l, "\pi_i"]
\end{tikzcd} \tag{4.7}$$

We consider the class of equations $\mathscr{E}$ of the form $FX \to Q_{FX}$ for all sets X, and denote by $\mathscr{B}'$ the subcategory of $\mathscr{C}$ defined by these equations.

Step 2 ($\mathscr{B}$ is contained in the class defined by the equations $\mathscr{E}$): We show that if an object B of $\mathscr{A}$ lies in $\mathscr{B}$, then IB satisfies the equations in $\mathscr{E}$. Let B be an object of $\mathscr{B}$, and let $u : FX \to IB$ be an arbitrary morphism. For the adjoint morphism $u_\sharp : RFX \to B$, one can construct a morphism $g : RQ_{FX} \to B$, obtained as in diagram (4.7), such that $g \circ Re = u_\sharp$.

It is easy to see that $g^\sharp : Q_{FX} \to IB$ makes the following diagram commutative. This shows that IB satisfies the equation $e : FX \to Q_{FX}$:

$$\begin{tikzcd}
FX \arrow[r, "e"] \arrow[d, "u"'] & Q_{FX} \arrow[dl, "g^\sharp"] \\
IB &
\end{tikzcd} \tag{4.8}$$

Step 3 (The subcategory defined by the equations $\mathscr{E}$ is contained in $\mathscr{B}$): Let B be an object in $\mathscr{A}$ such that IB satisfies the equations in $\mathscr{E}$. In particular IB

satisfies $FUIB \to Q_{FUIB}$, so there exists $v : Q_{FUIB} \to IB$ such that $v \circ e = \varepsilon'_{IB}$, where ε' is the counit of the adjunction $F \dashv U$. Since ε' is a regular epi, then v is also a regular epi. We have that the composition $\varepsilon_B \circ Rv : RQ_{FUIB} \to B$ is a regular epi in $\mathscr{A}$. Since the codomain of v is in the image of I, IRv is also a regular epi, therefore so is $I(\varepsilon_B \circ Rv)$. Using the fact that $\mathscr{B}$ is closed under H, and that RQ_{FUIB} is already in $\mathscr{B}$, we can conclude that $B \in \mathscr{B}$. □

The next proposition allows us to lift an adjunction between two categories to an adjunction between categories of algebras for functors satisfying some additional conditions.

Proposition 4.1.6. Let $\langle R, I, \eta, \varepsilon \rangle : \mathscr{A} \rightharpoonup \mathscr{B}$ be an adjunction. Let K and L be endofunctors on $\mathscr{A}$ and $\mathscr{B}$, respectively. Suppose there exist natural transformations: $\kappa : KR \to RL$ and $\lambda : LI \to IK$ making the following diagrams commute:

$$\begin{tikzcd}
LIR \arrow[r, "\lambda R"] & IKR \arrow[r, "I\kappa"] & IRL & KRI \arrow[r, "\kappa I"] \arrow[dr, "K\varepsilon"'] & RLI \arrow[r, "R\lambda"] & RIK \arrow[dl, "\varepsilon K"] \\
 & L \arrow[ul, "L\eta"] \arrow[ur, "\eta L"'] & & & K & \\
 & (C_1) & & & (C_2) &
\end{tikzcd} \tag{4.9}$$

Then there exists an adjunction $\langle \overline{R}, \overline{I}, \overline{\eta}, \overline{\varepsilon} \rangle : \mathsf{Alg}(K) \rightharpoonup \mathsf{Alg}(L)$, such that $U_{\mathscr{A}} \overline{R} = RU_{\mathscr{B}}$ and $IU_{\mathscr{A}} = U_{\mathscr{B}} \overline{I}$, where $U_{\mathscr{A}}$ and $U_{\mathscr{B}}$ denote the forgetful functors as in the next diagram:

$$\begin{tikzcd}
\mathsf{Alg}(K) \arrow[r, bend right, "\overline{I}"'] \arrow[r, phantom, "\perp"] \arrow[d, "U_{\mathscr{A}}"'] & \mathsf{Alg}(L) \arrow[l, bend right, "\overline{R}"'] \arrow[d, "U_{\mathscr{B}}"] \\
\mathscr{A} \arrow[loop left, "K"] \arrow[r, bend right, "I"'] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend right, "R"'] \arrow[loop right, "L"]
\end{tikzcd} \tag{4.10}$$

Proof. First let us define the functor $\overline{I}$. Let $f : KA \to A$ be a K-algebra. We define $\overline{I}(A, f) := (IA, If \circ \lambda_A)$. For an arbitrary morphism of K-algebras $u : (A, f) \to (A', f')$, we define $\overline{I}(u) = Iu$. The fact that Iu is a morphism of L-algebras follows from the commutativity of the outer square of the next diagram:

$$\begin{tikzcd}
LIA \arrow[r, "\lambda_A"] \arrow[d, "LIu"'] & IKA \arrow[r, "If"] \arrow[d, "IKu"'] & IA \arrow[d, "Iu"'] \\
LIA' \arrow[r, "\lambda'_A"] & IKA' \arrow[r, "If'"] & IA'
\end{tikzcd} \tag{4.11}$$

But the small squares commute, the former because λ is a natural transformation, and the latter because u is a K-algebra morphism. It is obvious that $\overline{I}$ is a functor and that $IU_{\mathscr{A}} = U_{\mathscr{B}}\overline{I}$. The functor $\overline{R}$ is defined similarly: if $g : LB \to B$ is a L-algebra, we define $\overline{R}(B,g) = (RB, Rg \circ \kappa_B)$. If $v : (B,g) \to (B',g')$ is a L-algebra morphism, we define $\overline{R}(v) = Rv$. The fact that Rv is indeed a K-algebra morphism is verified easily, using the naturality of κ and the fact that v is a L-algebra morphism.

In order to prove that $\overline{R}$ is left adjoint to $\overline{I}$, we will show that the unit η and the counit ε of the adjunction $R \dashv I$ are L-algebra and K-algebra morphisms respectively. This follows from the hypothesis (4.9) and the naturality of η and ε respectively.

Once this is achieved, η can be lifted to a natural transformations $\overline{\eta} : \mathrm{id} \to \overline{IR}$, and similarly ε can be lifted to a natural transformation $\overline{\varepsilon} : \overline{RI} \to \mathrm{id}$. But η and ε are the unit and the counit of the adjunction $R \dashv I$, therefore they satisfy the usual triangular equalities. Therefore, $\overline{\eta}$ and $\overline{\varepsilon}$ satisfy the triangular equalities for $\overline{R}$ and $\overline{I}$. □

Remark 4.1.7. The proposition has some useful special cases, under the additional assumption that I is full and faithful. For each of them, the commutativity of the diagrams (4.9) is straightforward to verify, using that the counit ε is iso.

1. Suppose L is given and we want to find an appropriate K. Then it follows from the theorem that we can do this, provided there is a natural transformation $\alpha : LIR \to IRL$ such that the diagram below commutes:

$$\begin{tikzcd}
LIR \arrow[rr, "\alpha"] & & IRL \\
& L \arrow[ul, "L\eta"] \arrow[ur, "\eta L"'] &
\end{tikzcd} \tag{4.12}$$

   If this is the case, one defines $K = RLI$, $\kappa : KR = RLIR \to RL$ as the composition $\varepsilon_{RL} \circ R\alpha$ and $\lambda : LI \to IRLI = IK$ as η_{LI}. Moreover, when α is an isomorphism, we have that $R\alpha$ and hence $KR \to RL$ are isomorphisms.

2. More generally, suppose we have given an isomorphism $\kappa : KR \to RL$. Then we define $\lambda = IK\varepsilon \circ I\kappa^{-1}I \circ \eta LI$. (Given K we can always find such a κ: Let $L = IKR$ and $\kappa = (\varepsilon KR)^{-1} : KR \cong RIKR$.)

Let Σ be a polynomial functor $\Sigma X = \coprod_{i \in J} X^{n_i}$. If $\mathscr{A}$ and $\mathscr{B}$ have (co)products, then Σ is defined on both categories, so it makes sense to write $R\Sigma \cong \Sigma R$. The following corollary says that for polynomial functors Σ the adjunction always lifts from the base categories to the categories of Σ-algebras.

Corollary 4.1.8. Let $\langle R, I, \eta, \varepsilon\rangle : \mathscr{A} \rightharpoonup \mathscr{B}$ be an adjunction such that I is full and faithful. Further assume that both categories have coproducts and finite products and that R preserves finite products. Consider an endofunctor $\Sigma X = \coprod_{i \in J} X^{n_i}$ on $\mathscr{A}$ and on $\mathscr{B}$. Then the adjunction lifts to an adjunction $\langle \bar{R}, \bar{I}, \eta, \varepsilon\rangle : \mathsf{Alg}(\Sigma) \rightharpoonup \mathsf{Alg}(\Sigma)$.

Proof. We use item 2 of the remark above and calculate $R\Sigma A = R(\coprod_{i \in J} A^{n_i}) \cong \coprod_{i \in J} (RA)^{n_i} = \Sigma RA$. □

Remark 4.1.9. $\overline{R}$ preserves finite limits whenever R does.

Proof. Assume $(B, g) = \varprojlim(B_i, g_i)$ is a finite limit in $\mathsf{Alg}(L)$. Since $U_{\mathscr{B}}$ preserves all limits by [ARV10, Remark 12.17], we have that $B = \varprojlim(B_i)$ in $\mathscr{B}$, and therefore $RB = \varprojlim(RB_i)$. Denote $\overline{R}(B_i, g_i)$ by (RB_i, f_i) and by $\pi_i : RB \to RB_i$ the morphisms of the limiting cone. For each index i we have a map $p_i : KRB \to RB_i$ obtained as the composition $f_i \circ K\pi_i$. From the universal property, we obtain a map $f : KRB \to RB$, such that each π_i is a K-algebra morphism from (RB, f) to (RB_i, f_i). We prove next that $(RB, f) = \varprojlim(RB_i, f_i)$ in $\mathsf{Alg}(K)$. Assume that we have a cone $q_i : (C, h) \to (RB_i, g_i)$. Since RB is a limit in $\mathscr{A}$ we get a unique map $k : C \to RB$, such that $\pi_i \circ k = q_i$. We need to show that k is a K-algebra morphism. To this end we will use the uniqueness of a morphism from $KC \to RB$ that makes the relevant diagrams commutative. □

4.2 HSP theorem for nominal sets and sheaf algebras

In this section we will prove an HSP theorem for algebras over $\mathrm{Sh}(\mathbb{I}^{op})$. We will call these algebras 'sheaf algebras'. Some of them, given by particular signatures, correspond, in a sense that will be made precise in Section 4.5.1, to nominal algebras, [GM09]. The signature will be given by a functor L on $\mathsf{Set}^{\mathbb{I}}$ that has a finitary presentation. On $\mathrm{Sh}(\mathbb{I}^{op})$ we can define the functor $\tilde{L}$ as I^*LI_*. Throughout this section we will use the notations from Diagram (4.1). The goal of this section is to derive an HSP theorem for $\mathsf{Alg}(\tilde{L})$ from Theorem 4.1.5. To this end we will need to impose some reasonable conditions on the functor L.

Theorem 4.2.1. HSP theorem for 'sheaf algebras'. In the situation of Diagram (4.1), let L be an endofunctor with a finitary presentation on $\mathsf{Set}^{\mathbb{I}}$ and assume there exists a natural transformation $\alpha : LI_*I^* \longrightarrow I_*I^*L$ such that $\alpha \circ L\eta = \eta L$. Let $\tilde{L}$ be the endofunctor on $\mathrm{Sh}(\mathbb{I}^{op})$ defined as I^*LI_*. Then a full subcategory of $\mathsf{Alg}(\tilde{L})$ is closed under HSP if and only if it is equationally definable.

Proof. By Theorem 3.2.4, we have that $\mathsf{Alg}(L)$ is monadic over $\mathsf{Set}^{|\mathbb{I}|}$, so it has a regular factorisation system. By point 1 of Remark 4.1.7 and Proposition 4.1.6, we can lift the adjunction $I^* \dashv I_*$ of Proposition 2.4.8 to an adjunction $\overline{I^*} \dashv \overline{I_*}$ between the categories of $\tilde{L}$-algebras and L-algebras. Since $\overline{I^*}$ preserves finite limits, it preserves monomorphisms, so we can apply Theorem 4.1.5. □

Remark 4.2.2. In the case of algebras over $\mathsf{Set}^{\mathbb{I}}$, we can assume that the equations defining a subcategory closed under HSP involve only finitely many variables. It is enough to prove that if $\mathscr{A} \subseteq \mathsf{Alg}(L)$ is closed under HSP then $\mathscr{A}$ is also closed under directed unions, see also Remark 4.1.1. The idea of the proof is to construct the directed union of a directed family $(X_i)_{i\in J}$ of algebras in $\mathscr{A}$, as a homomorphic image of a subalgebra of a product of algebras X_i, as in the proof of [AR94, Theorem 3.9]. The subtlety here is that the product considered there may be empty in the general many-sorted case, even if some of the X_i are not. However, in the case of algebras over $\mathsf{Set}^{\mathbb{I}}$ we can see that if one algebra X_{i_0} has the underlying presheaf non-empty, say, for example $X_{i_0}(S) \neq \emptyset$, then for all $j \geq i_0$ and for all sets T of cardinality larger than that of S we also have that $X_j(T)$ is non-empty. We can consider the product of the X_j for $j \geq i_0$, and this is non-empty.

4.3 Concrete syntax

This section illustrates the concrete syntax obtained from the abstract category-theoretic treatment of Section 4.2.

Specifying additional operations by a functor has the advantage that the initial algebra of terms comes equipped with an inductive principle. For an example see how λ-terms form the initial algebra for a functor in [FPT99, GP99, Hof99].

Recall from Section 3.3 the presentation for the category $\mathsf{Set}^{\mathbb{I}}$ given in Theorem 3.3.1 and the 'shift' functor δ on $\mathsf{Set}^{\mathbb{I}}$ from Definition 3.3.2, which corresponds to the abstraction operator of [GP99] and to the 'shift' functor on $\mathsf{Set}^{\mathbb{F}}$ from [FPT99]. Theorem 3.3.3 gives a presentation of δ by operations and equations.

Using these results, we illustrate the concrete syntax obtained in our setting, by giving a theory for the λ-calculus. We consider an endofunctor L on $\mathsf{Set}^{\mathbb{I}}$

given by

$$LX = \mathscr{N} + \delta X + X \times X \tag{4.13}$$

where $\mathscr{N}$ denotes the inclusion functor $\mathscr{N} : \mathbb{I} \to \mathsf{Set}$. In order to show that the HSP theorem holds for $\tilde{L}$-algebras, we need to prove that L satisfies the conditions in Theorem 4.2.1. This is actually a particular case of Proposition 4.4.14 that will be proved in the next section. In Section 4.3.1 we prove that the sheaf of λ-terms up to α-equivalence is the initial algebra for $\tilde{L} = I^* L I_*$ and we give the equations that characterise the subalgebra of λ-terms modulo $\alpha\beta\eta$-equivalence.

4.3.1 Axioms for the λ-calculus

The α-equivalence classes of λ-terms over $\mathbb{A}$ form a sheaf Λ_α in Sh($\mathbb{I}^{op}$). Indeed, we can define $\Lambda_\alpha(S)$ as the set of α-equivalence classes of λ-terms with free variables in S. On functions Λ_α acts by renaming the free variables.

We consider the endofunctor L on $\mathsf{Set}^{\mathbb{I}}$, defined by (4.13) and the endofunctor $\tilde{L}$ on Sh($\mathbb{I}^{op}$) defined as $I^* L I_*$. In a similar fashion as in [FPT99], we will show that Λ_α is isomorphic to the initial algebra $\mathscr{I}_{\tilde{L}}$ for $\tilde{L}$.

First let us notice that the underlying presheaf of $\mathscr{I}_{\tilde{L}}$ is the initial algebra $\mathscr{I}_L$ for L. Indeed, one can prove

Lemma 4.3.1. We have $I_* \mathscr{I}_{\tilde{L}} = \mathscr{I}_L$.

Proof. We can check that $\tilde{L}$ preserves ω-chains, so the initial algebra $\mathscr{I}_{\tilde{L}}$ is computed as the colimit of the sequence

$$\tilde{0} \to \tilde{L}\tilde{0} \to \tilde{L}^2\tilde{0} \to \cdots \to \mathscr{I}_{\tilde{L}} \tag{4.14}$$

where $\tilde{0}$ is just the empty sheaf. Let us denote by 0 the empty presheaf. Similarly $\mathscr{I}_L$ is the colimit of the initial sequence for L:

$$0 \to L0 \to L^2 0 \to \cdots \to \mathscr{I}_L. \tag{4.15}$$

Using the observation that L preserves sheaves and the fact that $I_*\tilde{0} = 0$ we can easily verify that $I_*\tilde{L}^n\tilde{0} \simeq L^n 0$ for all natural numbers n. But I_* preserves filtered colimits, so we have that $I_*\mathscr{I}_{\tilde{L}} \simeq \mathscr{I}_L$. □

We consider a functor $\Sigma : \mathsf{Set}^{\mathbb{I}} \to \mathsf{Set}^{\mathbb{I}}$, defined by

$$\Sigma X = \mathscr{N} + \mathscr{N} \times X + X \times X. \tag{4.16}$$

Notice that Σ preserves sheaves and that the initial algebra for Σ, let us denote it by $\mathscr{I}_\Sigma$, is just the presheaf of all λ-terms. Using a similar argument as above we can see that $I_*\mathscr{I}_{\tilde{\Sigma}} = \mathscr{I}_{\Sigma}$. $\mathscr{I}_{\tilde{\Sigma}}$ is the sheaf of all λ-terms. We will prove the isomorphism between Λ_α and $\mathscr{I}_{\tilde{L}}$ by constructing an epimorphism $\mathscr{I}_{\tilde{\Sigma}} \to \mathscr{I}_{\tilde{L}}$ in $\mathrm{Sh}(\mathbb{I}^{op})$ that identifies exactly α-equivalent terms.

First let us prove the next lemma:

Lemma 4.3.2. There exists a natural transformation $\theta : \mathscr{N} \times - \to \delta$ such that for any presheaf $X \in \mathsf{Set}^{\mathbb{I}}$ and any finite set of names $S \subseteq \mathbb{A}$ we have that $\theta(a,x) = \theta(b,y)$ for some $a,b \in S$ and $x,y \in X(S)$ if and only if for all $c \notin S$ we have

$$X(w_{S\setminus\{a\}\cup\{c\},a}(c/a)_{S\setminus\{a\}})(x) = X(w_{S\setminus\{b\}\cup\{c\},b}(c/b)_{S\setminus\{b\}})(x). \tag{4.17}$$

Moreover if X is a sheaf, then $\theta_X : \mathscr{N} \times X \to \delta X$ is a sheaf epimorphism.

Proof. We define $\theta_X(S) : \mathscr{N}(S) \times X(S) \to (\delta X)(S)$ by

$$(a,x) \mapsto (\delta X)(w_{S\setminus\{a\},a})(\{[a]_{S\setminus\{a\}}x\}_{\delta X}) \tag{4.18}$$

where $w_{S\setminus\{a\},a}$ is the inclusion of $S\setminus\{a\}$ into S. It is not difficult to check that this is indeed a natural transformation. Assume now that (a,x), $(b,y) \in \mathscr{N}(S) \times X(S)$ and $c \in \mathbb{A} \setminus S$. We have that $\theta(a,x) = \theta(b,x)$ is equivalent to

$$(\delta X)(w_{S\setminus\{a\},a})(\{[a]_{S\setminus\{a\}}x\}_{\delta X}) = (\delta X)(w_{S\setminus\{b\},b})(\{[b]_{S\setminus\{b\}}y\}_{\delta X}) \tag{4.19}$$

But

$$\begin{aligned}(\delta X)(w_{S\setminus\{a\},a})(\{[a]_{S\setminus\{a\}}x\}_{\delta X}) &= (\delta X)(w_{S\setminus\{a\},a})(\{[c]_{S\setminus\{a\}}(c/a)_{S\setminus\{a\}}x\}_{\delta X})\\ &= \{[c]_S X(w_{S\setminus\{a\},a}+c)((c/a)_{S\setminus\{a\}}(x))\}_{\delta X}\\ &= \{[c]_S X(w_{S\setminus\{a\}\cup\{c\},a})((c/a)_{S\setminus\{a\}}(x))\}_{\delta X}\end{aligned} \tag{4.20}$$

Similarly, $(\delta X)(w_{S\setminus\{b\},b})(\{[b]_{S\setminus\{b\}}y\}_{\delta X}) = \{[c]_S X(w_{S\setminus\{b\}\cup\{c\},b})((c/b)_{S\setminus\{b\}}(y))\}_{\delta X}$. Since $c \notin S$ we have an isomorphism $\delta X(S) \simeq X(S \cup \{c\})$. Therefore (4.19) is equivalent to (4.17).

In order to prove the last statement of the lemma, we use the characterisation of sheaf epimorphisms given in Proposition 2.4.16. Let $\{[c]_S y\}_{\delta X}$ be an arbitrary element of $(\delta X)(S)$. We have that $c \notin S$ and $y \in X(S \cup \{c\})$. The conclusion follows from the fact that $\theta_X(S \cup \{c\})(c,y) = (\delta X)(w_{S,c})(\{[c]_S y\}_{\delta X})$. □

Proposition 4.3.3. The sheaf of α-equivalence classes of λ-terms is isomorphic to the initial $\tilde{L}$-algebra $\mathscr{I}_{\tilde{L}}$.

Proof. This can be obtained by an inductive argument on the structure of the λ-terms. Using the natural transformation θ defined above we can construct a natural transformation $\vartheta : \Sigma \to L$ defined as $\vartheta_X = \mathrm{id}_{\mathbb{A}} + \theta + \mathrm{id}_X \times \mathrm{id}_X$. Now we can define inductively a natural transformation $\zeta^{(n)} : \Sigma^n \to L^n$. Explicitly, $\zeta^{(0)} = \mathrm{id}_0$ and $\zeta^{(n+1)}_X = L^n(\vartheta_X)\zeta^{(n)}_{\Sigma X}$. We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\zeta^{(0)}"] & \Sigma 0 \arrow[r] \arrow[d, "\zeta^{(1)}"] & \Sigma^2 0 \arrow[r] \arrow[d, "\zeta^{(2)}"] & \cdots \arrow[r] & \mathscr{I}_\Sigma \arrow[d, "\zeta"] \\
0 \arrow[r] & L0 \arrow[r] & L^2 0 \arrow[r] & \cdots \arrow[r] & \mathscr{I}_L
\end{tikzcd} \tag{4.21}$$

where ζ is obtained by taking the colimit. As seen above, $\mathscr{I}_L$ and $\mathscr{I}_\Sigma$ are the underlying presheaves for the sheaves $\mathscr{I}_{\tilde{L}}$ and $\mathscr{I}_{\tilde{\Sigma}}$, respectively.

Using Lemma 4.3.2, we can argue inductively that $\zeta^{(n)}_X$ is a sheaf epimorphism for all n and for all sheaves X. One can verify that this implies that ζ is a sheaf epimorphism. If two terms in $\mathscr{I}_\Sigma$ are identified by ζ, then they must be identified at some stage n by $\zeta^{(n)}$. Using Lemma 4.3.2 again, we can show by induction that two terms are in the kernel of $\zeta^{(n)}$ if and only if they are α-equivalent. □

To illustrate the concrete syntax appearing in our setting, we give a presentation for the functor L and a theory over the signature given by L for $\alpha\beta\eta$-equivalence of λ-terms.

Proposition 4.3.4. The endofunctor L is presented by a set of operation symbols

$$\begin{aligned}
a_S &: S \cup \{a\} \\
\mathsf{app}_S &: S \times S \to S \\
[a]_S &: S \cup \{a\} \to S
\end{aligned}$$

where S is a finite set of names and $a \notin S$, and the following set of equations:

$$\begin{array}{rcll}
 & \vdash & (b/a)_S a_S = b_S & (\mathrm{E}_0) \\
 & \vdash & w_{S\cup\{a\},b}\, a_S = a_{S\cup\{b\}} & (\mathrm{E}_1) \\
t : S \cup \{a,b\} & \vdash & (c/b)_S([a]_{S\cup\{b\}} t) = \mathrm{id}_{S\cup\{c\}}([a]_{S\cup\{c\}}(c/b)_{S\cup\{a\}} t) & (\mathrm{E}_2) \\
t : S \cup \{a\} & \vdash & [a]_S t = [b]_S (b/a)_S t & (\mathrm{E}_3) \\
t : S \cup \{a\} & \vdash & w_{S,b}[a]_S t = [a]_{S\cup\{b\}} w_{S\cup\{a\},b}\, t & (\mathrm{E}_4) \\
t_1, t_2 : S & \vdash & w_{S,a}\mathsf{app}_S(t_1,t_2) = \mathsf{app}_{S\cup\{a\}}(w_{S,a} t_1, w_{S,a} t_2) & (\mathrm{E}_5) \\
t_1, t_2 : S \cup \{a\} & \vdash & (b/a)_S \mathsf{app}_{S\cup\{a\}}(t_1,t_2) = \mathsf{app}_{S\cup\{b\}}((b/a)_S t_1, (b/a)_S t_2) & (\mathrm{E}_6)
\end{array}$$

Proof. The operation symbols a_S and the equations (E_0) and (E_1) present the constant functor $\mathscr{N}$. The operation symbols $[a]_S$ and the equations (E_2), (E_3) and (E_4) give a presentation of δ, recall Theorem 3.3.3. Finally, the operations of the form app_S and equations (E_5) and (E_6) form a presentation for $X \times X$, just as in Example 3.2.3. □

Example 4.3.5. The subalgebra of $\mathsf{Alg}(\tilde{L})$ of λ-terms modulo $\alpha\beta\eta$-equivalence is definable by the following equations, similar to [CP07, Fig. 4].

$$
\begin{array}{rll}
X : S\cup\{a\};\ Y : S \vdash & \mathsf{app}_{S\cup\{a\}}(w_a[a]_S(w_a Y), X) = w_a Y \ : S\cup\{a\} & (\beta\text{-}1)\\
X : S \vdash & \mathsf{app}_S([a]_S(a_S), X) = X \ : S & (\beta\text{-}2)\\
X : S\cup\{a,b\};\ Y : S \vdash & \mathsf{app}_S([a]_S([b]_{S\cup\{a\}}(X)), Y) = & \\
 & [b]_S(\mathsf{app}_{S\cup\{b\}}([a]_{S\cup\{b\}}(X), w_a Y)) \ : S & (\beta\text{-}3)\\
X, Y : S\cup\{a\};\ Z : S \vdash & \mathsf{app}_S([a]_S(\mathsf{app}_{S\cup\{a\}}(X, Y)), Z) = & \\
 & \mathsf{app}_S(\mathsf{app}_S([a]_S(X), Z), \mathsf{app}_S([a]_S(Y), Z)) \ : S & (\beta\text{-}4)\\
X : S\cup\{a\} \vdash & \mathsf{app}_{S\cup\{b\}}(w_b[a]_S(X), b_S) = (b/a)X \ : S\cup\{b\} & (\beta\text{-}5)\\
X : S \vdash & [a]_S(\mathsf{app}_{S\cup\{a\}}(w_a X, a_S)) = X \ : S & (\eta)
\end{array}
$$

The theory above is obtained by adapting the nominal equational theory of [CP07, Fig. 4]. In Section 4.5.2 we make this precise, by giving an algorithm for translating arbitrary theories of the nominal equational logic of [CP07] into many-sorted theories having the same models. Hence the fact that the above equations define λ-terms modulo $\alpha\beta\eta$-equivalence follows from [CP07, Example 9.5] and Theorem 4.5.10.

4.4 Uniform Theories

Gabbay [Gab08] proves an HSP theorem for nominal algebras, or rather an HSPA-theorem: A class of nominal algebras is definable by a theory of nominal algebra iff it is closed under HSP and under abstraction.

Our equational logic is more expressive than Gabbay's in the sense that more classes are equationally definable, namely all those closed under HSP where H refers not to closure under all quotients as in [Gab08], but to the weaker property of closure under support-preserving quotients (i.e. quotients in the presheaf category). Of course, it is a question whether this additional expressivity is wanted. We therefore isolate a fragment of standard equational logic, which we call the *uniform fragment*, and define notions of uniform signature, uniform terms and uniform equations. The main idea is that a uniform equation $\Gamma \vdash t = u : T$, where $\Gamma = \{X_1 : T_1, \dots, X_n : T_n\}$ is a context of variables and T a finite subset of $\mathbb{A}$, has an interpretation uniform in all sorts S containing T.

For this uniform fragment we are able to extend Theorem 3.11 to an HSPA theorem in the style of [Gab08]: classes of sheaf algebras are definable by uniform equations if and only if they are closed under quotients, subalgebras, products and under abstraction.

Let us give an intuitive motivation for the notions introduced in this section. Assume we want to investigate algebraic theories over nominal sets by studying their transport to $[\mathbb{I}, \mathsf{Set}]$. Suppose we have some notion of signature and equations over nominal sets, such as the nominal equational logics of [GM09, CP07]. A nominal set X satisfies an equation, if for any instantiation of the variables, possibly respecting some freshness constraints, we get equality in X. Notice that the support of the elements of X used to instantiate the variables can be arbitrarily large. Let us think what this means in terms of the corresponding presheaf IX. For a finite set of names S, $IX(S)$ is the set of elements of X supported by S. So IX should satisfy not one, but a set of 'uniform' equations. This means that we should be able to extend in a 'uniform' way the operation symbols together with their arities, the sort of the equations and the sort of the variables. We formalize this below, following the same lines as in [KP10b].

We start with the observation that the theory of the λ-calculus up to $\alpha\beta\eta$-equivalence (Example 4.3.5) uses only particular operations: names (atoms in [Gab08]), abstraction, and operations $f_S : A^n(S) \to A(S)$ that are 'uniform' in S. This motivates us to consider sheaf algebras for signatures given by a particular class of functors, and specified by 'uniform' equations.

Definition 4.4.1. A **uniform signature** is an $|\mathbb{I}|$-sorted signature such that the operation symbols form a sheaf in $\mathsf{Set}^{\mathbb{I}}$, say $\mathscr{O}$, and such that all the operation symbols $f \in \mathscr{O}(T)$ have arity of the form

$$T \times \cdots \times T \to T_0,$$

where $T_0 \subseteq T$. We will use the notation

$$\mathsf{bind}(f) = T \setminus T_0 \tag{4.22}$$

Additionally, we assume that if an operation symbol $f \in \mathscr{O}(T)$ has arity $T \times \cdots \times T \to T_0$ and $j : T \to S$ is an injective map, then $\mathscr{O}(j)(f)$ has arity

$$S \times \cdots \times S \to S \setminus j[\mathsf{bind}(f)],$$

where $j[\mathsf{bind}(f)]$ denotes the direct image of $\mathsf{bind}(f)$ under j. For an injective map $j : T \to S$ we write $j \cdot f$ for $\mathscr{O}(j)(f)$.

Looking at the theory of the λ-calculus in Example 4.3.5, we find that all operations are equivariant and that the equations are 'uniform' in S. To formalise the uniformity of an equational specification we first describe uniform terms, given by the set of rules in Figure 4.1.

In Figure 4.1, there are four schemas of rules: one for each operation $f : T \times \cdots \times T \to T_0$, two for the operations in $\mathbb{I}$ (weakenings, substitutions), and one for variables. Each rule can be instantiated in an infinite number of ways: T

$$\frac{\Gamma_1 \vdash t_1 : T, \; \dots, \; \Gamma_n \vdash t_n : T}{\Gamma_1 \cup \ldots \cup \Gamma_n \vdash f(t_1, \ldots t_n) : T_0} \quad (f : T \times \cdots \times T \to T_0 \in \mathcal{O}(T))$$

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash w_a t : T \uplus \{a\}} \qquad \frac{\Gamma \vdash t : T \uplus \{a\}}{\Gamma \vdash (b/a)t : T \uplus \{b\}} \qquad \frac{}{\Gamma \vdash X : T} \quad (X : T \in \Gamma)$$

Figure 4.1: Uniform terms

ranges over finite sets of names and a, b over names. Γ and Γ_i denote finite sets of sorted variables. Additionally, in the first rule we assume that if $X : T_i \in \Gamma_i$ and $X : T_j \in \Gamma_j$ then $T_i = T_j$, that is, a variable has the same sort in all the contexts Γ_i. The notation $T \uplus \{a\}$ indicates that an instantiation of the schema is only allowed for those sets T and those atoms a where $a \notin T$.

Definition 4.4.2. A **uniform term** $\Gamma \vdash t : T$ for a uniform signature is a term t of type T using variables from the context Γ formed according to the rules in Figure 4.1. A **uniform equation** is a pair of uniform terms of the same sort $\Gamma \vdash u = v : T$. A **uniform theory** consists of a set of uniform equations.

Example 4.4.3. A uniform theory for λ-calculus consists of the following uniform equations in the uniform signature given by the functor $\delta + \mathcal{N} \times - + - \times -$:

$$\begin{array}{rcll}
X : \emptyset, Y : \emptyset \vdash \mathsf{app}([a](w_a Y), X) & = & Y : \emptyset & (\beta\text{-}1) \\
X : \emptyset \vdash \mathsf{app}([a]a, X) & = & X : \emptyset & (\beta\text{-}2) \\
X : \{a, b\}, Y : \emptyset \vdash \mathsf{app}([a]([b]_{\{a\}}(X)), Y) & = & & \\
{[b]}(\mathsf{app}_{\{b\}}([a]_{\{b\}}(X), w_a Y)) : \emptyset & & & (\beta\text{-}3) \\
X : \{a\}, Y : \{a\}, Z : \emptyset \vdash \mathsf{app}([a](\mathsf{app}_{\{a\}}(X, Y)), Z) & = & & \\
\mathsf{app}(\mathsf{app}([a](X), Z), \mathsf{app}([a](Y), Z)) : \emptyset & & & (\beta\text{-}4) \\
X : \{a\} \vdash \mathsf{app}_{\{b\}}(w_b [a](X), b) & = & (b/a)X : \{b\} & (\beta\text{-}5) \\
X : \emptyset \vdash [a](\mathsf{app}_{\{a\}}(w_a X, a)) & = & X : \emptyset & (\eta)
\end{array}$$

where app, $[a]$ and a stand for $\mathsf{app}_\emptyset$, $[a]_\emptyset$ and $a_\emptyset$, respectively.

The idea is that a uniform equation $\Gamma \vdash u = v : T$ translates to a set of equations in the sense of standard many-sorted universal algebra: $\Gamma_S \vdash u_S = v_S : T \cup S$ where S ranges over the finite subsets of $\mathcal{N}$ with $S \cap T = \emptyset$. If we want to extend the sort of the equation, we might also have to change the sort of the variables. There is a subtlety here: do we raise the type of the variables or do we add weakenings? We prefer the former if, for example, we want to raise the type of the equation $X : \emptyset, Y : \emptyset \vdash X = Y : \emptyset$ by a set S. This becomes $X : S, Y : S \vdash X = Y : S$.

Similarly, if we want to translate the equation

$$X : \emptyset \vdash [a] w_a X = X : \emptyset \tag{4.23}$$

by a set $\{b\}$, where b is a different name than a, we should get

$$X:\{b\}\vdash [a]w_aX = X\ :\{b\}.$$

However, we should be able to translate (4.23) to a standard equation of sort $\{a\}$. We expect all the appearances of X within the translated equation to have the same sort. If, as above, we change the sort of X from $\emptyset$ to $\{a\}$, then on the left hand side we would get $[a]w_aX$, and this is not a well-formed term. In this example, the left hand side of the equation has an implicit freshness constraint on the variable X. Because of the weakening w_a appearing in front of X, we will not be able to instantiate X with elements whose sorts contain a. So a is 'fresh' for X. The solution is to define the translation of this equation as

$$X:\emptyset\vdash w_a[a]w_aX = w_aX:\{a\}$$

So we have to distinguish between the cases when we simply need to add some weakenings and the cases when we have to extend the sort of the variable. We formalise these observations in the next definitions.

Definition 4.4.4. Consider a uniform equation E

$$\Gamma\vdash u = v : T.$$

The **freshness set** of a variable $X : T_X \in \Gamma$ with respect to E is the set

$$\mathsf{Fr}_E(X) = \bigcup_{t:T}(T\setminus T_X)$$

where the union is taken over all sub-terms t of either u or v that contain the variable X.

Example 4.4.5. In the uniform equation for the uniform signature given by the functor defined in (4.13)

$$X:\emptyset\vdash [a]_\emptyset\mathsf{app}_{\{a\}}(w_aX, a_\emptyset) = X:\emptyset \tag{4.24}$$

X has sort $\emptyset$ and $\mathsf{Fr}_E(X) = \{a\}$.

Definition 4.4.6. The translation of an equation E of the form

$$\Gamma\vdash u = v : T_E$$

by $a\notin T_E$, is

$$tr_a(\Gamma) : tr_a(u) = tr_a(v) : T_E\cup\{a\},$$

where $tr_a(\Gamma)$, $tr_a(u)$, $tr_a(v)$ are defined as follows. The translation of the context Γ by a is

$$tr_a(\Gamma) = \{X : T \cup (\{a\} \setminus \mathsf{Fr}_E(X)) \quad | \quad X : T \in \Gamma\}. \tag{4.25}$$

The translation $tr_S(t:T)$ of a sub-term t of either u or v is defined by

$$\begin{array}{rcll}
tr_a(f(t_1,\ldots,t_n):T_0) & = & (w_a \cdot f)(tr_a(t_1),\ldots,tr_a(t_n)) & \text{if } a \notin T \\
tr_a(f(t_1,\ldots,t_n):T_0) & = & w_a f(t_1,\ldots,t_n) & \text{if } a \in T \\
tr_a(w_b t : T \uplus \{b\}) & = & w_{S\cup\{a\},b}\, tr_a(t:T) & \\
tr_a((b/c)t : T \uplus \{b\}) & = & (b/c)_{T\cup\{a\}}\, tr_a(t : T \uplus \{c\}) & \text{if } c \neq a \\
tr_a((b/a)t : T \uplus \{b\}) & = & w_a (b/a)_T\, t & \\
tr_a(X:T) & = & w_a X & \text{if } a \in \mathsf{Fr}_E(X) \\
tr_a(X:T) & = & X & \text{if } a \notin \mathsf{Fr}_E(X)
\end{array} \tag{4.26}$$

where in the first two conditions $f \in \mathscr{O}(T)$ has arity $T \times \cdots \times T \to T_0$.

For a finite set $S = \{a_1,\ldots,a_k\}$ disjoint from T_E we define $tr_S(E)$ as $tr_{a_1} \ldots tr_{a_k}(E)$.

Remark 4.4.7. In the definition of $tr_S(E)$ the order of the elements is not important, since we can prove that $tr_a\, tr_b(E) = tr_b\, tr_a(E)$. We only define $tr_S(t:T)$ for finite sets S such that $S \cap T = \emptyset$.

The definition of $tr_a(E)$ is sound, because initially we chose a set $a \notin T_E$, and then we can prove inductively that whenever we compute $tr_a(t:T')$ for some subterm $t : T'$ of either u or v we have $a \notin T'$.

A more subtle point is that the variable X in the translated equation always has the sort specified in $tr_a(\Gamma)$, namely $T \cup (\{a\} \setminus \mathsf{Fr}_E(X))$. We will prove this in Lemma 4.4.10.

Notice that the translation $tr_S(t)$ depends on the equation for which t is a sub-term, and if t has sort T then the sort of $tr_S(t)$ is $S \cup T$.

In the first condition of (4.26), we have that $a \notin T_0$, but a is one of the elements fresh for the output of f, that is, $a \in \mathsf{bind}(f)$.

Example 4.4.8. For $b \neq a$, the translation by a set $\{b\}$ of the uniform equation (4.24) is

$$X : \{b\} \vdash [a]_\emptyset \mathsf{app}_{\{a\}}(w_a X, a_\emptyset) = X : \{b\}.$$

But the translation of the same equation by $\{a,b\}$ is

$$X : \{b\} \vdash w_a [a]_\emptyset \mathsf{app}_{\{a,b\}}(w_a X, a_{\{b\}}) = w_a X : \{a,b\}.$$

We can do this translation, because the set $\{a,b\}$ is disjoint from the type of the uniform equation, which is the empty set. On the left hand side we use the weakening w_a because $\{a,b\} \cap \mathsf{bind}([a]_\emptyset) = \{a\}$. On the right hand side we use the weakening w_a because $\mathsf{Fr}_E(X) \cap \{a,b\} = \{a\}$.

Example 4.4.9. Translating the uniform theory given in Example 4.4.3, we get a standard many-sorted theory equivalent to that given in Example 4.3.5.

The above definition of $tr_S(X : T_X)$ is also justified by the next property that one expects for the set of standard equations obtained from a uniform equation.

Lemma 4.4.10. Let E be a uniform equation $\Gamma \vdash u = v : T$ and let $a \notin T$. Consider a variable X of sort T_X in Γ, that is, $X : T_X \in \Gamma$. All the occurrences of X in the standard equation $tr_a(\Gamma) \vdash tr_a(u) = tr_a(v) : T_E \cup \{a\}$ have the same sort $T_X \cup (\{a\} \setminus \mathsf{Fr}_E(X))$.

Proof. Note that, while applying the algorithm described in Definition 4.4.6, as we traverse the syntax tree of the term, some subterms may not be translated by a. Explicitly, we stop when we reach a term $f(t_1, \ldots, t_n)$ with $a \in \mathsf{bind}(f)$ or if we reach $(b/a)t$. If the variable X appears in such a term then those instances of X will have sort T_X in the translated equation. Any other instance of X has the sort $T_X \cup (\{a\} \setminus \mathsf{Fr}_E(X))$. So, we have to show that $T_X = T_X \cup (\{a\} \setminus \mathsf{Fr}_E(X))$. Since X appears in a term of the form $f(t_1, \ldots, t_n)$ with $a \in \mathsf{bind}(f)$ or $(b/a)t$, we can use Definition 4.4.4 to prove that $a \in \mathsf{Fr}_E(X) \cup T_X$.

□

Definition 4.4.11. We will say that a functor $L : \mathsf{Set}^{\mathbb{I}} \to \mathsf{Set}^{\mathbb{I}}$ has a *uniform presentation* if L is presented in the sense of Definition 3.2.2 by a uniform signature and a set of equations of rank one generated by a set of uniform equations and containing the following:

$$\begin{aligned}
&w_a f(x_1, \ldots, x_n) = (w_a \cdot f)(w_a x_1, \ldots, w_a x_n) \\
&f(x_1, \ldots, x_n) = ((b/a)_{T\setminus\{a\}} \cdot f)((b/a)_{T\setminus\{a\}} x_1, \ldots, (b/a)_{T\setminus\{a\}} x_n) \\
&(b/a)_{T_0\setminus\{a\}} f(x_1, \ldots, x_n) = ((b/a)_{T\setminus\{a\}} \cdot f_T)((b/a)_{T\setminus\{a\}} x_1, \ldots, (b/a)_{T\setminus\{a\}} x_n)
\end{aligned} \tag{4.27}$$

where $f \in \mathcal{O}(T)$ has arity $f : T \times \ldots \times T \to T_0$, and in the second equation $a \in \mathsf{bind}(f)$ and $b \notin T$, while in the third $a \in T_0$ and $b \notin T$.

Recall that an equation is of rank one when each variable is under the scope of exactly one operation symbol.

Intuitively, the equations (4.27) state that the operations are 'equivariant'. If X is a presheaf, elements of $LX(T_0)$ will be denoted by

$$\{f(x_1, \ldots, x_n)\}_X,$$

where $f \in \mathcal{O}(T)$ has arity $f : T \times \ldots \times T \to T_0$ and $x_i \in X(T)$ for $1 \leq i \leq n$.

Remark 4.4.12. Using the equations of Definition 4.4.11 we can deduce that

$$L(i)(\{f(x_1,\ldots,x_n)\}_X) = \{((i+\mathrm{id})\cdot f)(L(i+\mathrm{id})(x_1),\ldots,L(i+\mathrm{id})(x_n)\}_X$$

provided that $f : T\times\ldots\times T \to T_0$ and $i : T_0 \to U$ is an injective map, $U\cap\mathsf{bind}(f) = \emptyset$ and id is the identity map on $\mathsf{bind}(f)$.

Example 4.4.13.

1. The functor δ has a uniform presentation, with the operation symbols given in Section 3.3 structured as a presheaf as follows:

   $$\begin{aligned} &[a]_S \in \mathcal{O}(S\cup\{a\}) \\ &\mathcal{O}(w_b)([a]_S) = [a]_{S\cup\{b\}} \\ &\mathcal{O}((b/a)_S)([a]_S) = [b]_S \end{aligned}$$

   We have that $\mathsf{bind}([a]_S) = \{a\}$, so the equations (3.19) are of the form (4.27).

2. The presentation of the functor used for the axiomatisation of λ-calculus, defined in (4.13), is also uniform. Indeed, the equations appearing in Proposition 4.3.4 are of the form (4.27), because $\mathsf{bind}(a_S) = \mathsf{bind}(\mathsf{app}_S) = \emptyset$.

3. More generally, functors constructed from $\mathcal{N}$, $+$, $\times$ and δ have uniform presentations.

Proposition 4.4.14. If $L : \mathsf{Set}^{\mathbb{I}} \to \mathsf{Set}^{\mathbb{I}}$ has a uniform presentation, then:

1. there exists a natural transformation $\alpha : LI_*I^* \to I_*I^*L$ such that the diagram below commutes:

$$\begin{tikzcd} LI_*I^* \arrow[rr, "\alpha"] & & I_*I^*L \\ & L \arrow[ul, "L\eta"] \arrow[ur, "\eta L"'] & \end{tikzcd} \tag{4.28}$$

2. $\tilde{L} = I^*LI_*$ preserves sheaf epimorphisms.

Proof. Recall the discussion on how to compute the sheafification of a presheaf from Section 2.4 by twice applying the $(-)^+$ functor. It is enough to prove that we can find a natural transformation α such that the diagram below commutes:

$$\begin{tikzcd} L(-)^+ \arrow[rr, "\alpha"] & & (-)^+L \\ & L \arrow[ul, "L\eta^+"] \arrow[ur, "\eta^+L"'] & \end{tikzcd} \tag{4.29}$$

For then, the next diagram commutes:

$$\begin{tikzcd}
L(P^+)^+ \arrow[r, "="] & L(P^+)^+ \arrow[rr, "\alpha_{P^+}"] & & (LP^+)^+ \arrow[r, "(\alpha_P)^+"] & (LP^+)^+ \\
 & LP^+ \arrow[ul, "L\eta^+_{P^+}"] \arrow[u, "L(\eta^+_P)^+"'] \arrow[rr, "\alpha_P"] & & (LP)^+ \arrow[u, "(L\eta^+_P)^+"] \arrow[ur, "(\eta^+_{LP})^+"'] & \\
 & & LP \arrow[ul, "L\eta^+_P"] \arrow[ur, "\eta^+_{LP}"'] & &
\arrow[from=1-2, to=2-4, phantom, "(3)"]
\arrow[from=2-4, to=1-5, phantom, "(2)"]
\arrow[from=3-3, to=2-3, phantom, "(1)"]
\end{tikzcd} \tag{4.30}$$

The upper-right square (3) commutes by naturality of α and the smaller triangles (1) and (2) by (4.29). Finally $L\eta^+_{P^+}$ and $L(\eta^+_P)^+$ equalise $L\eta^+_P$ by naturality of η^+. Thus the outer triangle commutes. But this is exactly diagram (4.28).

We need to define a natural transformation α making diagram (4.29) commutative. Let X be a presheaf. Elements of $LX^+(T')$ will be of the form

$$\{f(\overline{x_1},\ldots,\overline{x_n})\}_{X^+},$$

where $f \in \mathcal{O}(T)$ has arity $f : T \times \ldots \times T \to T'$ and we can assume without loss of generality that x_i are elements supported by T of the set $X(S)$, for some $T \subseteq S$.

We put $\alpha_{T'}(\{f_T(\overline{x_1},\ldots,\overline{x_n})\}_{X^+}) = \overline{\{f_S(x_1,\ldots,x_n)\}_X}$. In order to show that this is well-defined, we have to prove that $\{f_S(x_1,\ldots,x_n)\}_L$ is supported by T'. Let $i, j : S \setminus \mathsf{bind}(f) \to U$ be two injective maps that agree on T'. We have to show that

$$L(i)(\{f_S(x_1,\ldots,x_n)\}_X) = L(j)(\{f_S(x_1,\ldots,x_n)\}_X)$$

If $U \cap \mathsf{bind}(f) = \emptyset$ this follows easily from Remark 4.4.12 and the fact that the x_k are supported by T. If this is not the case, say for example, if $U \cap \mathsf{bind}(f) = \{a\}$, then we can apply the second equation of Definition 4.4.11 for some name $b \notin S \cup U$:

$$\{f_S(x_1,\ldots,x_n)\}_X = \{\mathcal{O}((b/a)_{S\setminus\{a\}})(f_S)((b/a)_{S\setminus\{a\}}x_1,\ldots,(b/a)_{S\setminus\{a\}}x_n)\}_X$$

and we can use again Remark 4.4.12, plus the fact that $(b/a)_{S\setminus\{a\}}x_k$ is supported by $T \setminus \{a\} \cup \{b\}$.

It is now easy to see that α makes diagram (4.12) commutative. It remains to check that $\tilde{L}$ preserves sheaf epimorphisms. It is enough to prove that whenever $e : X \to Y$ is a sheaf epimorphism, $LI_*e : LI_*X \to LI_*Y$ has the property stated in Proposition 2.4.16. Let $y = \{f_S(y_1,\ldots,y_n)\}_L$ be an element in $(LI_*Y)(S')$, for some operation symbol $f_S : S \times \cdots \times S \to S'$ and $y_1,\ldots,y_n \in Y(S)$. We prove that there exists an inclusion $w' : S' \to T'$ and $x \in LI_*X(T')$ such that $LI_*Y(w')(y) = (LI_*e)_{T'}(x)$. Because $e : X \to Y$ is a sheaf epimorphism, there exists an inclusion $w : S \to T$ and $x_k \in X(T)$ for all $1 \leq k \leq n$ such that $Y(w)(y_k) = e_T(x_k)$ for all k. Let w' denote the inclusion of S' into $T' = T \setminus (S \setminus S')$ and let $x \in LI_*X(T')$ be $\{f_T(x_1,\ldots,x_n)\}_{LX}$. Using the first equation of Definition 4.4.11 we can derive $LI_*Y(w')(y) = (LI_*e)_{T'}(x)$. □

Corollary 4.4.15. If $L : \mathsf{Set}^{\mathbb{I}} \to \mathsf{Set}^{\mathbb{I}}$ has a uniform presentation, then the HSP theorem 4.2.1 holds for $\tilde{L}$-algebras.

Notice that an $\tilde{L}$-algebra $\tilde{L}\mathbb{A} \to \mathbb{A}$ corresponds via the adjunction $I^* \dashv I_*$ to an L-algebra $LI_*\mathbb{A} \to I_*\mathbb{A}$.

Notation 4.4.16. A sheaf algebra $\mathbb{A}$ for a uniform signature given by a functor L is an L-algebra $\alpha : LA \to A$. In subsequent proofs and calculations, we will use the following notation: for $f_T \in \Sigma$ having arity $T \times \cdots \times T \to T'$, α maps $(x_1,\ldots x_n) \in A^n(T)$ to $f_T^{\mathbb{A}}(x_1,\ldots x_n) \in A(T')$. For each algebra $\mathbb{A}$ and each valuation v sending variables $X : T_X$ to elements of $A(T_X)$, a term $t : T$ of type T evaluates to an element $[\![t]\!]_{\mathbb{A},v,T}$ in $A(T)$.

Definition 4.4.17. An algebra $\mathbb{A}$ satisfies the uniform equation $t = u : T$ iff for all $S \cap T = \emptyset$ and all valuations v of variables, we have that $\mathbb{A}, v \models t_S = u_S$, that is, $[\![t]\!]_{\mathbb{A},v,S\cup T}$ and $[\![u]\!]_{\mathbb{A},v,S\cup T}$ denote the same element of $A(S \cup T)$.

In the remainder of the section, we are going to show that classes of sheaf algebras defined by uniform equations are precisely those closed under sheaf quotients, sub-algebras, products and abstraction. In our setting, abstraction (which corresponds to atoms-abstraction [Gab08]) maps an algebra with carrier A to an algebra with carrier δA. To describe this notion, we need to recall the definition of δ from Section 3.3. For $c \notin S$, there is an isomorphism

$$\begin{array}{rcl} A(S \cup \{c\}) & \to & \delta A(S) \\ x & \mapsto & \{[c]_S x\}_{\delta A} \end{array} \tag{4.31}$$

Definition 4.4.18. Given a sheaf algebra $\mathbb{A}$ for a uniform signature with structure $LA \to A$ its abstraction $\delta\mathbb{A}$ with structure $L(\delta A) \to \delta A$ is given by

$$f_T^{\delta\mathbb{A}}(\{[c]x_1\}_{\delta\mathbb{A}},\ldots\{[c]x_n\}_{\delta\mathbb{A}}) = \{[c]f_{T\cup\{c\}}\mathbb{A}(x_1,\ldots x_n)\}_{\delta\mathbb{A}}$$

where $c \notin T$.

The next lemmas establish a connection between the evaluation of a uniform term $t : T$ in $\delta\mathbb{A}$ and the evaluation of $t_{\{a\}} : T \cup \{a\}$ in $\mathbb{A}$, for $a \notin T$. Note that this is possible for uniform terms, but not for terms. Recall from Definition 4.4.2 that a uniform term is not a term (in the sense of set-based universal algebra) but a family of terms.

Lemma 4.4.19. Consider a uniform subterm $t : T$ within an equation E. For all atoms $a \notin T$ and for all valuations $v_{\mathbb{A}}$ in $\mathbb{A}$ of the variables in $t_{\{a\}} = tr_{\{a\}}(t)$, there exists a valuation $v_{\delta\mathbb{A}}$ in $\delta\mathbb{A}$ of the variables in t such that

$$[\![t]\!]_{\delta\mathbb{A},v_{\delta\mathbb{A}},T} = \{[a]_T\, [\![t_{\{a\}}]\!]_{\mathbb{A},v_{\mathbb{A}},T\cup\{a\}}\}_{\delta A} \tag{4.32}$$

Proof. Note that a variable $X : T_X$ in t will have sort $T_X \cup \{a\} \setminus \mathsf{Fr}_E(X)$ in $t_{\{a\}}$. We know that $v_{\mathbb{A}}(X) \in A(T_X \cup \{a\} \setminus \mathsf{Fr}_E(X))$. We define $v_{\delta\mathbb{A}}(X) \in \delta\mathbb{A}(T_X)$ by

$$v_{\delta\mathbb{A}}(X) = \begin{cases} \{[a]_T v_{\mathbb{A}}(X)\}_{\delta A} & \text{if } a \notin T_X \cup \mathsf{Fr}_E(X) \\ \{[b] A(w_b) v_{\mathbb{A}}(X)\}_{\delta A} & \text{if } a \in T_X \cup \mathsf{Fr}_E(X) \end{cases}$$

We can prove that if a belongs to the sort U of a subterm $u : U$ of t, then

$$[\![u]\!]_{\delta\mathbb{A},v_{\delta\mathbb{A}},U} = \{[b]_U w_b\, [\![u]\!]_{\mathbb{A},v_{\mathbb{A}},U}\}_{\delta A} \tag{4.33}$$

Now we can prove (4.32) by induction on the structure of terms. For example, let $f_T : T \times \cdots \times T \to T'$ be an operation symbol such that $a \notin T'$. If $a \notin T$ then the proof follows by induction. But, if $a \in T \setminus T'$, then we have

$$\begin{aligned} [\![f_T(t_1,\ldots,t_n)]\!]_{\delta\mathbb{A},v_{\delta\mathbb{A}},T'} &= f_T^{\delta\mathbb{A}}([\![t_1]\!]_{\delta\mathbb{A},v_{\delta\mathbb{A}},T},\ldots,[\![t_n]\!]_{\delta\mathbb{A},v_{\delta\mathbb{A}},T}) \\ &= f_T^{\delta\mathbb{A}}(\{[b] w_b [\![t_1]\!]_{\mathbb{A},v_{\mathbb{A}},T}\}_{\delta A},\ldots,\{[b] w_b [\![t_n]\!]_{\mathbb{A},v_{\mathbb{A}},T}\}_{\delta A}) \\ &= \{[b] w_b f_T^{\mathbb{A}}([\![t_1]\!]_{\mathbb{A},v_{\mathbb{A}},T},\ldots,[\![t_n]\!]_{\mathbb{A},v_{\mathbb{A}},T})\}_{\delta A} \\ &= \{[a] w_a f_T^{\mathbb{A}}([\![t_1]\!]_{\mathbb{A},v_{\mathbb{A}},T},\ldots,[\![t_n]\!]_{\mathbb{A},v_{\mathbb{A}},T})\}_{\delta A} \\ &= \{[a] f_{T\cup\{a\}}^{\mathbb{A}}([\![t_1]\!]_{\mathbb{A},v_{\mathbb{A}},T},\ldots,[\![t_n]\!]_{\mathbb{A},v_{\mathbb{A}},T})\}_{\delta A} \\ &= \{[a] [\![t_{\{a\}}]\!]_{\mathbb{A},v_{\mathbb{A}},T\cup\{a\}}\}_{\delta A} \end{aligned}$$

□

For illustration, consider the uniform term $t = \mathsf{app}_{\emptyset}(X, Y)$ within an equation E, such that X, Y are variables of sort $\emptyset$, with $\mathsf{Fr}_E(X) = \{a\}$ and $\mathsf{Fr}_E(Y) = \emptyset$. We have that $t_{\{a\}} = \mathsf{app}_{\{a\}}(w_a X, Y_{\{a\}})$. Then $v_{\delta\mathbb{A}}(X) \in \delta\mathbb{A}(\emptyset)$ is defined as $v_{\delta\mathbb{A}}(X) = \{[b] A(w_b) v_{\mathbb{A}}(X)\}_{\delta A}$, for some fresh b, whilst $v_{\delta\mathbb{A}}(Y) \in \delta\mathbb{A}(\emptyset)$ is defined as $v_{\delta\mathbb{A}}(Y) = \{[a] v_{\mathbb{A}}(Y_{\{a\}})\}_{\delta A}$.

Lemma 4.4.20. Consider a uniform term $t : T$ within an equation E and let a be a name such that $a \notin T_X \cup \mathsf{Fr}_E(X)$ for all variables X occurring in E. For all valuations $v_{\delta\mathbb{A}}$ in $\delta\mathbb{A}$ of the variables in t there exists a valuation in $\mathbb{A}$ of the variables in $tr_{\{a\}}(t)$, such that (4.32) holds.

Proof. Note that if X has type T_X in t, it has type $T_X \cup \{a\} \setminus \mathsf{Fr}_E(X) = T_X \cup \{a\}$ in $tr_{\{a\}}(t)$. We define $v_{\mathbb{A}}(X)$ as the unique element of $A(T_X \cup \{a\})$ such that $v_{\delta\mathbb{A}}(X) = \{[a]_{T_X} v_{\mathbb{A}}(X)\}_{\delta A}$. The proof is by induction on the structure of terms. □

Proposition 4.4.21. If a class $\mathscr{B}$ of sheaf algebras is defined by a uniform set of equations, then $\mathscr{B}$ is closed under abstraction.

Proof. Assume that the sheaf algebra $\mathbb{A}$ satisfies a uniform equation $t = u : T$. Consider a valuation v of the variables of t_S, u_S in the algebra $\delta\mathbb{A}$. We need to show $\delta\mathbb{A}, v \models t_S = u_S$ for all finite sets of names S, disjoint from T. Choose a name a such that $a \notin S \cup T$ and $a \notin T_X \cup \mathsf{Fr}_E(X)$ for all variables X. Consider the valuation $v_{\mathbb{A}}$ as in Lemma 4.4.20. Since $\mathbb{A}$ satisfies $t_{S\cup\{a\}} = u_{S\cup\{a\}} : T \cup S \cup \{a\}$, we have $[[t_{S\cup\{a\}}]]_{\mathbb{A},v_{\mathbb{A}},S\cup T\cup\{a\}} = [[u_{S\cup\{a\}}]]_{\mathbb{A},v_{\mathbb{A}},S\cup T\cup\{a\}}$. Then $[[t_S]]_{\delta\mathbb{A},v,S\cup T} = [[u_S]]_{\delta\mathbb{A},v,S\cup T}$ follows from Lemma 4.4.20 applied for t_S and u_S. □

Proposition 4.4.22. If a class $\mathscr{B}$ of sheaf algebras is defined by a uniform set of equations, then $\mathscr{B}$ is closed under quotients.

Proof. Consider a quotient of sheaves $f : \mathbb{A} \to \mathbb{B}$, such that $\mathbb{A}$ satisfies the uniform equations. Consider the uniform equation $t = u : T$ and choose S disjoint from T and a valuation v in $\mathbb{B}$ of the variables in t_S and u_S. We have to show $[[t_S]]_{\mathbb{B},v,S\cup T} = [[u_S]]_{\mathbb{B},v,S\cup T}$. If a variable X has sort T_X in the uniform equation, its translation has sort $T_X \cup S \setminus \mathsf{Fr}_E(X)$ in $t_S = u_S$, so $v(X) \in \mathbb{B}(T_X \cup S \setminus \mathsf{Fr}_E(X))$. Using Proposition 2.4.16, we can find a finite set of names S', such that $S \subseteq S'$ and for all variables X appearing in the equation there exists $v_{\mathbb{A}}(X) \in A(T_X \cup S' \setminus \mathsf{Fr}_E(X))$ such that $f_{T_X \cup S' \setminus \mathsf{Fr}_E(X)}(v_{\mathbb{A}}(X)) = \mathbb{B}(w_X)(v(X))$, where w_X denotes the inclusion $w_X : T_X \cup S \setminus \mathsf{Fr}_E(X) \to T_X \cup S' \setminus \mathsf{Fr}_E(X)$. $v_{\mathbb{A}}$ is a valuation of the variables in $t_{S'} = u_{S'}$. From this, we prove by induction on the structure of t that $f_{S'\cup T}([[t_{S'}]]_{\mathbb{A},v_{\mathbb{A}},S'\cup T}) = B(w)([[t_S]]_{\mathbb{B},v,S\cup T})$. Since $B(w)$ is injective, this concludes the proof. □

Theorem 4.4.23. A class $\mathscr{B}$ of sheaf algebras for a uniform signature is definable by uniform equations if and only if it is closed under sheaf quotients, subalgebras, products and abstraction.

Proof. Assume that a class of sheaf algebras for a uniform signature is defined by uniform equations. Using Corollary 4.4.15, we can derive closure under subalgebras and products. Closure under abstraction and sheaf quotients follows from Propositions 4.4.21 and 4.4.22, respectively. Conversely, from closure under HSPA we derive closure under presheaf epimorphisms, subalgebras and products, hence the class of sheaf-algebras is definable by a set of equations $\mathscr{E}$ in the sense of standard many-sorted universal algebra. We have to show that these equations come from a uniform theory. It is enough to show that whenever $t =$

$u : T$ is in $\mathscr{E}$, then for all $a \notin T$ and $B \in \mathscr{B}$ we have that $B \models t_{\{a\}} = u_{\{a\}} : T \cup \{a\}$. This follows from closure under abstraction and Lemma 4.4.19. □

4.5 Comparison with other nominal logics

We now show that uniform theories have the same expressive power as nominal algebra [GM09], nominal equational logic [CP07] and synthetic nominal equational logic [FH08]. Explicitly, we show how to translate theories of nominal algebra and nominal equational logic into uniform theories (Section 4.5.1, respectively Section 4.5.2) and how to translate uniform theories into synthetic nominal theories (Section 4.5.3). In each case, we prove that the translations are semantically invariant.

4.5.1 Comparison with nominal algebra

In this subsection we show how to translate the syntax and theories of the nominal algebra [GM09] to uniform signatures and uniform theories. Then we prove semantic invariance, that is, there is a correspondence between models for a nominal algebra theory and models for the uniform theory obtained via this translation.

Translation of syntax. We refer the reader to [GM09] for the syntax and semantics of nominal algebra. To each nominal algebra signature there corresponds a uniform signature given by the functor $\mathscr{N} + \delta + \Sigma$, where δ is as in Section 4.3 and Σ is a polynomial functor on $\mathsf{Set}^{\mathbb{I}}$, given by

$$\Sigma A = \coprod \{f_{n_i}\} \times A^{n_i}$$

where the coproduct is taken after all the operation symbols f_{n_i} with arity n_i in the nominal algebra signature.

Translations of equational judgements. Assume $\Delta \vdash t = u$ is an equality judgement in the sense of [GM09]. It is reasonable to ask that the uniform equation obtained by translating such an equality judgement has to satisfy the following:

1. All occurrences of X in the uniform equation have the same sort.
2. If $a\#X$ is in Δ, then in the uniform equation we can only instantiate X with elements whose support does not contain a.
3. We can prove semantic invariance of this translation, that is, a nominal set satisfies an equational judgement iff the corresponding sheaf satisfies the translated uniform equation.

In order to address 1. and 2. above, for each unknown X appearing in this judgement, we have to consider the following sets: $\mathsf{anc}(X)$ defined as the set of names a for which there is an occurrence of πX, for some permutation π, such that $[a]$ is an ancestor of πX in the syntax tree of the equation,[2] and

$$\mathsf{fresh}(X) = \{a \in \mathcal{N} \mid a\#X \in \Delta\}.$$

Before giving the actual translation, we will first find the type T_E of the uniform equation E, obtained by translating $\Delta \vdash t = u$. This is done recursively:

$$\begin{aligned} \mathsf{type}(t=u) &= \mathsf{type}(t) \cup \mathsf{type}(u) \\ \mathsf{type}(f(t_1,\dots,t_n)) &= \cup \mathsf{type}(t_i) \\ \mathsf{type}([a]t) &= \mathsf{type}(t) \setminus \{a\} \\ \mathsf{type}(a) &= \{a\} \\ \mathsf{type}(\pi X) &= (\mathsf{anc}(X) \setminus \mathsf{fresh}(X)) \cup \mathsf{supp}(\pi) \end{aligned} \tag{4.34}$$

We define

$$T_E = \mathsf{type}(t=u) \cup (\bigcup_{X \in E} (\mathsf{fresh}(X) \setminus \mathsf{anc}(X)).$$

The reason for adding $\bigcup_{X \in E} (\mathsf{fresh}(X) \setminus \mathsf{anc}(X)$, is that we want to be able to retrieve the names in $\mathsf{fresh}(X)$ from the uniform equation obtained, even if they do not appear in any subterms. For example, the type of the translation of $b\#X \vdash X = Y$ should be $\{b\}$, and not the empty set.

The actual translation is the uniform equation $\mathcal{T}_{T_E}(t) = \mathcal{T}_{T_E}(u) : T_E$, where $\mathcal{T}_T(t) : T$ is a uniform term of type T, defined recursively by:

$$\begin{aligned} \mathcal{T}_T(f(t_1,\dots,t_n)) &= f(\mathcal{T}_T(t_1),\dots,\mathcal{T}_T(t_n)) : T \\ \mathcal{T}_T(a) &= a_T : T \\ \mathcal{T}_T([a]t) &= [a]_T \mathcal{T}_{T \cup \{a\}}(t) : T \qquad \text{if } a \notin T \\ \mathcal{T}_T([a]t) &= w_a [a]_T \mathcal{T}_T(t) : T \qquad \text{if } a \in T \\ \mathcal{T}_T(\pi X) &= \pi_T w_{a_1} \dots w_{a_k} X_{T \setminus \mathsf{fresh}(X)} : T, \end{aligned} \tag{4.35}$$

where in the last condition $\{a_1,\dots,a_k\} = T \cap \mathsf{fresh}(X)$ and $\pi_T : T \to T$ is the restriction of π to T. This restriction makes sense because by (4.34) we have that $\mathsf{supp}(\pi) \subseteq T$. On the right hand side of the above equations, we have nominal terms for the uniform signature given by $\mathcal{N} + \delta + \Sigma$, obtained according to the rules in Figure 4.1. Recall from Proposition 4.3.4 that $\mathcal{N}$ is presented by the operation symbols a_T, while δ is presented by the operation symbols $[a]_T$.

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[2] We say that $[a]$ ia an ancestor of πX rather than of X, because, in the definition of nominal terms, X is not a nominal subterm of the moderated unknown πX.

Example 4.5.1. Consider the following judgement in nominal algebra:

$$a\#X \vdash [a]\mathsf{app}(X,a) = X$$

We have that $\mathsf{fresh}(X) = \{a\}$, $\mathsf{anc}(X) = \{a\}$ and that the type of the translated uniform equation is $\emptyset$. The translation is the uniform equation

$$[a]_\emptyset \mathsf{app}_{\{a\}}(w_a X, a_\emptyset) = X$$

and this corresponds to the set of equations (η) of Example 4.3.5, that is indexed by all finite sets S that do not contain a.

Lemma 4.5.2. If X is an unknown appearing in the equality judgement E, then all the instances of the variable X have the same sort $(T_E \cup \mathsf{anc}(X)) \setminus \mathsf{fresh}(X)$ in the translated uniform equation.

Proof. X may appear more than once in the equality judgement. When traversing the syntax tree, the subscript of $\mathscr{T}$ may change, so we have to prove that whenever we have to translate $\mathscr{T}_T(\pi X)$, the set T has the property that

$$T \setminus \mathsf{fresh}(X) = (T_E \cup \mathsf{anc}(X)) \setminus \mathsf{fresh}(X).$$

Note that first we apply the translation with index T_E, and as we traverse the tree, this sort will only increase by a name a when we reach a subterm of the form $[a]t$. If we eventually reach a leaf containing the unknown X, such an a must be in the set $\mathsf{anc}(X)$. Therefore $T \subseteq T_E \cup \mathsf{anc}(X)$, so we know that $T \setminus \mathsf{fresh}(X) \subseteq (T_E \cup \mathsf{anc}(X)) \setminus \mathsf{fresh}(X)$. Conversely, let $a \in (T_E \cup \mathsf{anc}(X)) \setminus \mathsf{fresh}(X)$. If $a \in T_E \setminus \mathsf{fresh}(X)$, then $a \in T \setminus \mathsf{fresh}(X)$ because $T_E \subseteq T$. It remains to consider the case when $a \in \mathsf{anc}(X) \setminus \mathsf{fresh}(X)$. We distinguish two cases, depending on whether this particular instance of X has $[a]$ as an ancestor. If this is the case, the set T must contain the name a. If this is not the case, then we have that $a \in \mathsf{type}(t = u) \subseteq T_E$, hence $a \in T$. □

Lemma 4.5.3. $\mathsf{Fr}_E(X) = \mathsf{fresh}(X)$.

Proof. Consider $a \in \mathsf{fresh}(X)$. Let us denote by T_X the sort of the variable X in $\mathscr{T}(E)$. We know that $T_X = (T_E \cup \mathsf{anc}(X)) \setminus \mathsf{fresh}(X)$, so $a \notin T_X$. We have two cases:

1. If $a \in \mathsf{anc}(X)$, then there exists a subterm $[a]v$, such that X occurs in v. The sort of $\mathscr{T}(v)$ must contain the name a, so $a \in \mathsf{Fr}_E(X)$.

2. If $a \in \mathsf{fresh}(X) \setminus \mathsf{anc}(X)$, then $a \in T_E$, so again we obtain that $a \in \mathsf{Fr}_E(X)$.

Conversely, if $a \in \mathsf{Fr}_E(X)$ there exists a subterm v in E containing πX, for some permutation π, such that the variable $X : T_X$ occurs in $\mathscr{T}_T(v) : T$ and $a \in T \setminus T_X$. If the sort of $\mathscr{T}(\pi X)$ is S, we have that $T \subseteq S$, hence we get that $a \in S \setminus T_X = S \cap \mathsf{fresh}(X)$, so $a \in \mathsf{fresh}(X)$. □

Translation of semantics. Let $\mathbb{X} = (|\mathbb{X}|, \cdot, \mathbb{X}_{atm}, \mathbb{X}_{abs}, \{\mathbb{X}_f \mid f \in \mathsf{S}\})$ be a nominal algebra for a nominal signature S. Let $\mathscr{N} + \delta + \Sigma$ be the functor corresponding to this signature. We consider the sheaf $\mathfrak{X}$ corresponding to the nominal set $(|\mathbb{X}|, \cdot)$. The translation of $\mathbb{X}$ is the sheaf algebra $\mathscr{N} + \delta\mathfrak{X} + \Sigma\mathfrak{X} \to \mathfrak{X}$ given by

$$\begin{array}{rcl} a_S & \mapsto & \mathbb{X}_{atm}(a) \\ \{[a]_S x\}_{\delta\mathfrak{X}} & \mapsto & \mathbb{X}_{abs}(\mathbb{X}_{atm}(a), x) \\ (f, x_1, \ldots, x_n) & \mapsto & \mathbb{X}_f(x_1, \ldots, x_n) \end{array}$$

That this is well defined follows from the equivariance of $\mathbb{X}_{atm}, \mathbb{X}_{abs}, \mathbb{X}_f$.

Theorem 4.5.4. [semantic invariance] Let $\mathbb{X}$ be a nominal algebra for a nominal signature. Let E be the uniform equation of type T_E, obtained by translating an equality judgement $\Delta \vdash u = v$. Then $\mathbb{X}$ satisfies $\Delta \vdash u = v$ if and only if $\mathfrak{X}$ satisfies the uniform equation E.

Proof. First assume that $[[\Delta \vdash u = v]]^{\mathbb{X}}$ holds. We need to prove that $\mathfrak{X} \models tr_S\, E$ for all finite sets S that are disjoint from T_E. Consider a valuation ς of the variables appearing in $tr_S(E)$ in $\mathfrak{X}$. If X is a variable of sort T_X in E, then X has sort $S \setminus \mathsf{Fr}_E(X) \cup T_X$ in $tr_S(E)$. So $\varsigma(X) \in \mathfrak{X}(S \setminus \mathsf{Fr}_E(X) \cup T_X)$ is an element of the nominal set $\mathbb{X}$ supported by $S \setminus \mathsf{Fr}_E(X) \cup T_X$. Using Lemma 4.5.2 and Lemma 4.5.3, we know that $S \setminus \mathsf{Fr}_E(X) \cup T_X$ is disjoint from $\mathsf{fresh}(X)$, so whenever the freshness primitive $a\#X$ is in Δ, $a\#\varsigma(X)$. Let ς' be a valuation in $\mathbb{X}$ of the unknowns in $\Delta \vdash u = v$ that maps X to $\varsigma(X)$. We have to prove that $[[tr_S(\mathscr{T} u)]]^{\mathfrak{X}}_{\varsigma} = [[tr_S(\mathscr{T} v)]]^{\mathfrak{X}}_{\varsigma}$. This follows from the claim below, which can be proved by induction on the structure of the terms.

Claim 4.5.5. For all subterms t of either u or v, we have $[[t]]^{\mathbb{X}}_{\varsigma'} = [[\mathscr{T} t]]^{\mathfrak{X}}_{\varsigma}$.

Conversely, assume that $\mathfrak{X}$ satisfies the uniform equation E. Consider a valuation ς' in $\mathbb{X}$ of the unknowns of $\Delta \vdash u = v$, such that $a\#\varsigma'(X)$ whenever $a\#X \in \Delta$. We consider the finite set of atoms $S := \bigcup_X \mathsf{supp}(\varsigma'(X)) \setminus T_E$. We can define a valuation ς of the variables occurring in $tr_S(E)$ in $\mathfrak{X}$, simply by taking $\varsigma(X) = \varsigma'(X)$. In order to prove that this is well defined we can check that $\varsigma'(X)$ is supported by $S \setminus \mathsf{Fr}_E(X) \cup T_X$. Since ς' can be obtained from ς as before, we can finalize the proof by applying again the Claim 4.5.5. □

Corollary 4.5.6. Theorem 4.5.4 and Theorem 4.4.23 give a new proof for Gabbay's HSPA theorem, [Gab08, Theorem 9.3].

4.5.2 Comparison with NEL

This section compares uniform theories to the nominal equational logic of [CP07]. For simplicity we only consider the one-sorted version of NEL,[3] although extending our work to many-sortedness over sheaves is not difficult. We only consider theories for which the axioms are of the form

$$\begin{aligned} &\Delta \vdash \bar{a} \mathrel{\#\!\!\!\!\approx} t \\ &\Delta \vdash t \approx t'. \end{aligned} \tag{4.36}$$

This fragment has the same expressive power as the entire NEL.

Translation of syntax. Recall that a signature for NEL is given by a nominal set $\mathsf{Op} = (|\mathsf{Op}|, \cdot)$ of operation symbols. Consider a theory for this signature, consisting of axioms as in (4.36). We will construct a presheaf of operations $\mathcal{O}$ as in Definition 4.4.11, which is almost the sheaf corresponding to Op via the isomorphism between nominal sets and $\mathrm{Sh}(\mathbb{I}^{op})$: for all finite sets of names S, $\mathcal{O}(S)$ contains the operation symbols whose support is contained in S. But we also add more information about the arity of these operation symbols.

Definition 4.5.7. Consider a NEL theory for a signature that contains an operation symbol f. The set $\mathsf{bind}(f)$ is defined as the set of names a such that there is an axiom in the theory of the form $\{X_1, \ldots, X_n\} \vdash \bar{b} \mathrel{\#\!\!\!\!\approx} (\pi \cdot f)(X_1, \ldots, X_n)$ for a finite set of names $\bar{b}$ and a permutation π, such that $\pi(b) = a$ for some $b \in \bar{b}$.

If $f \in \mathsf{Op}$ is an n-ary operation symbol, such that $\mathsf{supp}(f) = T$ and $T \subseteq S$, we consider an operation symbol in $f_S \in \mathcal{O}(S)$, with arity $f_S : S \times \cdots \times S \to S \setminus \mathsf{bind}(f)$. The definition above implies that $\mathsf{bind}(\pi \cdot f) = \pi[\mathsf{bind}(f)]$. We also obtain that $\mathsf{bind}(f) \subseteq \mathsf{supp}(f)$. So we can derive that, for any injective map $j : S \to S'$, $\mathcal{O}(j)(f_S)$ has the arity $S' \times \cdots \times S' \to S' \setminus j[\mathsf{bind}(f)]$. So $\mathcal{O}$ is a presheaf as in Definition 4.4.11. Note that the arity of an operation symbol in $\mathcal{O}$ depends not only on the nominal signature, but also on the theory, because of the way freshness constraints are expressed in NEL, see (4.36). The translation of a NEL signature is the uniform signature given by the functor L with a uniform presentation given by $\mathcal{O}$: see Definition 4.4.11.

Example 4.5.8. If L_a is an operation symbol as in the NEL signature for λ-calculus of [CP07, Example 3.1] then $\mathsf{bind}(L_a) = \{a\}$. The translation of this NEL signature is the uniform signature given by the functor defined in (4.13).

Translation of a theory. From each axiom in a theory in the sense of [CP07], having the form

$$\Delta \vdash t \approx t'$$

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[3] Note that we will not need the sorting environments of [CP07] in this case.

we will obtain a uniform equation E of sort T_E. As in the previous section we first describe a way of finding the sort T_E. Again, all occurrences of a variable X are expected to have the same sort in the translation, so we need to pay attention to the bound names of the terms that contain X and to the names that should be fresh for X. To this end we define the set $\mathsf{anc}(X)$ by

$$\mathsf{anc}(X) = \bigcup \mathsf{bind}(f)$$

taken over all operations f such that X appears in a subterm of either t or t' of the form $f(t_1,\dots,t_n)$. Similarly we define

$$\mathsf{fresh}(X) = \bar{a} \quad \text{iff} \quad \bar{a} \# X \in \Delta$$

The fact that $a \# X$ is in the freshness environment will be expressed in the uniform equation by adding a weakening w_a in front of X.

In order to find T_E we define a function type recursively:

$$\begin{aligned} \mathsf{type}(t = u) &= \mathsf{type}(t) \cup \mathsf{type}(u) \\ \mathsf{type}(f(t_1,\dots,t_n)) &= (\cup \mathsf{type}(t_i) \cup \mathsf{supp}(f)) \setminus \mathsf{bind}(f) \\ \mathsf{type}(\pi X) &= (\mathsf{anc}(X) \setminus \mathsf{fresh}(X)) \cup \mathsf{supp}(\pi) \end{aligned} \tag{4.37}$$

We define $T_E = \mathsf{type}(t = u) \cup (\bigcup_{X \in E} (\mathsf{fresh}(X) \setminus \mathsf{anc}(X))$

The translation of the axiom $\Delta \vdash t \approx t'$ is the uniform equation $\mathscr{T}_{T_E}(t) = \mathscr{T}_{T_E}(u) : T_E$, where $\mathscr{T}_T(t)$ is a term of sort T, defined recursively by

$$\begin{aligned} \mathscr{T}_T(f(t_1,\dots,t_n)) &= w_{T \cap \mathsf{bind} f} f_{T \cup \mathsf{bind}(f)}(\mathscr{T}_{T \cup \mathsf{bind}(f)}(t_1),\dots,\mathscr{T}_{T \cup \mathsf{bind}(f)}(t_n)) \\ \mathscr{T}_T(\pi X) &= \pi_T w_{T \cap \mathsf{fresh}(X)} X_{T \setminus \mathsf{fresh}(X)} \end{aligned} \tag{4.38}$$

The permutation π has its support included in T and π_T is the restriction of π to T. As in the previous section we can prove that all instances of a variable X have the same sort in the uniform equation, namely $(T_E \cup \mathsf{anc}(X)) \setminus \mathsf{fresh}(X)$. The proof of this is analogous to that of Lemma 4.5.2. The only difference is that now instead of reasoning only about abstractions $[a]$, we allow more general operation symbols. Similarly to Lemma 4.5.3, we get that $\mathsf{Fr}_{\mathscr{T}(E)}(X) = \mathsf{fresh}(X)$.

Similarly, from each axiom in a theory in the sense of [CP07], having the form

$$\Delta \vdash \bar{a} \# t,$$

we obtain a uniform equation of the form

$$\Gamma_E \vdash w_{\bar{b}} \mathscr{T}_{T_E \cup \bar{a}}(t) = w_{\bar{a}} (\bar{b}/\bar{a}) \mathscr{T}_{T_E \cup \bar{a}}(t) : T_E \cup \bar{a} \cup \bar{b}, \tag{4.39}$$

where $\bar{b}$ is a set of fresh variables having the same cardinality as $\bar{a}$ and $(\bar{b}/\bar{a})$ denotes the composition $(b_1/a_1)\ldots(b_n/a_n)$ when $\bar{a}=\{a_1,\ldots,a_n\}$ and $\bar{b}=\{b_1,\ldots,b_n\}$.

For each variable X appearing in t we define $\mathsf{anc}(X)$ and $\mathsf{fresh}(X)$ as above. Then, we put

$$\Gamma_E=\{X:(T_E\cup\mathsf{anc}(X))\setminus\mathsf{fresh}(X)\mid X\in t\}.$$

We define $\mathsf{type}(t)$ recursively as in (4.37) and we put $T_E=\mathsf{type}(t)$. Then $\mathscr{T}_{T_E\cup\bar{a}}(t)$ is defined recursively as in (4.38).

Example 4.5.9. The η rule of the NEL theory for $\alpha\beta\eta$-equivalence of untyped λ-terms [CP07, Example 6.2]

$$a\not\#\, x\vdash L_a(A\ x\ V_a)\approx x$$

translates to

$$[a](\mathsf{app}(w_aX,a))=X\ :\emptyset$$

Translation of semantics. Consider a NEL theory as in (4.36) for a signature Op. Let $\mathbb{X}$ be an algebra for this theory, that is, a nominal set $|\mathbb{X}|$, equipped with equivariant functions $\mathsf{Op}_n\times|\mathbb{X}|^n\to|\mathbb{X}|$, for all arities n. (Op_n is the set of operation symbols of arity n, and is a nominal subset of Op.) We consider the sheaf $\mathfrak{X}$ corresponding to $|\mathbb{X}|$. The sheaf algebra corresponding to $\mathbb{X}$ is an algebra for the functor L with a uniform presentation obtained from Op, see the translation of syntax above. This sheaf algebra $L\mathfrak{X}\to\mathfrak{X}$ maps

$$\{f_T(x_1,\ldots,x_n)\}_{\mathfrak{X}}\mapsto\mathbb{X}[\![f]\!](x_1,\ldots,x_n)$$

where $\mathbb{X}[\![f]\!]$ is as in [CP07]. This map is well-defined because of the equivariance of the functions $\mathsf{Op}_n\times|\mathbb{X}|^n\to|\mathbb{X}|$.

Theorem 4.5.10. [semantic invariance] A structure $\mathbb{X}$ for a NEL signature is a structure for a NEL theory as in (4.36) if and only if the sheaf algebra $\mathfrak{X}$ obtained as above is an algebra for the translated uniform theory.

Proof. First we can check that a structure $\mathbb{X}$ for a nominal signature satisfies a judgement $\Delta\vdash t\approx t'$ iff the sheaf algebra $\mathfrak{X}$, constructed as above, satisfies the uniform equation $E:T_E$ obtained as the translation of $\Delta\vdash t\approx t'$. The proof for this follows the same lines as the proof for Theorem 4.5.4. From a valuation ς' in $\mathbb{X}$ of the variables in the freshness environment Δ we get a valuation ς in $\mathfrak{X}$ of the variables in some $tr_S(E)$, for $S\cap T_E=\emptyset$, and vice-versa. It remains to check that we have $[\![t]\!]^{\mathbb{X}}_{\varsigma'}=[\![\mathscr{T}t]\!]^{\mathfrak{X}}_{\varsigma}$, and this goes by induction on the structure of the terms. The next equalities hold in the underlying nominal set of $\mathbb{X}$.

$$\begin{array}{rcl}
\llbracket \mathscr{T}_T(f(t_1,\dots,t_n))\rrbracket^{\mathfrak{X}}_{\varsigma} & = & \llbracket w_{T\cap\mathsf{bind}f}\, f_{T\cup\mathsf{bind}(f)}(\mathscr{T}_{T\cup\mathsf{bind}(f)}(t_1),\dots,\mathscr{T}_{T\cup\mathsf{bind}(f)}(t_n))\rrbracket^{\mathfrak{X}}_{\varsigma} \\
& = & \mathfrak{X}(w_{T\cap\mathsf{bind}f})(\llbracket f_{T\cup\mathsf{bind}(f)}(\mathscr{T}_{T\cup\mathsf{bind}(f)}(t_1),\dots,\mathscr{T}_{T\cup\mathsf{bind}(f)}(t_n))\rrbracket^{\mathfrak{X}}_{\varsigma}) \\
& = & \llbracket f_{T\cup\mathsf{bind}(f)}(\mathscr{T}_{T\cup\mathsf{bind}(f)}(t_1),\dots,\mathscr{T}_{T\cup\mathsf{bind}(f)}(t_n))\rrbracket^{\mathfrak{X}}_{\varsigma} \\
& = & \{f_{T\cup\mathsf{bind}(f)}(\llbracket t_1\rrbracket^{\mathfrak{X}}_{\varsigma},\dots,\llbracket t_n\rrbracket^{\mathfrak{X}}_{\varsigma})\}_{\mathfrak{X}} \\
& = & \{f_{T\cup\mathsf{bind}(f)}(\llbracket t_1\rrbracket^{\mathbb{X}}_{\varsigma'},\dots,\llbracket t_n\rrbracket^{\mathbb{X}}_{\varsigma'})\}_{\mathfrak{X}} \\
& = & \mathbb{X}\llbracket f\rrbracket(\llbracket t_1\rrbracket^{\mathbb{X}}_{\varsigma'},\dots,\llbracket t_n\rrbracket^{\mathbb{X}}_{\varsigma'}) \\
& = & \llbracket f(t_1,\dots,t_n)\rrbracket^{\mathbb{X}}_{\varsigma'}
\end{array}$$

For axioms of the form $X_1,\dots,X_n \vdash a \not\# f(X_1,\dots,X_n)$, semantical invariance follows since the operation symbol f corresponds on the side of uniform signatures to operation symbols whose arities have the property that a does not belong to the result. So, for any valuation of the variables X_i in $\mathfrak{X}$, a translation of $f(X_1,\dots,X_n)$ is evaluated to an element y of $\mathfrak{X}(S)$ for a finite set S, with $a \notin S$. This means that, if $\mathfrak{X}$ comes from a nominal set $|\mathbb{X}|$, we have that a is fresh for y in $|\mathbb{X}|$.

More generally, we can check that a structure $\mathbb{X}$ for a nominal signature satisfies a judgement $\Delta \vdash \bar{a} \not\# t$ if and only if the sheaf algebra $\mathfrak{X}$, satisfies the translation $E : T_E$ obtained in (4.39). From a valuation ς' in $\mathbb{X}$ of the variables in the freshness environment Δ we get a valuation ς in $\mathfrak{X}$ of the variables in some $tr_S(E)$, for $S \cap T_E = \emptyset$, and vice-versa. As above we have that $\llbracket t\rrbracket^{\mathbb{X}}_{\varsigma'} = \llbracket \mathscr{T} t\rrbracket^{\mathfrak{X}}_{\varsigma}$. It remains to check that $\bar{a}\#\llbracket t\rrbracket^{\mathbb{X}}_{\varsigma'}$ if and only if $\mathfrak{X}(w_{\bar{b}})\llbracket \mathscr{T} t\rrbracket^{\mathfrak{X}}_{\varsigma} = \mathfrak{X}(w_{\bar{a}})\mathfrak{X}(\bar{b}/\bar{a})\llbracket \mathscr{T} t\rrbracket^{\mathfrak{X}}_{\varsigma}$ for some set $\bar{b}$ of fresh variables. The latter is equivalent to

$$\llbracket \mathscr{T} t\rrbracket^{\mathfrak{X}}_{\varsigma} = (\bar{a}\ \bar{b})\cdot \llbracket \mathscr{T} t\rrbracket^{\mathfrak{X}}_{\varsigma} \tag{4.40}$$

for some fresh $\bar{b}$, where $(\bar{a}\ \bar{b})$ denotes the product of transposition $(a_i\ b_i)$ for $\bar{a} = \{a_1,\dots,a_n\}$ and $\bar{b} = \{b_1,\dots,b_n\}$. Using Remark 2.1.17 and Theorem 2.1.16 this is equivalent to $\bar{a}\#\llbracket \mathscr{T} t\rrbracket^{\mathfrak{X}}_{\varsigma}$.

□

Corollary 4.5.11. Theorem 4.4.23 and Theorem 4.5.10 give an HSPA theorem for models of NEL.

4.5.3 Comparison with SNEL

Fiore and Hur [FH08] introduced an abstract notion of systems of equations between terms in the setting of enriched category and showed how sound and complete equational logics can be obtained in this abstract setting. Their case study consists in developing a nominal equational logic which is logically equivalent to those introduced by Gabbay and Mathijssen, respectively Clouston and Pitts.

Hur [Hur10, Section 8.2.6] translated the nominal syntax of [FH08] into NEL syntax and stated that this translation is semantically invariant. Thus, using the results from the previous section, we conclude that the SNEL can be translated into uniform theories. To close the circle, we show in this section that uniform theories give rise to nominal theories in the sense of [FH08] having the same models.

Central to the approach of Fiore and Hur is the fact that $(\mathsf{Nom}, 1, \otimes)$ is a symmetric monoidal category, where $\otimes$ is the separating tensor described in (2.37). Their nominal equational reasoning applies to signatures given by Nom-enriched functors (and the free monads which they generate). A functor $F : \mathsf{Nom} \to \mathsf{Nom}$ is Nom-enriched when there exists a natural transformation, called strength

$$\tau_{\mathbb{X},\mathbb{Y}} : F\mathbb{X} \otimes \mathbb{Y} \to F(\mathbb{X} \otimes \mathbb{Y}).$$

Abstract terms in the setting of [FH08] are morphisms of the form

$$t : \mathbb{A}^{\otimes n} \to F(\coprod_{i=1}^{l} \mathbb{A}^{\otimes n_i}) \tag{4.41}$$

where $\mathbb{A}^{\otimes k}$ denotes $\mathbb{A} \otimes \ldots \otimes \mathbb{A}$ (k-times). We refer the reader to [FH08, Section 5, pp. 15-16] for the definitions of the nominal syntax, nominal theories and the corresponding models. Terms in this syntax are built using the rules

$$\frac{}{[\overline{a}]V \vdash X(\overline{b})} \quad (X(\overline{b}) \in \textstyle\coprod_{X \in |V|} \mathbb{A}^{\otimes V(X)})$$

$$\frac{[\overline{a}]V \vdash t_i (1 \leq i \leq k)}{[\overline{a}]V \vdash f(t_1 \ldots t_k)}$$

where

1. $|V|$ is a finite set of variables and V is a function which associates a natural number to each variable in $|V|$;

2. $\overline{a} \in \mathbb{A}^{\otimes n}$ for some n;

3. $\overline{b} \in \mathbb{A}^{\otimes V(X)}$ and $X(\overline{b})$ is just a notation for the injection of the tuple of distinct names $\overline{b}$ in the X-component of the coproduct $\coprod_{X \in |V|} \mathbb{A}^{\otimes V(X)}$.

Each term in the above syntax corresponds to an abstract term in the sense of (4.41). Consider a uniform theory for a uniform signature given by a sheaf of operation symbols $\mathscr{O}$. Consider the nominal set $\mathbb{O}$ corresponding to $\mathscr{O}$. If $f \in \mathscr{O}(T)$ has arity $f : \underbrace{T \times \ldots \times T}_{n} \to T'$ we assign arity n to the element $\overline{f} \in \mathbb{O}$ that corresponds to f. Then the functor $F : \mathsf{Nom} \to \mathsf{Nom}$ given by

$$F\mathbb{X} = \coprod_{n} \mathbb{O}_n \times \mathbb{X}^n \tag{4.42}$$

is Nom-enriched, where $\mathbb{O}_n$ is the subset of operation symbols of $\mathbb{O}$ of arity n. Notice that, by considering only the functor F we have lost some information, namely the set of names that each operation symbols f binds. However, we can show that sheaf algebras for the uniform signature $\mathscr{O}$ correspond to those F-algebras that additionally satisfy the following SNEL equations. For each $f \in \mathscr{O}(T)$ and $a \in \mathsf{bind}(f)$ we consider the abstract equation

$$[\overline{c}, a, b]\{X_1 : l, \ldots, X_n : l\} \vdash \overline{f}(X_1(\overline{c}, a), \ldots, X_n(\overline{c}, a)) = ((a\ b)\cdot\overline{f})(X_1(\overline{c}, b), \ldots, X_n(\overline{c}, b)), \tag{4.43}$$

where $\{\overline{c}\} = T \setminus \{a\}$, $b \notin T$ and l is the cardinal of T plus one.

Example 4.5.12. Recall the uniform signature for the λ-calculus described in Example 4.4.3. This corresponds to the nominal signature of [FH08, Example 5.1]. We have $\overline{\mathsf{app}_S} = A$, $\overline{[a]_S} = L_a$ and $\overline{a_S} = V_a$. The fact $\mathsf{bind}([a]_S) = \{a\}$ is expressed in SNEL by the abstract equation

$$[a, b]\{X : 1\} \vdash L_a X(a) = L_b X(b).$$

Next we show how to translate uniform equations in the uniform signature given by $\mathscr{O}$ into abstract equations for the functor F described in (4.42). Consider a uniform equation E of the form

$$\Gamma \vdash t = t' : T_E.$$

1. Let $\overline{a}$ denote a tuple of all the distinct names that appear in the equation E in an arbitrary order. That is, the names in $\overline{a}$ are obtained by considering the union of all the finite sets T such that there exists $X : T$ in Γ or there exists a subterm $u : T$ of either t or t'.

2. Let $|V|$ be the finite set of variables from the context Γ.

3. For each $X : T \in \Gamma$ consider the set $\mathsf{Fr}_E(X)$, see Definition 4.4.4. Let $V(X)$ be the cardinality of the set $\{\overline{a}\} \setminus \mathsf{Fr}_E(X)$, where $\{\overline{a}\}$ denotes the set of the names appearing in the tuple $\overline{a}$.

4. We inductively define $\mathscr{T}_E(u)$ for uniform subterms u of either t or t' by

$$\begin{array}{lcl}\mathscr{T}_E(f(u_1,\dots,u_n)) & = & \overline{f}(\mathscr{T}_E(u_1),\dots,\mathscr{T}_E(u_n))\\ \mathscr{T}_E(w_a u) & = & \mathscr{T}_E(u)\\ \mathscr{T}_E(\sigma f(u_1,\dots,u_n)) & = & \mathscr{T}_E((\sigma\cdot f)(\sigma u_1,\dots,\sigma u_n))\\ \mathscr{T}_E(\sigma X) & = & X(\langle\sigma\rangle(\overline{a}\setminus \mathsf{Fr}_E(X)))\end{array}$$

   where σ stands for a composition of the form $(b_1/a_1)\dots(b_n/a_n)$ and $\overline{a}\setminus \mathsf{Fr}_E(X)$ stands for the tuple obtained from $\overline{a}$ by deleting the names from $\mathsf{Fr}_E(X)$. Given a tuple of distinct names $\overline{c}$ and $\sigma=(b_1/a_1)\dots(b_n/a_n)$ the tuple $\langle\sigma\rangle\overline{c}$ is obtained from $\overline{c}$ by successively replacing a_i by b_i.

5. The translation $\mathscr{T}(E)$ is defined as

$$[\overline{a}]V\vdash \mathscr{T}_E(t)=\mathscr{T}_E(t').$$

Example 4.5.13. Consider the uniform equation

$$X:\{a\}\ \vdash\ \mathsf{app}_{\{b\}}(w_b[a](X),b)=(b/a)X\ :\{b\} \tag{4.44}$$

from Example 4.3.5. We have that $\overline{a}=(a,b)$ and $\mathsf{Fr}_E(X)=\{b\}$. So $|V|=\{X\}$ and $V(X)=1$. We compute the term on the right hand side of the translated equation by

$$\begin{array}{lcl}\mathscr{T}_E((b/a)X) & = & X(\langle(b/a)\rangle\{a\})\\ & = & X(b).\end{array}$$

Similarly, on the left hand side we get

$$\begin{array}{lcl}\mathscr{T}_E(\mathsf{app}_{\{b\}}(w_b[a](X),b)) & = & \mathsf{app}(\mathscr{T}_E(w_b[a](X)),V_b)\\ & = & \mathsf{app}(\mathscr{T}_E([a](X)),V_b)\\ & = & \mathsf{app}([a]\mathscr{T}_E(X),V_b)\\ & = & \mathsf{app}([a]X(a),V_b).\end{array}$$

Hence the corresponding equation in SNEL is

$$[a,b]\{X:1\}\vdash \mathsf{app}([a]X(a),V_b)=X(b).$$

This is precisely the equation (β_ε) of [FH08, Example 5.1].

Using the model theory of synthetic nominal equational logic, one can prove that a sheaf algebra satisfies a uniform equation if and only if the corresponding F-algebra satisfies the abstract equation obtained using the above algorithm.

Let $(\mathbb{X},\xi)$ be an F-algebra satisfying the additional equations (4.43). Then the sheaf $\mathfrak{X}$ corresponding to the nominal set $\mathbb{X}$ can be equipped with an algebra structure for the uniform signature given by $\mathcal{O}$. Let E be a uniform equation of the form

$$\Gamma \vdash t = t' : T.$$

Theorem 4.5.14 (semantic invariance)**.** $(\mathbb{X},\xi) \vDash \mathscr{T}_E(E)$ if and only if $\mathfrak{X} \vDash E$.

Proof. Recall the model theory and the notations for SNEL from [FH08]. We have that $(\mathbb{X},\xi) \vDash \mathscr{T}_E(E)$ when

$$[\![[\overline{a}]V \vdash \mathscr{T}_E(t)]\!]_{(\mathbb{X},\xi)}(\overline{b},(\langle\overline{c}_Y\rangle s_y)_{y\in|V|}) = [\![[\overline{a}]V \vdash \mathscr{T}_E(t')]\!]_{(\mathbb{X},\xi)}(\overline{b},(\langle\overline{c}_Y\rangle s_y)_{y\in|V|})$$

for all $(\overline{b},(\langle\overline{c}_Y\rangle s_y)_{y\in|V|} \in \mathbb{A}^{\#|\overline{a}|}\#\prod_{Y\in|V|}[\mathbb{A}^{\#V(Y)},\mathbb{X}]$.

For the left-to-right direction consider a valuation v of the variables in Γ and consider the abstract equation obtained from E

$$[\overline{a}]V \vdash \mathscr{T}(t) = \mathscr{T}(t').$$

For each variable Y in $|V|$ let $\overline{b}_Y$ denote the tuple $\overline{a} \setminus \mathsf{Fr}_E(Y)$. Since all the elements of $\mathsf{Fr}_E(Y)$ are fresh for $v(Y)$ we have $\overline{a}\#\langle\overline{b}_Y\rangle v(Y)$.

We show that for all uniform subterms u of t or t' we have that

$$[\![[\overline{a}]V \vdash \mathscr{T}_E(u)]\!]_{\mathbb{X},\xi}(\overline{a},(\langle\overline{b}_Y\rangle v(Y))_{Y\in|V|}) = [\![u]\!]_{\mathfrak{X},v} \tag{4.45}$$

The proof is by induction on the structure of u. For example, when $u = (c/d)X$ we can prove that

$$\begin{aligned}
[\![[\overline{a}]V \vdash X(\langle(c/d)\rangle\overline{b}_X)]\!]_{\mathbb{X},\xi}(\overline{a},(\langle\overline{b}_Y\rangle v(Y))_{Y\in|V|}) &= (\langle(c/d)\rangle\overline{b}_X\,\overline{b}_X)\cdot v(X)\\
&= (c\ d)\cdot v(X)\\
&= \mathfrak{X}(c/d)(v(X))\\
&= [\![(c/d)X]\!]_{\mathfrak{X},v}
\end{aligned}$$

We used that $\mathfrak{X}(S) = \{x \in \mathbb{X} \mid \mathsf{supp}(x) \subseteq S\}$ and that $\mathfrak{X}((c/d)_S)(x) = (c\ d)\cdot x$, recall (2.31) for details. The other cases use $\mathfrak{X}(w_a)(x) = x$ and the inductive definitions of the semantics of the abstract terms. It follows that $[\![t]\!]_{\mathfrak{X},v} = [\![t']\!]_{\mathfrak{X},v}$. We can show similarly that $\mathfrak{S}$ satisfies all the equations $tr_S(E)$ for $S \cap T = \emptyset$.

For the right-to-left implication we have to show that whenever $\mathfrak{X} \vDash E$ we have $[\![[\overline{a}]V \vdash \mathscr{T}_E(t)]\!]_{(\mathbb{X},\xi)} = [\![[\overline{a}]V \vdash \mathscr{T}_E(t')]\!]_{(\mathbb{X},\xi)}$. It is enough to prove that their evaluation at $(\overline{a},(\langle\overline{c}_Y\rangle s_y)_{y\in|V|})$ coincides for all $(\langle\overline{c}_Y\rangle s_y)_{y\in|V|}$ with the support disjoint from $\{\overline{a}\}$. Given such $(\langle\overline{c}_Y\rangle s_y)_{y\in|V|}$, for each $Y \in |V|$ we have $\overline{b}_Y\#s_Y$, hence we can find $u_Y \in \mathbb{X}$ such that $\langle\overline{c}_Y\rangle s_y = \langle\overline{b}_Y\rangle u_y$. Moreover $\mathsf{Fr}_E(X)\#u_Y$. Hence we can consider a valuation v in $\mathfrak{X}$ of all the variables in an extension $tr_S(\Gamma)$. To finalize the proof, we use that $\mathfrak{X} \vDash tr_S(E)$ and (4.45). □

4.6 Conclusions and further work

In this chapter we have studied a way of transferring universal algebra from $\mathsf{Set}^{\mathbb{I}}$ to $\mathrm{Sh}(\mathbb{I}^{op})$. We focused on algebras on nominal sets for functors $\tilde{L} : \mathrm{Sh}(\mathbb{I}^{op}) \to \mathrm{Sh}(\mathbb{I}^{op})$ that correspond to sifted colimit preserving functors on $\mathsf{Set}^{\mathbb{I}}$. We have seen that under some mild assumptions on $L : \mathsf{Set}^{\mathbb{I}} \to \mathsf{Set}^{\mathbb{I}}$ we can obtain an appropriate $\tilde{L}$ as $I^* L I_*$. Some questions were left open though, for example what are the properties of the functors $\tilde{L}$ obtained in this fashion? Or what are the requirements on a functor $\tilde{L}$ on sheaves so that we obtain an appropriate L on $\mathsf{Set}^{\mathbb{I}}$? Another task is to characterise the uniform theories from a categorical perspective. We believe we can provide answers to some of these questions and characterise the functors on nominal sets obtained by transferring sifted colimit preserving functors across the adjunction between Nom and $\mathsf{Set}^{\mathbb{I}}$ as exactly the functors that are determined by their action on the strong nominal sets. Central to this development is the fact that the adjunction between Nom and $\mathsf{Set}^{|\mathbb{I}|}$ is of descent type. The main category theoretic tool used is the notion of finitely-based functors, developed in [VK11], as a generalisation of functors presented by operations and equations in the more general setting of locally finitely presentable categories.

More results from universal algebra can be transferred to nominal setting. For example we can obtain a quasi-variety theorem, applying the methodology described in this chapter.

Another motivation for this study is to give a foundation for the work on logics for nominal calculi in the style of [BK07]. For example, the models for the π-calculus of [Sta08] should fit in the realm of universal algebra over nominal sets.

Chapter 5

Nominal Stone type dualities

In this chapter we present a nominal version of Stone type dualities. In Section 5.1 we recall classical Stone dualities for Boolean algebras and distributive lattices. In Section 5.2 we observe that a simple attempt to internalise Stone duality in nominal sets fails. We also show that the power object of a nominal set is a nominal-complete atomic Boolean algebra, but has a richer structure: one can define an operation n that 'corresponds' to the И quantifier and that is a restriction operation in the sense of [Pit11, Pit10]. In Section 5.4 we introduce distributive lattices equipped with a restriction operation and in section 5.5 we prove a duality theorem with certain nominal bitopological spaces. This duality restricts to the nominal Stone duality we obtained in [GLP11].

5.1 Preliminaries on Stone type dualities

Stone [Sto36] showed that each Boolean algebra is isomorphic to a subalgebra of the powerset of its ultrafilters. This subalgebra is obtained by considering only those sets that are both open and closed with respect to a certain topology on the set of ultrafilters. He then generalised his representation theorem to distributive lattices [Sto37]. Stone type dualities have numerous applications in theoretical computer science : Abramsky's domain theory in logical form [Abr91] takes its cue from a duality between program logics and denotations. Coalgebraic modal logic uses Stone duality as the bridge between systems and logics [BK05].

Let us first look at Stone duality from a rather category-theoretical perspective. We have the functor $P : \mathsf{Set} \to \mathsf{BA}^{op}$ that maps a set to its powerset regarded as a Boolean algebra and for each function $f : X \to Y$ the map $Pf : PY \to PX$ is defined as the inverse image and is a Boolean algebra morphism. The functor P has a right adjoint $S : \mathsf{BA}^{op} \to \mathsf{Set}$ where SB is the set of ultrafilters of a

Boolean algebra B and for any $h : B \to B'$ the map $Sh : SB' \to SB$ is defined by $Sh(\mathfrak{f}') = h^{-1}(\mathfrak{f}')$. Notice that SB can be regarded as the set of Boolean algebra morphisms from B to 2.

The adjunction $P \dashv S$ yields a monad β on Set. The category of Eilenberg-Moore algebras for β is isomorphic to the category of compact Hausdorff spaces CHaus, see [Man69, Joh82]. Moreover the adjunction $P \dashv S$ is of descent type, or equivalently, the comparison functor $K : \mathsf{BA}^{op} \to \mathsf{CHaus}$ is full and faithful. Its essential image is exactly the full subcategory Stone of CHaus consisting of the spaces that are zero-dimensional, that is, spaces for which the sets that are both open and closed form a basis for the topology. The compact Hausdorff spaces that are zero-dimensional are called Stone spaces. Thus we have Stone duality:[1]

Theorem 5.1.1. The categories BA and Stone are dually equivalent.

Let us note that an important part of the proof of this theorem hinges on the adjunction $P \dashv S$ being of descent type. This is equivalent to the counit of the adjunction being a regular epimorphism, or equivalently, the map $\varepsilon_B^{op} : B \to PSB$ defined by

$$b \mapsto \{F \in SB \mid b \in F\} \tag{5.1}$$

being a regular monomorphism. This in turn is equivalent to the Ultrafilter Theorem:

Theorem 5.1.2. Any filter of a Boolean algebra can be extended to an ultrafilter.

The ultrafilter theorem and Stone's representation theorem are equivalent, see [Jec73]. The proof of the ultrafilter theorem uses the Axiom of Choice. Nevertheless, there are models of ZF set theory for which the Axiom of Choice fails, but the ultrafilter theorem still holds.

There are several duality theorems for distributive lattices. Stone [Sto37] represented them as spectral spaces, that is, spaces that are compact, T_0, coherent and sober. This representation theorem may seem not as satisfactory as that for Boolean algebras. One reason is that on the topological side there is a lack of symmetry. Given a distributive lattice D, we can construct the dual lattice D^{op}, with the same carrier set, but with top and bottom, and meets and joins swapped. One could ask, how can we obtain the spectral space corresponding to D^{op} from the dual (X, τ) of D. The answer may seem a bit complicated: One has to equip X with the topology generated by the complements of sets that are both open and compact in τ.

---

[1] Recall that two categories are dually equivalent, when one is equivalent to the opposite of the other, see [ML71].

Priestley's reformulation of Stone duality for distributive lattices [Pri70] restores the symmetry on the topological side. She proved that distributive lattices are dually equivalent to what we now call Priestley spaces. They are compact topological spaces with an order that satisfies a separation axiom: if $x \not\leq y$ there exists a set U that is both open and closed— for short clopen—such that $x \in U$ and $y \notin U$, see [Pri70]. If $(X, \tau, \leq)$ is the Priestley space dual to a distributive lattice D, then the dual space of D^{op} is just $(X, \tau, \geq)$. The spectral space corresponding to D is (X, τ_+), where τ_+ is the topology generated by opens that are upper sets with respect to $\leq$. The lack of symmetry in the spectral spaces approach appears because τ_-, the topology generated by lower opens, is not taken into account. Thus, distributive lattices can be represented as bitopological spaces (X, τ_+, τ_-). The bitopological spaces arising in this fashion are described in [BBGK10] and are called pairwise Stone spaces. The duals of Boolean algebras are precisely the pairwise Stone spaces for which the two topologies coincide, and these are isomorphic to Stone spaces.

5.2 Stone duality fails in nominal sets

In this section we will prove that an internal version of Stone's representation theorem fails in nominal sets. Let us see how much of the theory can be replayed internally in nominal sets.

First, let us consider the category nBA of nominal Boolean algebras. Objects are Boolean algebra objects in Nom, that is tuples $(\mathbb{B}, \cdot, \wedge, \neg)$ where $(\mathbb{B}, \cdot)$ is a nominal set, $(\mathbb{B}, \wedge, \neg)$ is a Boolean algebra and the operations $\wedge, \neg$ are equivariant. Morphisms are equivariant Boolean algebra morphisms.

We have a functor $\mathscr{P} : \mathsf{Nom} \to \mathsf{nBA}^{op}$. On objects, $\mathscr{P}\mathbb{X}$ is the power object of the nominal set $\mathbb{X}$ equipped with a Boolean algebra structure, with $\wedge$ interpreted as intersection and $\neg$ as complement. One can easily check that this is well defined. If $f : \mathbb{X} \to \mathbb{Y}$ is a nominal sets morphism, then $\mathscr{P} f(Y) = f^{-1}(Y)$ for all $Y \in \mathscr{P}\mathbb{Y}$.

We can also consider a functor $\mathscr{S} : \mathsf{nBA}^{op} \to \mathsf{Nom}$ defined as follows. Given a nominal Boolean algebra $\mathbb{B}$, consider the set $\mathscr{S}\mathbb{B}$ of finitely-supported ultrafilters of $\mathbb{B}$. Notice that finitely supported ultrafilters correspond to finitely-supported Boolean algebra morphisms from $\mathbb{B}$ to the nominal Boolean algebra 2. If $\mathfrak{f} \subseteq \mathbb{B}$ is a finitely-supported ultrafilter, then so is $\pi \cdot \mathfrak{f} = \{\pi \cdot b \mid b \in \mathfrak{f}\}$. So $\mathscr{S}\mathbb{B}$ is a nominal set. For a nominal Boolean algebra morphism $h : \mathbb{B} \to \mathbb{B}'$ we define an equivariant function $\mathscr{S}(h^{op}) : S\mathbb{B}' \to S\mathbb{B}$ by $S(h^{op})(\mathfrak{f}') = h^{-1}(\mathfrak{f}')$.

Proposition 5.2.1. We have an adjunction $\mathscr{P} \dashv \mathscr{S} : \mathsf{nBA}^{op} \to \mathsf{Nom}$.

Proof. For a nominal set $\mathbb{X}$, the unit $\eta_{\mathbb{X}} : \mathbb{X} \to \mathscr{S}\mathscr{P}\mathbb{X}$ is given by

$$x \mapsto \{Y \in \mathscr{P}\mathbb{X} \mid x \in Y\} \tag{5.2}$$

It is easy to check that for all $x \in \mathbb{X}$ we have that $\eta(x)$ is supported by $\mathsf{supp}(x)$ and is an ultrafilter in $\mathscr{P}\mathbb{X}$.

For a nominal Boolean algebra $\mathbb{B}$, the counit $\varepsilon_{\mathbb{B}} : \mathscr{P}\mathscr{S}\mathbb{B} \to \mathbb{B}$ in nBA^{op} is given by the nominal Boolean algebra morphism $\varepsilon^{op}_{\mathbb{B}}$ that maps

$$b \mapsto \{\mathfrak{f} \in \mathscr{S}\mathbb{B} \mid b \in \mathfrak{f}\} \tag{5.3}$$

It is easy to check that for all $b \in \mathbb{B}$ we have that $\varepsilon^{op}(b)$ is supported by $\mathsf{supp}(b)$. Moreover, η and ε satisfy the usual triangular identities. $\square$

However this adjunction is not of descent type. This boils down to the fact that the map ε^{op} defined in (5.3) is not a monomorphism. We will prove in fact that the ultrafilter theorem does not hold in nominal sets. The classical theorem uses the Axiom of Choice. It is known that the Axiom of Choice fails for nominal sets, but there are models of ZF that satisfy the ultrafilter theorem, but not the Axiom of Choice, see [Jec73]. The example we provide below is inspired by [Jec73, Problem 4.6.3, pp 52], stating that the ordering principle fails in the basic Fraenkel model. And classically the ultrafilter theorem implies the ordering principle, [Jec73, Section 2.3.2].

Proposition 5.2.2. There exists a nominal Boolean algebra having a finitely-supported filter that cannot be extended to a finitely-supported ultrafilter.

Proof. Consider the nominal set $\mathscr{P}_{fin}(\mathbb{A})$ and for each $P \in \mathscr{P}_{fin}(\mathbb{A})$ consider the set M_P of total orders on P. Put $\mathbb{M} = \bigcup_{P \in \mathscr{P}_{fin}(\mathbb{A})} M_P$ and define a $\mathfrak{S}(\mathbb{A})$-action on $\mathbb{M}$ given as follows. If $\leq_P$ is an order on P define $\pi \cdot \leq_P$ to be an order on $\pi \cdot P$ given by

$$u\ (\pi \cdot \leq_P)\ v \quad \Longleftrightarrow \quad (\pi^{-1} \cdot u) \leq_P (\pi^{-1} \cdot v). \tag{5.4}$$

Then $(\mathbb{M}, \cdot)$ is a nominal set as for all $P \in \mathscr{P}_{fin}(\mathbb{A})$ each order on P is supported by P.

We will consider next finitely-supported partial maps from $\mathscr{P}_{fin}(\mathbb{A})$ to $\mathbb{M}$. Formally these are elements of the nominal set $[\mathscr{P}_{fin}(\mathbb{A}), \mathbb{M}+1]$, where $\mathbb{M}+1$ is the coproduct of $\mathbb{M}$ and the terminal object 1 of Nom.

Let $\mathbb{X}$ be the subset of $[\mathscr{P}_{fin}(\mathbb{A}), \mathbb{M}+1]$ consisting of those maps x such that

1. for all $P \in \mathscr{P}_{fin}(\mathbb{A})$ either $x(P) = 1$ or $x(P) \in M_P$ and
2. for all $P, Q \in \mathscr{P}_{fin}(\mathbb{A})$ such that $x(P) \in M_P$ and $x(Q) \in M_Q$ the orders $x(P)$ and $x(Q)$ coincide when restricted to $P \cap Q$.

It is easy to check that $\mathbb{X}$ is a nominal subset of $[\mathscr{P}_{fin}(\mathbb{A}), \mathbb{M}+1]$. Consider $x \in \mathbb{X}$ and recall from (2.10) that $(\pi \cdot x)(P) = \pi \cdot x(\pi^{-1} \cdot P)$. If $x(\pi^{-1} \cdot P) = 1$ then $(\pi \cdot x)(P) = 1$. Otherwise, if $x(\pi^{-1} \cdot P) \in M_{\pi^{-1} \cdot P}$ then $(\pi \cdot x)(P) \in M_P$, so condition 1. above is satisfied.

As for condition 2, if $(\pi \cdot x)(P)$ and $(\pi \cdot x)(Q)$ are orders on P, respectively on Q, then they agree on $P \cap Q$ because $x(\pi^{-1} \cdot P)$ and $x(\pi^{-1} \cdot Q)$ are orders on $\pi^{-1} \cdot P$, respectively $\pi^{-1} \cdot Q$, that agree on $(\pi^{-1} \cdot P) \cap (\pi^{-1} \cdot Q) = \pi^{-1} \cdot (P \cap Q)$.

Next we consider the nominal Boolean algebra $\mathscr{P}(\mathbb{X})$ and we construct a finitely-supported filter $\mathfrak{f}$. For each $P \in \mathscr{P}_{fin}(\mathbb{A})$ consider the set

$$X_P = \{x \in \mathbb{X} \mid x(P) \in M_P\} \tag{5.5}$$

Clearly X_P is supported by P, so is in $\mathscr{P}(\mathbb{X})$. Moreover, for all natural numbers n and $P_1, \ldots, P_n \in \mathscr{P}_{fin}(\mathbb{A})$ we have that $X_{P_1} \cap \ldots \cap X_{P_n}$ is non-empty. To see this, consider an order $\leq$ on $P_1 \cup \ldots \cup P_n$ and define $x \in [\mathscr{P}_{fin}(\mathbb{A}), \mathbb{M}+1]$ as follows. Put $x(P_i)$ to be the restriction of $\leq$ to P_i and $x(Q) = 1$ for all $Q \notin \{P_1, \ldots, P_n\}$. One can check that x is supported by $P_1 \cup \ldots \cup P_n$, and that $x(P_i)$ are pairwise compatible on intersections. Hence x is in $X_{P_1} \cap \ldots \cap X_{P_n}$.

Put $\mathfrak{f} = \{Y \in \mathscr{P}(\mathbb{X}) \mid \exists P_1, \ldots, P_n \in \mathscr{P}_{fin}(\mathbb{A}).\, X_{P_1} \cap \ldots \cap X_{P_n} \subseteq Y\}$. It is easy to check that $\mathfrak{f}$ is a finitely-supported (in fact equivariant) filter of $\mathscr{P}(\mathbb{X})$.

Assume by contradiction that $\mathfrak{f}$ can be extended to a finitely-supported ultrafilter $\mathfrak{u}$. For each $P \in \mathscr{P}_{fin}(\mathbb{A})$ we have that $X_P \in \mathfrak{u}$ and X_P is equal to a disjoint union $\bigcup\limits_{\leq \in M_P} \{x \in \mathbb{X} \mid x(P) = \leq\}$. It follows that for each $P \in \mathscr{P}_{fin}(\mathbb{A})$ there exists a unique total order $\prec_P$ on P such that

$$Y_P = \{x \in \mathbb{X} \mid x(P) = \prec_P\} \in \mathfrak{u}.$$

One can show that whenever $\pi \in \mathit{fix}(\mathsf{supp}(\mathfrak{u}))$ we have

$$\pi \cdot \prec_P = \prec_{\pi \cdot P} . \tag{5.6}$$

The orders $(\prec_P)_P$ are pairwise compatible, because for all P, Q we have $Y_P \cap Y_Q \in \mathfrak{u}$. Thus we can define a total order $\prec$ on $\mathbb{A}$ such that the restriction of $\prec$ to each finite P is $\prec_P$. Using (5.6) we can show that $\prec$ is a finitely-supported order on $\mathbb{A}$. Indeed, for $\pi \in \mathit{fix}(\mathsf{supp}(\mathfrak{u}))$ we have

$$\begin{aligned}
u \prec v &\iff u \prec_{\{u,v\}} v \\
&\iff \pi \cdot u (\pi \cdot \prec_{\{u,v\}}) \pi \cdot v && \text{using (5.4)} \\
&\iff \pi \cdot u \prec_{\pi \cdot \{u,v\}} \pi \cdot v && \text{using (5.6)} \\
&\iff \pi \cdot u \prec \pi \cdot v
\end{aligned}$$

But the existence of a finitely-supported total order on $\mathbb{A}$ leads to a contradiction. Assume $a, b, c \notin \mathsf{supp}(\prec)$ and $a \prec b$. Then $(a\ c) \cdot a \prec (a\ c) \cdot b$, so $c \prec b$. On the other hand $(a\ b)(b\ c) \cdot a \prec (a\ b)(b\ c) \cdot b$, hence $b \prec c$, contradiction! Hence the filter $\mathfrak{f}$ cannot be extended to a finitely-supported ultrafilter. □

5.3 The power object of a nominal set

In this section we explore the structure of the power object of a nominal set. We have seen that the functor $\mathscr{P} : \mathsf{Nom}^{op} \to \mathsf{Nom}$ is monadic and $\mathscr{P}^{op}$ is its left adjoint. The fact that the equivariant elements in the power object of a nominal set $\mathscr{P}X$ form a complete atomic Boolean algebra is a consequence of a theorem of Barr and Diaconescu for atomic toposes [BD80]. Recall that Nom is equivalent to the Grothendieck topos for the atomic topology on $\mathbb{I}^{op}$.

In fact more can be proved: the category of Eilenberg-Moore algebras for the monad corresponding to this adjunction is the category of complete atomic nominal Boolean algebras. Here, by 'complete' we mean 'internally complete in nominal sets.'

Definition 5.3.1. Let $\mathbb{B}$ be a nominal Boolean algebra. An element $x \in \mathbb{B}$ is called an *atom* if $x \neq \bot$ and for all $y \in \mathbb{B}$, $y \leq x$ implies $y = \bot$ or $y = x$.

Let $\mathsf{At}(\mathbb{B})$ denote the set of atoms of the nominal Boolean algebra $\mathbb{B}$.

Lemma 5.3.2. $\mathsf{At}(\mathbb{B})$ is a nominal subset of $\mathbb{B}$.

Proof. Consider π a finitely-supported permutation and an atom $x \in \mathsf{At}(\mathbb{B})$. We can prove that $\pi \cdot x$ is also an atom. Consider $y \in \mathbb{B}$ such that $y \leq \pi \cdot x$. Since $\leq$ is equivariant, we have $\pi^{-1} \cdot y \leq x$, hence $\pi^{-1} \cdot y = \bot$ or $\pi^{-1} \cdot y = x$. Or equivalently, $y = \bot$ or $y = \pi \cdot x$. This shows that $\mathsf{At}(\mathbb{B})$ is an equivariant subset of $\mathbb{B}$. □

Definition 5.3.3. A nominal Boolean algebra $\mathbb{B}$ is called *atomic* if for all $x \in \mathbb{B}$ such that $x \neq \bot$ there exists an atom $y \in \mathsf{At}(\mathbb{B})$ such that $y \leq x$.

Definition 5.3.4. A nominal Boolean algebra is called *nominal-complete* if each finitely-supported subset $X \subseteq \mathbb{B}$ has a supremum $\bigvee X \in \mathbb{B}$.

Definition 5.3.5. A complete atomic nominal Boolean algebra is a nominal Boolean algebra that is atomic (Definition 5.3.3) and nominal-complete (Definition 5.3.4). A morphism of complete atomic nominal Boolean algebras is an equivariant Boolean algebra morphism that preserves suprema of finitely-supported sets.

Let nCABA denote the category of complete atomic nominal Boolean algebras.

Example 5.3.6. If $\mathbb{X}$ is a nominal set then $\mathscr{P}\mathbb{X}$ is a complete atomic nominal Boolean algebra. Indeed, the atoms of $\mathscr{P}\mathbb{X}$ are the singletons. Given a finitely-supported family $(Y_i)_i$ in $\mathscr{P}\mathbb{X}$, we have that $\bigcup Y_i$ is also finitely-supported. Below we will show that any complete atomic nominal Boolean algebra is actually isomorphic to the powerset of a nominal set.

From now on $\mathbb{B}$ will denote a complete atomic nominal Boolean algebra.

Lemma 5.3.7. Given $x \in \mathbb{B}$, the set $\{y \in \mathsf{At}(\mathbb{B}) \mid y \le x\}$ is finitely-supported and its support is included in $\mathsf{supp}(x)$.

Proof. Assume $a, b \notin \mathsf{supp}(x)$ and y is an atom such that $y \le x$. Then $(a\ b) \cdot y \le (a\ b) \cdot x$, by equivariance of $\le$. But $(a\ b) \cdot x = x$, so $(a\ b) \cdot y \le x$. By Lemma 5.3.2 $(a\ b) \cdot y$ is a an atom. This shows that

$$(a\ b) \cdot \{y \in \mathsf{At}(\mathbb{B}) \mid y \le x\} = \{y \in \mathsf{At}(\mathbb{B}) \mid y \le x\}. \tag{5.7}$$

Hence $\mathsf{supp}(\{y \in \mathsf{At}(\mathbb{B}) \mid y \le x\}) \subseteq \mathsf{supp}(x)$. □

Lemma 5.3.8. Given $x \in \mathbb{B}$ we have that $x = \bigvee\{y \in \mathsf{At}(\mathbb{B}) \mid y \le x\}$

Proof. By Lemma 5.3.7 the set $\{y \in \mathsf{At}(\mathbb{B}) \mid y \le x\}$ is finitely-supported, hence using Definition 5.3.4 it has a supremum $x_0 \in \mathsf{B}$. Clearly $x_0 \le x$. Assume the converse inequality does not hold. Then $\neg x_0 \wedge x \ne \bot$. By Definition 5.3.3, there exits $z \in \mathsf{At}(\mathbb{B})$ such that $z \le \neg x_0 \wedge x$. But $z \le x$ implies $z \in \{y \in \mathsf{At}(\mathbb{B}) \mid y \le x\}$, hence $z \le x_0$. Since $z \le \neg x_0$ we get $z = \bot$, contradiction. So $x = x_0$ and we are done. □

Lemma 5.3.9. The map $\varphi : \mathbb{B} \to \mathscr{P}(\mathsf{At}(\mathbb{B}))$ defined by

$$x \mapsto \{y \in \mathsf{At}(\mathbb{B}) \mid y \le x\}.$$

is a morphism of complete atomic nominal Boolean algebras.

Proof. By Example 5.3.6 and Lemma 5.3.2 we have that $\mathscr{P}(\mathsf{At}(\mathbb{B}))$ is a complete atomic nominal Boolean algebra. By Lemma 5.3.7 φ is well defined. The map φ is equivariant. Indeed

$$\begin{aligned}(a\ b) \cdot \{y \in \mathsf{At}(\mathbb{B}) \mid y \le x\} &= \{(a\ b) \cdot y \mid y \in \mathsf{At}(\mathbb{B}) \text{ and } y \le x\}\\ &= \{y \in \mathsf{At}(\mathbb{B}) \mid (a\ b) \cdot y \le x\}\\ &= \{y \in \mathsf{At}(\mathbb{B}) \mid y \le (a\ b) \cdot x\}\end{aligned}$$

The proof for the fact that φ preserves finitely-supported suprema and negation is just as in the classical case:

Let x denote an atom of $\mathbb{B}$, and $\{x_i | i \in I\}$ a finitely-supported subset of $\mathbb{B}$. Assume that $x \leq \bigvee_i x_i$. We have that $x \wedge \bigvee_i x_i = \bigvee_i (x \wedge x_i)$. Since for all i we have $x \wedge x_i \in \{\perp, x\}$ it follows that there exists $i \in I$ such that $x \leq x_i$. This proves that φ preserves finitely-supported suprema. □

The map $\psi : \mathscr{P}(\mathsf{At}(\mathbb{B})) \to \mathbb{B}$ defined by

$$X \mapsto \bigvee X.$$

is well defined by Definition 5.3.4 and is equivariant. We can now show that φ is an isomorphism and ψ is its inverse:

Proposition 5.3.10. $\mathbb{B}$ and $\mathscr{P}(\mathsf{At}(\mathbb{B}))$ are isomorphic complete atomic nominal Boolean algebras.

Proof. We have $\psi \circ \varphi = id_{\mathbb{B}}$ by Lemma 5.3.8. Let us show that $\varphi \circ \psi = id_{\mathscr{P}(\mathsf{At}(\mathbb{B}))}$. Given $X \in \mathscr{P}(\mathsf{At}(\mathbb{B}))$, we have to prove that $\{x \in \mathsf{At}(\mathbb{B}) \mid x \leq \bigvee X\} = X$. The right-to-left inclusion is clear. Consider an atom x such that $x \leq \bigvee X$. It follows that there exists $y \in X$ such that $x \leq y$. Since both x and y are atoms, this implies $x = y$, hence $x \in X$, so the left-to-right inclusion also holds.

Both maps φ and ψ are equivariant, hence the direct image of any finitely-supported subset of $\mathbb{B}$, respectively $\mathscr{P}(\mathsf{At}(\mathbb{B}))$, is a finitely-supported subset of $\mathscr{P}(\mathsf{At}(\mathbb{B}))$, respectively $\mathbb{B}$. Then we can easily prove that ψ also preserves finitely-supported suprema and negation. Thus ψ is an isomorphism inverse to φ. □

Proposition 5.3.11. The categories Nom and nCABA are dually equivalent.

Proof. We define the contravariant functor $\mathscr{P} : \mathsf{Nom} \to \mathsf{nCABA}$ such that $\mathscr{P}\mathbb{X}$ is the powerset of $\mathbb{X}$, and for equivariant $f : \mathbb{X} \to \mathbb{Y}$ we put $\mathscr{P}(f)(B) = f^{-1}(B)$. This is well defined. By Proposition 5.3.10 we know that $\mathscr{P}$ is surjective on objects.

$\mathscr{P}$ is clearly faithful: Assume $f, g : \mathbb{X} \to \mathbb{Y}$ are equivariant maps such that $\mathscr{P}f = \mathscr{P}g$. Then, for all $y \in Y$ we have $f^{-1}(\{y\}) = g^{-1}(\{y\})$, hence for all $x \in \mathsf{X}$ we have $f(x) = y$ iff $g(x) = y$, or equivalently, $f = g$.

$\mathscr{P}$ is full: If $\tau : \mathscr{P}\mathbb{Y} \to \mathscr{P}\mathbb{X}$ is a morphism in nCABA, then the sets $(\tau(\{y\}))_{y \in \mathbb{Y}}$ are pairwise disjoint and, since τ preserves finitely-supported suprema, their union is $\tau(\mathbb{Y}) = \mathbb{X}$. So for all $x \in \mathbb{X}$ there exists a unique $y_x \in \mathbb{Y}$ such that $x \in \tau(\{y_x\})$. Set $f : \mathbb{X} \to \mathbb{Y}$ given by $f(x) = y_x$. We have $\tau = \mathscr{P}f$. □

In the previous section we have seen that the Boolean algebra structure is not enough to provide a representation theorem. In the remainder of this section we will define a restriction operation on $\mathscr{P}\mathbb{X}$. Recall the $\reflectbox{N}$ quantifier from Definiton 2.1.15.

Definition 5.3.12. Define a restriction operation $\mathsf{n} : \mathbb{A} \times \mathscr{P}\mathbb{X} \to \mathscr{P}\mathbb{X}$ by

$$\mathsf{n}a.X = \{x \in \mathbb{X} \mid \reflectbox{N} b.(a\ b)\cdot x \in X\} \tag{5.8}$$

for all $a \in \mathbb{A}$ and $X \in \mathscr{P}\mathbb{X}$.

Using Theorem 2.1.16 we have that

$$\begin{aligned} \mathsf{n}a.X &= \{x \in \mathbb{X} \mid \exists b\#a,x,X.\ (a\ b)\cdot x \in X\} \\ &= \{x \in \mathbb{X} \mid \forall b\#a,x,X.\ (a\ b)\cdot x \in X\} \end{aligned}$$

It is easy to verify that $\mathsf{n} : \mathbb{A} \times \mathscr{P}\mathbb{X} \to \mathscr{P}\mathbb{X}$ is equivariant. The operation n can be regarded as a semantic interpretation of the new quantifier $\reflectbox{N}$. We illustrate this point in the next lemma.

Lemma 5.3.13. Consider a nominal set $\mathbb{X}$ and an equivariant relation $R \subseteq \mathbb{A} \times \mathbb{X}$. Then

$$\{x \in \mathbb{X} \mid \reflectbox{N} a.R(a,x)\} = \mathsf{n}a.\{x \in \mathbb{X} \mid R(a,x)\} \tag{5.9}$$

Proof. Given $x \in \mathbb{X}$ we have the equivalences

$$\begin{aligned} \reflectbox{N} a.R(a,x) &\iff \reflectbox{N} b.(a\ b)\cdot R(a,x) \\ &\iff \reflectbox{N} b.R(b,(a\ b)\cdot x) \\ &\iff \reflectbox{N} b.(a\ b)\cdot x \in \{y \in \mathbb{X} \mid R(a,y)\} \\ &\iff x \in \mathsf{n}a.\{y \in \mathbb{X} \mid R(a,y)\} \end{aligned} \tag{5.10}$$

□

Next we summarise some of the properties of n, see [GLP11] for proofs.

Proposition 5.3.14. Let $a \in \mathbb{A}$ and $X \in \mathscr{P}\mathbb{X}$. The restriction operation n has the following properties:

1. $a\#\mathsf{n}a.X$.
2. $\mathsf{n}a.\mathsf{n}b.X = \mathsf{n}b.\mathsf{n}a.X$.
3. If $a\#X$ then $\mathsf{n}a.X = X$.
4. $\mathsf{n}a.(X \cap Y) = (\mathsf{n}a.X) \cap (\mathsf{n}a.Y)$.
5. $\mathsf{n}a.(\mathbb{X} \setminus X) = \mathbb{X} \setminus \mathsf{n}a.X$.
6. If $a\#x$ then $x \in X$ if and only if $x \in \mathsf{n}a.X$.
7. If $f : \mathbb{X} \to \mathbb{Y}$ is an equivariant function then $f^{-1}(\mathsf{n}a.U) = \mathsf{n}a.f^{-1}(U)$.

5.4 Nominal distributive lattice with restriction

The category of *nominal restriction sets* was introduced in Pitts' 'Structural Recursion with Locally Scoped Names', see [Pit11, Definition 2.6].

Definition 5.4.1. A nominal restriction set $\mathbb{X}$ is a nominal set equipped with a name-restriction operation, that is an equivariant map $\text{И} : \mathbb{A} \times \mathbb{X} \to \mathbb{X}$ that satisfies:

$$\begin{array}{lrl} \mathsf{Swap} & & \text{И}a.\text{И}b.x = \text{И}b.\text{И}a.x \\ \mathsf{Garbage} & a\#x \Rightarrow & \text{И}a.x = x \\ \mathsf{Alpha} & b\#x \Rightarrow & \text{И}a.x = \text{И}b.(b\ a)\cdot x \end{array}$$

Morphisms are equivariant maps that preserve И.

Definition 5.4.2. A **nominal bounded distributive lattice with restriction** is a bounded distributive lattice object in *Res*, that is, a nominal restriction set equipped with equivariant operations $\bot, \top, \vee, \wedge$ that satisfy the following additional axioms:

$$\begin{array}{ll} \mathsf{Distrib}\text{ - }\wedge & \text{И}a.(x \wedge y) = (\text{И}a.x) \wedge (\text{И}a.y) \\ \mathsf{Distrib}\text{ - }\vee & \text{И}a.(x \vee y) = (\text{И}a.x) \vee (\text{И}a.y) \end{array}$$

Lemma 5.4.3. $a\#\text{И}a.x$. As a corollary, $\text{И}a.\text{И}a.x = \text{И}a.x$.

Proof. We have that

$$\begin{array}{rcll} a\#\text{И}a.x & \Leftrightarrow & \text{Л}b.\,(a\ b)\cdot \text{И}a.x = \text{И}a.x & \text{(by Remark 2.1.17)} \\ & \Leftrightarrow & \text{Л}b.\,\text{И}b.(a\ b)\cdot x = \text{И}a.x & \text{(И is equivariant)} \end{array}$$

The latter is true by Alpha and Theorem 2.1.16. The last part of the lemma follows using Garbage. □

Notation 5.4.4. Given a finite set of names $C = \{c_1, \ldots, c_n\} \subseteq \mathbb{A}$ we denote by by $\text{И}C.x$ the element $\text{И}c_1 \ldots \text{И}c_n.x$. Notice that the order in which we write the elements of C is not important by Swap.

Remark 5.4.5. It follows from Lemma 5.4.3 that $\text{И}a.\bot = \bot$ and $\text{И}a.\top = \top$, so we omitted these from the axioms of Definition 5.4.2.

The morphisms between two nominal distributive lattices with restriction, are *Res*-morphisms that preserve the lattice operations. Let $\mathsf{nDL_n}$ denote the category of nominal distributive lattices with restriction.

Definition 5.4.6. Let $(\mathbb{L}, \cdot, \vee, \wedge, \top, \bot, \text{И})$ be a nominal distributive lattice with restriction. An **n-filter** $\mathfrak{f}$ of $\mathbb{L}$ is a finitely-supported set $\mathfrak{f} \in \mathscr{P}(\mathbb{L})$ such that:

1. $\top \in \mathfrak{f}$
2. $x \in \mathfrak{f}$ and $y \in \mathfrak{f}$ if and only if $x \wedge y \in \mathfrak{f}$
3. $\mathsf{И}a.\forall x.(x \in \mathfrak{f} \iff и a.x \in \mathfrak{f})$

Definition 5.4.7. Call an n-filter $\mathfrak{f}$ **proper** when $\bot \notin \mathfrak{f}$.

Call an n-filter $\mathfrak{f}$ **prime** when $\mathfrak{f}$ is proper and for all $x, y \in \mathbb{L}$ we have $x \vee y \in \mathfrak{f}$ implies $x \in \mathfrak{f}$ or $y \in \mathfrak{f}$.

The notions of n-ideal, proper n-ideal and prime n-ideal are defined dually.

Lemma 5.4.8. If $\mathfrak{f}$ is an n-filter of $\mathbb{L}$ and $\mathfrak{i}$ is an n-ideal of $\mathbb{L}$, such that $\mathfrak{f} \cap \mathfrak{i} = \emptyset$, then the set

$$\mathscr{F} = \{\mathfrak{g} \mid \mathfrak{g} \text{ is an } \mathsf{n}\text{-filter}, \mathfrak{g} \cap \mathfrak{i} = \emptyset, \mathfrak{f} \subseteq \mathfrak{g}, \mathsf{supp}(\mathfrak{g}) \subseteq \mathsf{supp}(\mathfrak{i}) \cup \mathsf{supp}(\mathfrak{f})\}$$

has a maximal element with respect to inclusion.

Proof. Given a chain $(\mathfrak{f}_i)_i$ in $\mathscr{F}$ we have that $\bigcup \mathfrak{f}_i \in \mathscr{F}$. Indeed, $\bigcup \mathfrak{f}_i$ is supported by $\mathsf{supp}(\mathfrak{i}) \cup \mathsf{supp}(\mathfrak{f})$ because each $\mathfrak{f}_i$ is supported by $\mathsf{supp}(\mathfrak{i}) \cup \mathsf{supp}(\mathfrak{f})$. Moreover $\bigcup \mathfrak{f}_i$ is an n-filter. By Zorn's lemma $\mathscr{F}$ has a maximal element $\bar{\mathfrak{f}}$. □

Theorem 5.4.9. If $\mathfrak{f}$ is an n-filter of $\mathbb{L}$ and $\mathfrak{i}$ is an n-ideal of $\mathbb{L}$, such that $\mathfrak{f} \cap \mathfrak{i} = \emptyset$, then there exists a prime n-filter $\bar{\mathfrak{f}}$ such that $\mathfrak{f} \subseteq \bar{\mathfrak{f}}$, $\bar{\mathfrak{f}} \cap \mathfrak{i} = \emptyset$ and $\mathsf{supp}(\bar{\mathfrak{f}}) \subseteq \mathsf{supp}(\mathfrak{f}) \cup \mathsf{supp}(\mathfrak{i})$.

Proof. Consider an n-filter $\bar{\mathfrak{f}}$ as obtained in Lemma 5.4.8. We will prove that $\bar{\mathfrak{f}}$ is prime. Assume by contradiction that $x \vee y \in \bar{\mathfrak{f}}$, but neither x nor y is in $\bar{\mathfrak{f}}$.

We will use the following notation. For $u \in \mathbb{L}$ let C_u denote the set

$$\mathsf{supp}(u) \setminus (\mathsf{supp}(\mathfrak{f}) \cup \mathsf{supp}(\mathfrak{i})).$$

The first crucial observation is that we can assume without loss of generality that $\mathsf{supp}(x) \subseteq \mathsf{supp}(\mathfrak{f}) \cup \mathsf{supp}(\mathfrak{i})$ and $\mathsf{supp}(y) \subseteq \mathsf{supp}(\mathfrak{f}) \cup \mathsf{supp}(\mathfrak{i})$. This is because we can replace x by $x' = и C_x.x$ and similarly, y by $y' = и C_y.y$, recall Notation 5.4.4. Then $\mathsf{supp}(x'), \mathsf{supp}(y') \subseteq \mathsf{supp}(\mathfrak{f}) \cup \mathsf{supp}(\mathfrak{i})$. By condition 3 of Definition 5.4.6 we have that $x' \vee y' \in \bar{\mathfrak{f}}$, but $x', y' \notin \bar{\mathfrak{f}}$.

At least one of the sets $\{z \wedge x \mid z \in \bar{\mathfrak{f}}\}$ and $\{z \wedge y \mid z \in \bar{\mathfrak{f}}\}$ is disjoint from $\mathfrak{i}$. Otherwise there exist $z_1, z_2 \in \bar{\mathfrak{f}}$ such that $z_1 \wedge x \in \mathfrak{i}$ and $z_2 \wedge y \in \mathfrak{i}$. But then $(z_1 \wedge x) \vee (z_2 \wedge y) \in \mathfrak{i} \cap \bar{\mathfrak{f}}$ and this contradicts the fact that $\mathfrak{i} \cap \bar{\mathfrak{f}} = \emptyset$.

Assume that

$$\{z \wedge x \mid z \in \bar{\mathfrak{f}}\} \cap \mathfrak{i} = \emptyset. \tag{5.11}$$

Consider the set

$$\mathfrak{f}' = \{(z \wedge x) \vee u \mid z \in \bar{\mathfrak{f}}, u \in \mathbb{L}\}.$$

It is easy to see that $\mathfrak{f}'$ has the following properties:

1. $\top \in \mathfrak{f}'$, because $\top = (\top \wedge x) \vee \top$;
2. $u \in \mathfrak{f}'$ and $v \in \mathfrak{f}'$ iff $u \wedge v \in \mathfrak{f}'$;
3. If $a \# \mathfrak{f}, \mathfrak{i}$ then $u \in \mathfrak{f}'$ implies $\text{и} a.u \in \mathfrak{f}'$;
4. $\bar{\mathfrak{f}} \subseteq \mathfrak{f}'$, but $x \in \mathfrak{f}' \setminus \bar{\mathfrak{f}}$;
5. $\mathsf{supp}(\mathfrak{f}') \subseteq \mathsf{supp}(\mathfrak{f}) \cup \mathsf{supp}(\mathfrak{i})$. This follows because by the Conservation of support principle we have $\mathsf{supp}(\mathfrak{f}') \subseteq \mathsf{supp}(\bar{\mathfrak{f}}) \cup \mathsf{supp}(x)$. But $\mathsf{supp}(x) \subseteq \mathsf{supp}(\bar{\mathfrak{f}})$ and $\mathsf{supp}(\bar{\mathfrak{f}}) \subseteq \mathsf{supp}(\mathfrak{f}) \cup \mathsf{supp}(\mathfrak{i})$.
6. $\mathfrak{f} \subseteq \mathfrak{f}'$ and $\mathfrak{i} \cap \mathfrak{f}' = \emptyset$, the latter follows from (5.11).

So $\mathfrak{f}'$ is almost an n-filter, but for condition 3 of Definition 5.4.6. We can extend it to a n-filter by considering

$$\mathfrak{f}'' = \{u \in \mathbb{L} \mid \text{и} C_u.u \in \mathfrak{f}'\}$$

It is easy to check that $\mathfrak{f}''$ is an n-filter and $\mathsf{supp}(\mathfrak{f}'') \subseteq \mathsf{supp}(\mathfrak{f}) \cup \mathsf{supp}(\mathfrak{i})$. Using the fact that $\mathfrak{f}'$ satisfies property 3 above, it follows that $\mathfrak{f}' \subseteq \mathfrak{f}''$. Hence $\bar{\mathfrak{f}} \subsetneq \mathfrak{f}''$.

Let us show that $\mathfrak{f}'' \cap \mathfrak{i} = \emptyset$. Assume there exists $u \in \mathfrak{f}'' \cap \mathfrak{i}$. Since $\mathfrak{i}$ is an n-ideal, we have that $\text{и} C_u.u \in \mathfrak{i}$ and this contradicts that $\mathfrak{f}' \cap \mathfrak{i} = \emptyset$.

We have proved that $\mathfrak{f}'' \in \mathscr{F}$ and $\bar{\mathfrak{f}} \subsetneq \mathfrak{f}''$. This contradicts the fact that $\bar{\mathfrak{f}}$ is a maximal element of $\mathscr{F}$. Therefore $\bar{\mathfrak{f}}$ is a prime n-filter. □

We finish this section with a useful result concerning generation of prime n-filters.

Definition 5.4.10. Let $(\mathbb{L}, \cdot, \vee, \wedge, \top, \bot, \text{и})$ be a nominal distributive lattice with restriction. A finitely-supported set $A \subseteq \mathbb{L}$ is called и-stable when

$$\text{И} a.\forall x \in \mathbb{L}.\ x \in A \Rightarrow \text{и} a.x \in A \tag{5.12}$$

Lemma 5.4.11. Consider a finitely-supported subset A of a nominal distributive lattice that is и-stable and has the finite intersection property, that is, for all $a_1, \ldots, a_n \in A$ we have $a_1 \wedge \ldots \wedge a_n \neq \bot$. Then there exists a prime n-filter $\mathfrak{f}$ such that $A \subseteq \mathfrak{f}$.

Proof. For all $x \in \mathbb{L}$ let C_x denote the set $\mathsf{supp}(x) \setminus \mathsf{supp}(A)$. Consider the set

$$\mathfrak{p} = \{x \in \mathbb{L} \mid \exists a_1, \ldots, a_n \in A.\ \text{и} C_x.x \geq a_1 \wedge \ldots \wedge a_n\}$$

Using the axioms for и, we can check that $\mathfrak{p}$ is a n-filter. For example, to prove that condition 3 of Definition 5.4.6 is satisfied, notice that for $a \# A$ we have that $\text{и} C_x.x = \text{и} C_{\text{и} a.x}.(\text{и} a.x)$.

Since A is $\mathsf{И}$-stable we have that $A \subseteq \mathfrak{p}$. Since A has the finite intersection property, we have that $\perp \notin \mathfrak{p}$. The result follows applying Theorem 5.4.9 for $\mathfrak{p}$ and $\mathfrak{i} = \{\perp\}$. □

5.5 Stone duality for nominal distributive lattices with restriction

5.5.1 From nominal distributive lattices with restriction to nominal bitopological spaces

Next we will introduce a notion of topological space internal to nominal sets. A natural requirement is that the carrier of such a space is a nominal set and the open sets are finitely-supported and form a nominal set. In general topology, arbitrary unions of open sets are open. This condition needs to be slightly modified in the nominal setting, because arbitrary unions of finitely-supported sets might not be finitely-supported. Here is the formal definition:

Definition 5.5.1. A **nominal topological space** is a pair $(\mathbb{X}, \tau)$ consisting of a nominal sets $\mathbb{X}$ and a nominal subset τ of $\mathscr{P}(\mathbb{X})$ satisfying

1. $\emptyset, \mathbb{X} \in \tau$;
2. If $U, V \in \mathbb{X}$ then $U \cap V \in \mathbb{X}$;
3. If $\mathscr{U}$ is a finitely-supported subset of τ then $\bigcup \mathscr{U} \in \tau$.

Definition 5.5.2. Let $(\mathbb{X}, \tau)$ be a nominal topological space. A finitely supported subset $U \in \mathscr{P}(\mathbb{X})$ is called

1. **open** when $U \in \tau$;
2. **closed** when $\mathbb{X} \setminus U \in \tau$;
3. **clopen** when U is both open and closed.

Lemma 5.5.3. Let $(\mathbb{X}, \tau)$ be a nominal topological space and consider $a \in \mathbb{A}$. Recall the restriction operation n from Definition 5.3.12. Then the following hold:

1. If U is closed then $\mathsf{n}a.U$ is closed.
2. If U is open then $\mathsf{n}a.U$ is open.
3. If U is clopen then $\mathsf{n}a.U$ is clopen.

Proof. 1. By Definition 5.3.12 we have that $\mathsf{n}a.U = \bigcap\limits_{b\#U,a} (a\ b)\cdot U$. Since for all $b\#U, a$ the set $(a\ b)\cdot U$ is closed, we conclude that $\mathsf{n}a.U$ is closed.

2. follows from 1. and the fact that $\mathbb{X}\setminus \mathsf{n}a.U = \mathsf{n}a.(\mathbb{X}\setminus U)$.

3. is an immediate consequence of 1. and 2. □

We will represent nominal distributive lattices with restriction as nominal bitopological spaces.

Definition 5.5.4. A **nominal bitopological space** $(\mathbb{X}, \tau_1, \tau_2)$ is a nominal set $\mathbb{X}$ equipped with two nominal topologies τ_1 and τ_2. Morphism of nominal bitopological spaces are equivariant bicontinuous (that is, continuous in both topologies) functions. Let nBiTop denote the category of nominal bitopological spaces.

Next we will define a functor $F : \mathsf{nDL}_{\text{И}}^{op} \to \mathsf{nBiTop}$. Given $(\mathbb{L}, \cdot, \vee, \wedge, \top, \bot, \text{ᴎ})$ a nominal bounded distributive lattice with restriction, consider the set $\mathsf{pf}(\mathbb{L})$ of prime n-filters of $\mathbb{L}$. For each $x \in \mathbb{L}$, put

$$x^+ = \{\mathfrak{f} \in \mathsf{pf}(\mathbb{L}) \mid x \in \mathfrak{f}\} \tag{5.13}$$

and

$$x^- = \{\mathfrak{f} \in \mathsf{pf}(\mathbb{L}) \mid x \notin \mathfrak{f}\} \tag{5.14}$$

Notice that x^+ and x^- are subsets of $\mathsf{pf}(\mathbb{L})$ supported by $\mathsf{supp}(x)$. So we have two maps $(-)^+, (-)^- : \mathbb{L} \to \mathscr{P}(\mathsf{pf}(\mathbb{L}))$. Before giving the definition for the functor F we establish some of their properties.

Lemma 5.5.5. For all $x, y \in \mathbb{L}$ and $a \in \mathbb{A}$ the following equalities hold:

$$\begin{array}{ll}
x^+ \cap y^+ = (x \wedge y)^+ & x^- \cap y^- = (x \vee y)^- \\
x^+ \cup y^+ = (x \vee y)^+ & x^- \cup y^- = (x \wedge y)^- \\
\top^+ = \mathsf{pf}(\mathbb{L}) & \top^- = \emptyset \\
\bot^+ = \emptyset & \bot^- = \mathsf{pf}(\mathbb{L}) \\
\mathsf{n}a.x^+ = (\text{ᴎ}a.x)^+ & \mathsf{n}a.x^- = (\text{ᴎ}a.x)^-.
\end{array} \tag{5.15}$$

Proof. Most of these are easy verifications. We give a proof for $\mathsf{n}a.x^+ = (\text{ᴎ}a.x)^+$.

$$\begin{array}{rcll}
\mathfrak{f} \in \mathsf{n}a.x^+ & \iff & \text{И}b.(a\ b)\cdot \mathfrak{f} \in x^+ & \text{by Definition 5.8} \\
& \iff & \text{И}b.x \in (a\ b)\cdot \mathfrak{f} & \\
& \iff & \text{И}b.(a\ b)\cdot x \in \mathfrak{f} & \\
& \iff & \exists b\#a, x, \mathfrak{f}.\ (a\ b)\cdot x \in \mathfrak{f} & \text{by Theorem 2.1.16} \\
& \iff & \exists b\#a, x, \mathfrak{f}.\ \text{ᴎ}b.(a\ b)\cdot x \in \mathfrak{f} & \text{by Definiton 5.4.6} \\
& \iff & \text{ᴎ}a.x \in \mathfrak{f} & \\
& \iff & \mathfrak{f} \in (\text{ᴎ}a.x)^+ &
\end{array} \tag{5.16}$$

Analoguously we can show $\mathsf{n}a.x^- = (\text{ᴎ}a.x)^-$. □

Lemma 5.5.6. The maps $(-)^+, (-)^- : \mathbb{L} \to \mathscr{P}(\mathsf{pf}(\mathbb{L}))$ are equivariant and injective.

Proof. The first part is easy. To check injectivity of $(-)^+, (-)^-$ consider $x, y \in \mathbb{L}$ that are different. It is enough to prove the existence of a prime n-filter that contains only one of them.

We can assume without loss of generality that $x \not\leq y$. For $z \in \mathbb{L}'$ let C_z denote the set $\mathsf{supp}(z) \setminus (\mathsf{supp}(x) \cup (\mathsf{supp}(y))$. We consider the sets

$$\mathfrak{f} = \{z \in \mathbb{L}' \mid \text{И} C_z.z \geq x\}$$

and

$$\mathfrak{i} = \{z \in \mathbb{L}' \mid \text{И} C_z.z \leq y\}$$

We can check that $\mathfrak{f}$ is a n-filter containing x and $\mathfrak{i}$ is a n-ideal containing y. Moreover, we have that $\mathfrak{f} \cap \mathfrak{i} = \emptyset$, otherwise we would contradict the assumption that $x \not\leq y$. By Theorem 5.4.9, there exists a prime n-filter $\mathfrak{f}'$ such that $\mathfrak{f} \subseteq \mathfrak{f}'$ and $\mathfrak{f}' \cap \mathfrak{i} = \emptyset$. So $x \in \mathfrak{f}'$ and $y \notin \mathfrak{f}'$. □

Consider the nominal topology τ_+ generated by $\{x^+ \mid x \in \mathbb{L}\}$. Since this set is closed under finite intersections by (5.15), τ^+ is obtained by taking finitely-supported unions of x_i^+.

Similarly, let τ_- denote the nominal topology generated by $\{x^- \mid x \in \mathbb{L}\}$. That is, the open sets in τ_- are finitely-supported unions of x_i^-.

We put $F(\mathbb{L}) = (\mathsf{pf}(\mathbb{L}), \tau_+, \tau_-)$. Given $h : \mathbb{L} \to \mathbb{L}'$ a morphism in $\mathsf{nDL}_{\text{И}}$, we define $F(h^{op}) : \mathsf{pf}(\mathbb{L}') \to \mathsf{pf}(\mathbb{L})$ by $F(h^{op})(\mathfrak{f}') = h^{-1}(\mathfrak{f}')$.

Lemma 5.5.7. The functor $F : \mathsf{nDL}_{\text{И}}^{op} \to \mathsf{nBiTop}$ is well defined.

Proof. We need to check that for all morphisms $h : \mathbb{L} \to \mathbb{L}'$ in $\mathsf{nDL}_{\text{И}}$ the map $F(h^{op})$ is a morphism of bitopological spaces. It is easy to check that it is equivariant. Notice that $F(h^{op})^{-1}(x^+) = h(x)^+$ and $F(h^{op})^{-1}(x^-) = h(x)^-$ for all $x \in \mathbb{L}$. It follows that $F(h^{op})$ is bicontinuous. □

Lemma 5.5.8. The functor F is faithful.

Proof. Consider $h_1, h_2 : \mathbb{L} \to \mathbb{L}'$ two morphisms in $\mathsf{nDL}_{\text{И}}$ such that $F(h_1^{op}) = F(h_2^{op})$. Assume to the contrary that there exists $x \in \mathbb{L}$ such that $h_1(x) \neq h_2(x)$. By Lemma 5.5.6 there exists a prime n-filter $\mathfrak{f}$ such that $h_1(x) \in \mathfrak{f}$ and $h_2(x) \notin \mathfrak{f}$ or $h_2(x) \in \mathfrak{f}$ and $h_1(x) \notin \mathfrak{f}$. This contradicts the fact that $F(h_1^{op})(\mathfrak{f}) = F(h_2^{op})(\mathfrak{f})$. Hence $h_1 = h_2$. □

5.5.2 The duality

We prove that the functor F is a full embedding and describe its essential image. To this end we introduce nominal pairwise Stone bitopological spaces, and we show that they form a full subcategory of nBiTop dually equivalent to $\mathsf{nDL}_{\text{И}}$.

Definition 5.5.9. A nominal bitopological space is called **pairwise Hausdorff** when for all distinct $x, y \in \mathbb{X}$ there exist disjoint $U \in \tau_1$ and $V \in \tau_2$ such that $x \in U$ and $y \in V$, or there exists disjoint $U \in \tau_2$ and $V \in \tau_1$ such that $x \in U$ and $y \in V$.

Lemma 5.5.10. $(\mathsf{pf}(\mathbb{L}), \tau_+, \tau_-)$ is pairwise Hausdorff.

Proof. Let $\mathfrak{f}, \mathfrak{g}$ be distinct prime n-filters. Then there exists $x \in \mathfrak{f} \setminus \mathfrak{g}$ or there exists $x \in \mathfrak{g} \setminus \mathfrak{f}$. The sets $U = x^+$ and $V = x^-$ satisfy the required properties. $\square$

Definition 5.5.11. Given a nominal topology τ, a finitely-supported subset $\mathscr{U} \subseteq \tau$ is called n**-stable** when

$$\text{И}a.\forall U.(U \in \mathscr{U} \Rightarrow \mathsf{n}a.U \in \mathscr{U}).$$

Remark 5.5.12. 1. Any finite subset $\mathscr{U} \subseteq \tau$ is n-stable. If $\mathscr{U}$ is an equivariant subset of τ then $\mathscr{U}$ is n-stable iff for all $a \in \mathbb{A}$ we have $U \in \mathscr{U} \Rightarrow \mathsf{n}a.U \in \mathscr{U}$.

2. Notice that if $\mathscr{U}$ is a finitely-supported subset of τ, hence $\mathscr{U}$ is a finitely-supported subset of $\mathscr{P}\mathbb{X}$. Then $\mathscr{U}$ is n-stable if and only if it is И-stable in the sense of Definition 5.4.10.

Definition 5.5.13. A nominal bitopological space is called **pairwise** n**-compact** when for all n-stable $\mathscr{U}_1 \in \mathscr{P}(\tau_1)$ and n-stable $\mathscr{U}_2 \in \mathscr{P}(\tau_2)$ such that $\bigcup \mathscr{U}_1 \cup \bigcup \mathscr{U}_2$ covers $\mathbb{X}$ there exists a finite subset of $\mathscr{U}_1 \cup \mathscr{U}_2$ that covers $\mathbb{X}$.

Lemma 5.5.14. Given a n-compact nominal topological space $(\mathbb{X}, \tau_1, \tau_2)$ and $\mathscr{U}_1$, $\mathscr{U}_2$ finitely supported n-stable subsets of τ_1, respectively τ_2 such that $\mathscr{U}_1 \cup \mathscr{U}_2$ has the finite intersection property, (that is, any finite subset of $\mathscr{U}_1 \cup \mathscr{U}_2$ has non-empty intersection), then $\bigcap(\mathscr{U}_1 \cup \mathscr{U}_2) \neq \emptyset$.

Proof. Assume by the contrary that $\bigcap(\mathscr{U}_1 \cup \mathscr{U}_2) \neq \emptyset$. Then the finitely supported sets $\mathscr{V}_1 = \{\mathbb{X} \setminus U \mid U \in \mathscr{U}_1\}$ and $\mathscr{V}_2 = \{\mathbb{X} \setminus U \mid U \in \mathscr{U}_2\}$ are n-stable and their union cover $\mathbb{X}$. Hence there exists a finite subcover of $\mathscr{V}_1 \cup \mathscr{V}_2$. This contradicts the fact that $\mathscr{U}_1 \cup \mathscr{U}_2$ has the finite intersection property. $\square$

Lemma 5.5.15. $(\mathsf{pf}(\mathbb{L}), \tau_+, \tau_-)$ is pairwise n-compact.

Proof. Consider n-stable $\mathcal{U}_1 \in \mathcal{P}(\tau_+)$ and n-stable $\mathcal{U}_2 \in \mathcal{P}(\tau_-)$ such that $\bigcup \mathcal{U}_1 \cup \bigcup \mathcal{U}_2$ covers $\mathsf{pf}(\mathbb{L})$. Let

$$A_1 = \{x \in \mathbb{L} \mid \exists U \in \mathcal{U}_1 . x^+ \subseteq U\}$$

$$A_2 = \{x \in \mathbb{L} \mid \exists U \in \mathcal{U}_2 . x^- \subseteq U\}$$

The sets A_1 and A_2 have the following properties:

1. $\mathsf{supp}(A_i) \subseteq \mathsf{supp}(\mathcal{U}_i)$
2. $a \# \mathcal{U}_i$ and $x \in A_i$ implies $\mathsf{\Finv} a . x \in A_i$. Indeed, if $x \in A_1$ there exists $U \in \mathcal{U}_1$ such that $x^+ \in \mathcal{U}_1$. But since $a \# \mathcal{U}_1$ and $\mathcal{U}_1$ is n-stable we have $\mathsf{n} a . (x^+) \subseteq \mathsf{n} a . U \in \mathcal{U}_1$. By Lemma 5.5.5 $\mathsf{n} a . (x^+) = (\mathsf{\Finv} a . x)^+$, thus $\mathsf{\Finv} a . x \in A_1$.

The sets

$$\mathcal{V}_1 = \{x^+ \mid x \in A_1\}$$

$$\mathcal{V}_2 = \{x^- \mid x \in A_2\}$$

have the properties:

1. $\mathsf{supp}\mathcal{V}_1 \subseteq \mathsf{supp}\mathcal{U}_1$ and $\mathsf{supp}\mathcal{V}_2 \subseteq \mathsf{supp}\mathcal{U}_2$
2. $\mathcal{V}_1$ and $\mathcal{V}_2$ are n-stable.
3. $\mathcal{V}_1$ and $\mathcal{V}_2$ cover $\mathsf{pf}(\mathbb{L})$.

It is enough to show that $\mathsf{pf}(\mathbb{L})$ has a finite subcover of $\mathcal{V}_1 \cup \mathcal{V}_2$. For $u \in \mathbb{L}$ let C_u denote the set $\mathsf{supp}(u) \setminus (\mathsf{supp}(\mathcal{U}_1) \cup \mathsf{supp}(\mathcal{U}_2))$.

Consider the sets

$$\mathfrak{i} = \{u \in \mathbb{L} \mid \exists m \geq 1 . \exists x_1, \ldots, x_m \in A_1 . \quad \mathsf{\Finv} C_u . u \leq x_1 \vee \ldots \vee x_m\}.$$

$$\mathfrak{f} = \{u \in \mathbb{L} \mid \exists m \geq 1 . \exists y_1, \ldots, y_m \in A_2 . \quad \mathsf{\Finv} C_u . u \geq y_1 \wedge \ldots \wedge y_m\}.$$

Then $\mathfrak{i}$ is an n-ideal and $\mathfrak{f}$ is a n-filter. To check condition 3 of Definition 5.4.6 use that whenever $x \in A_i$ and $a \# \mathcal{U}_1$ we have $\mathsf{\Finv} a . x \in A_i$. Moreover $A_1 \subseteq \mathfrak{i}$ and $A_2 \subseteq \mathfrak{f}$.

Assume that $\mathfrak{i} \cap \mathfrak{f} = \emptyset$. By Theorem 5.4.9 there exists a prime n-filter $\mathfrak{f}'$ such that $\mathfrak{f} \subseteq \mathfrak{f}'$ and $\mathfrak{f}' \cap \mathfrak{i} = \emptyset$. But this implies $\mathfrak{f}' \notin x^+$ for all $x \in A_1$ and $\mathfrak{f}' \notin x^-$ for all $x \in A_2$, contradicting that $\bigcup \mathcal{V}_1 \cup \bigcup \mathcal{V}_2$ covers $\mathsf{pf}(\mathbb{L})$.

Hence there exists $u \in \mathfrak{i} \cap \mathfrak{f}$. We obtain $x_1 \vee \ldots \vee x_m \geq y_1 \wedge \ldots \wedge y_n$ for some $x_1, \ldots, x_m \in A_1$ and $y_1, \ldots, y_n \in A_2$. It is easy to check that $\{x_1^+, \ldots, x_m^+, y_1^-, \ldots, y_n^-\}$ covers $\mathsf{pf}(\mathbb{L})$. $\square$

Definition 5.5.16. A nominal bitopological space is called **pairwise zero dimensional** when the sets that are open in τ_1 and closed in τ_2 form a basis for τ_1 and the sets that are open in τ_2 and closed in τ_1 form a basis for τ_2.

Lemma 5.5.17. $(\mathsf{pf}(\mathbb{L}), \tau_+, \tau_-)$ is pairwise zero-dimensional.

Proof. We show that every set that U is τ_+-open and τ_--closed is of the form x^+ for some $x \in \mathbb{L}$. Since U is τ_+-open we have

$$U = \bigcup \{x^+ \mid x^+ \subseteq U\}$$

Since $\mathsf{pf}(\mathbb{L}) \setminus U$ is τ_--open we have that

$$\mathsf{pf}(\mathbb{L}) \setminus U = \bigcup \{x^- \mid x^- \subseteq \mathsf{pf}(\mathbb{L}) \setminus U\}$$

Notice that the sets $\{x^+ \mid x^+ \subseteq U\}$ and $\{x^- \mid x^- \subseteq \mathsf{pf}(\mathbb{L}) \setminus U\}$ are finitely-supported and n-stable. Their union is a cover for $\mathsf{pf}(\mathbb{L})$ and by Lemma 5.5.15 it has a finite subcover. So there exists $x_1, \ldots, x_n \in \mathbb{L}$ such that $U = \cup x_i^+$. By Lemma 5.5.5 we have $U = (x_1 \vee \ldots \vee x_n)^+$.

Similarly, τ_--opens that are τ_+-closed are of the form x^- for some $x \in \mathbb{L}$. The conclusion follows because $\{x^+ \mid x \in \mathbb{L}\}$ is a basis for τ_+ and $\{x^- \mid x \in \mathbb{L}\}$ is a basis for τ_-. ☐

Definition 5.5.18. A **nominal pairwise Stone space** is a nominal bitopological space that is pairwise Hausdorff, pairwise n-compact and pairwise zero-dimensional.

Let nBiSt denote the full subcategory of nominal bitpological spaces with objects nominal pairwise Stone spaces.

Lemma 5.5.19. The functor F is full.

Proof. Consider a morphism of nominal bitopological spaces $u : (\mathsf{pf}(\mathbb{L}'), \tau'_+, \tau'_-) \to (\mathsf{pf}(\mathbb{L}), \tau_+, \tau_-)$. We will define a morphism $h : \mathbb{L} \to \mathbb{L}'$ in $\mathsf{nDL}_{\mathsf{И}}$ such that $F(h^{op}) = u$. For all $x \in \mathbb{L}$ we have that $u^{-1}(x^+)$ is open in τ'_+ and closed in τ'_-. Using a similar argument as in Lemma 5.5.17 there exists $y_x \in \mathbb{L}'$ such that $u^{-1}(x^+) = y_x^+$. Note that y_x is unique, by Lemma 5.5.6. Define $h(x) = y_x$. The function h is equivariant, because u and $(-)^+$ are equivariant. From (5.15) it follows that h is a morphism of distributive lattices. We can easily check that h preserves $\mathsf{И}$:

$$\begin{aligned} (h(\mathsf{И}a.x))^+ &= u^{-1}((\mathsf{И}a.x)^+) \\ &= u^{-1}(\mathsf{n}a.(x^+)) && \text{by Lemma 5.5.5} \\ &= \mathsf{n}a.u^{-1}(x^+) \\ &= \mathsf{n}a.(h(x))^+ \\ &= (\mathsf{И}a.h(x))^+ \end{aligned} \tag{5.17}$$

By Lemma 5.5.6 $(-)^+$ is injective, hence $h(\text{И}a.x) = \text{И}a.h(x)$. So h is a morphism in $\mathsf{nDL}_{\text{И}}$.

Finally we can check that $F(h^{op})(\mathfrak{f}') = u(\mathfrak{f}')$. Indeed $x \in h^{-1}(\mathfrak{f}')$ if and only if $\mathfrak{f}' \in (h(x))^+$, or equivalently $x \in u(\mathfrak{f}')$. ☐

Theorem 5.5.20. The categories $\mathsf{nDL}_{\text{И}}$ and nBiSt are dually equivalent.

Proof. By Lemmas 5.5.8 and 5.5.19 the functor $F : \mathsf{nDL}_{\text{И}}^{op} \to \mathsf{nBiTop}$ is a full embedding. We have seen that for each nominal distributive lattice with restriction $F(\mathbb{L})$ is a nominal pairwise Stone space. It suffices to show that for each $(\mathbb{X}, \tau_1, \tau_2)$ in nBiSt there exists $G(\mathbb{X})$ in $\mathsf{nDL}_{\text{И}}$ such that $F(G(\mathbb{X}))$ is isomorphic to $(\mathbb{X}, \tau_1, \tau_2)$. Let $G(\mathbb{X})$ be the nominal set of sets that are open in τ_1 and closed in τ_2. By Lemma 5.5.3 and Proposition 5.3.14 we have that $(G(\mathbb{X}), \cap, \cup, \emptyset, \mathbb{X}, \mathsf{n})$ is a nominal distributive lattice with restriction. We define $\phi : \mathbb{X} \to F(G(\mathbb{X}))$ by

$$\phi(x) = \{U \in G(\mathbb{X}) \mid x \in U\}.$$

It is easy to check that $\phi(x)$ is a prime n-filter in $G(\mathbb{X})$. Condition 3 in Definition 5.4.6 is satisfied because for all $a \in \mathbb{A}$ such that $a\#x$ we have by point 6 of Proposition 5.3.14 that $U \in \phi(x)$ if and only if $\mathsf{n}a.U \in \phi(x)$. We can prove that ϕ is equivariant. Since $\mathbb{X}$ is pairwise Hausdorff it follows that ϕ is injective. To prove that it is onto, consider $\mathfrak{f}$ a prime n-filter of $G(\mathbb{X})$. Put $\mathfrak{g} = \{V \in \tau_2 \mid \mathbb{X} \setminus V \in \tau_1 \setminus \mathfrak{f}\}$. We can prove that $\mathfrak{f} \cup \mathfrak{g}$ has the finite intersection property and is n-stable. Hence, by Lemma 5.5.14, there exists $x \in \mathbb{X}$ such that $x \in \bigcap(\mathfrak{f} \cup \mathfrak{g})$. It follows that $\phi(x) = \mathfrak{f}$. Let ψ denote the inverse of ϕ.

It remains to prove that ϕ and ψ are bicontinuous. Observe that for all $U \in G(\mathbb{X})$ we have $\phi^{-1}(U^+) = U \in \tau_1$ and $\phi^{-1}(U^-) = \mathbb{X} \setminus U \in \tau_2$. Since $\{U^+ \mid U \in G(\mathbb{X})\}$ is a basis for τ_+ and $\{U^- \mid U \in G(\mathbb{X})\}$ is a basis for τ_-, it follows that ϕ is bicontinuous. Also for all $U \in G(\mathbb{X})$ we have $\psi^{-1}(U) = U^+$ and $\psi^{-1}(\mathbb{X} \setminus U) = U^-$. Since $\mathbb{X}$ is pairwise zero-dimensional, it follows that ψ is bicontinuous. ☐

5.6 Stone duality for nominal Boolean algebras with restriction

In this section we will show that the duality between nominal distributive lattices with restriction and nominal pairwise Stone spaces restricts to a duality between nominal Boolean algebras with restriction and nominal Stone spaces. This is exactly the duality we have obtained in [GLP11].

Definition 5.6.1. A nominal Boolean algebra with restriction $(\mathbb{B}, \cdot, \wedge, \neg, \text{И})$ is a nominal Boolean algebra equipped with a name-restriction operation $\text{И} : \mathbb{A} \times$

$\mathbb{B} \to \mathbb{B}$ satisfying

$$\begin{array}{ll} \mathsf{Distrib} - \wedge & \reflectbox{N}a.(x \wedge y) = (\reflectbox{N}a.x) \wedge (\reflectbox{N}a.y) \\ \mathsf{Distrib} - \neg & \reflectbox{N}a.(\neg x) = \neg(\reflectbox{N}a.x) \end{array}$$

A morphism of nominal Boolean algebras with restriction is a nominal Boolean algebra mophisms that preserves $\reflectbox{N}$. Let $\mathsf{nBA}_{\reflectbox{N}}$ denote the category of nominal Boolean algebras with restriction.

Nominal Boolean algebras with restriction are in particular nominal distributive lattices with restriction. We will see next the effect that the presence of negation has on the topological side. First let us see a generalisation of a classical result for Boolean algebras to nominal setting:

Lemma 5.6.2. Given $\mathbb{B}$ a nominal Boolean algebra with restriction and $\mathfrak{f}$ a finitely-supported subset of $\mathbb{B}$, the following are equivalent:

1. $\mathfrak{f}$ is a prime n-filter.
2. For all $x \in \mathbb{B}$ either $x \in \mathfrak{f}$ or $\neg x \in \mathfrak{f}$.
3. $\mathfrak{f}$ is a maximal proper n-filter.

Proof. $1 \Rightarrow 2$. This is similar to the classical case, using that for all x we have $\top = x \vee \neg x \in \mathfrak{f}$.

$2 \Rightarrow 3$. Assume there exists a proper n-filter $\mathfrak{p}$ such that $\mathfrak{f} \subsetneq \mathfrak{p}$. There exists $x \in \mathfrak{p} \setminus \mathfrak{f}$. Then $\neg x \in \mathfrak{f}$, hence $\neg x \in \mathfrak{p}$. It follows $\bot = x \wedge \neg x \in \mathfrak{p}$ and this contradicts the fact that $\mathfrak{p}$ is proper.

$3 \Rightarrow 1$. Assume $x \vee y \in \mathfrak{f}$, but $x, y \notin \mathfrak{f}$. Then $\mathfrak{f}' = \{z \in \mathbb{B} \mid z \vee x \in \mathfrak{f}\}$ is a n-filter such that $\mathfrak{f} \subsetneq \mathfrak{f}'$. Condition 3 of Definition 5.4.6 is satisfied because for $a\#\mathfrak{f}, x$ we have

$$\begin{array}{rcl} z \in \mathfrak{f}' & \iff & z \vee x \in \mathfrak{f} \\ & \iff & \reflectbox{N}a.(z \vee x) \in \mathfrak{f} \\ & \iff & (\reflectbox{N}a.z) \vee x \in \mathfrak{f} \\ & \iff & \reflectbox{N}a.z \in \mathfrak{f}' \end{array} \tag{5.18}$$

This contradicts the maximality of $\mathfrak{f}$. □

Given a nominal Boolean algebra with restriction, we will call its prime n-filters n-ultrafilters. We can show

Theorem 5.6.3. In a nominal Boolean algebra with restriction, every n-filter $\mathfrak{f}$ can be extended to n-ultrafilter $\bar{\mathfrak{f}}$ such that $\mathsf{supp}(\bar{\mathfrak{f}}) \subseteq \mathsf{supp}(\mathfrak{f})$.

Proof. Given a prime n-filter $\mathfrak{f}$, apply Theorem 5.4.9 for $\mathfrak{f}$ and $\mathfrak{i} = \{\bot\}$. □

Another consequence of Lemma 5.6.2 is:

Corollary 5.6.4. Given $\mathbb{B}$ a nominal Boolean algebra with restriction and $x \in \mathbb{B}$ we have $x^{+} = (\neg x)^{-}$ and $x^{-} = (\neg x)^{+}$.

Proposition 5.6.5. The duality between $\mathsf{nDL}_{\text{И}}$ and nBiSt restricts to a duality between $\mathsf{nBA}_{\text{И}}$ and the full subcategory of nBiSt of spaces for which the two topologies coincide.

Proof. Given $\mathbb{B}$ a nominal Boolean algebra with restriction, the corresponding nominal pairwise Stone space has the property that the two topologies coincide. This is because, by Corollary 5.6.4, the two basis $\{x^{+} \mid x \in \mathbb{B}\}$ and $\{x^{-} \mid x \in \mathbb{B}\}$ coincide.

Conversely, if $(\mathbb{X}, \tau, \tau)$ is a nominal pairwise Stone space then the nominal distributive lattice $G(\mathbb{X})$ constructed in the proof of Theorem 5.5.20 is in fact a nominal Boolean algebra with restriction. The negation of $U \in G(\mathbb{X})$ is $\mathbb{X} \setminus U$. This is well defined because the carrier set consists of clopens of τ (see Definition 5.5.2) and if $U \subseteq \mathbb{X}$ is a clopen so is $\mathbb{X} \setminus U$. □

The nominal pairwise Stone spaces for which the two topologies coincide are simply nominal topological spaces $(\mathbb{X}, \tau)$ that have the following properties:

1. are n-**compact**, that is, every n-stable finitely supported $\mathscr{U} \subseteq \tau$ that covers $\mathbb{X}$ has a finite subcover.

2. are **Hausdorff**, that is for all distinct $x, y \in \mathbb{X}$ there exist disjoint $U, V \in \tau$ such that $x \in U$ and $y \in V$.

3. are **zero-dimensional**, that is, the clopens in τ form a basis for τ.

Definition 5.6.6. We call the nominal topological spaces satisfying the above three properties **nominal Stone spaces**.

Let nSt denote the category of nominal Stone spaces and continuous equivariant maps. Using Proposition 5.6.5 we have obtained the duality theorem from [GLP11]:

Theorem 5.6.7. The categories $\mathsf{nBA}_{\text{И}}$ and nSt are dually equivalent.

The notions of n-compactness and pairwise n-compactness may seem rather ad-hoc, but, as we will see next, they arise naturally from the proofs. To close the circle, we prove that the duality in Theorem 5.6.7 is obtained using the same category theoretical machinery described in Section 5.1. We have an adjunction of descent type $\mathscr{P} \dashv \mathscr{S} : \mathsf{nBA}^{op}_{\text{И}} \to \mathsf{Nom}$. This yields a monad (β, μ, η) on Nom, and Theorem 5.6.13 below shows that the Eilenberg-Moore algebras for this monad are precisely the n-compact Hausdorff topological spaces. Theorem 5.6.7 actually says that the essential image of the comparison functor K is the category of nominal Stone spaces.

$$\begin{tikzcd}
\mathsf{nBA}^{op}_{\text{И}} \arrow[rr, bend left, "K"'] \arrow[dr, bend left, "\mathscr{S}"] & & \mathsf{Nom}^{\beta} \arrow[dl, bend left, "U^{\beta}"] \\
& \mathsf{Nom} \arrow[ul, bend left, "\mathscr{P}"] \arrow[ur, bend left, "F^{\beta}"] \arrow[loop right, "\beta"] &
\end{tikzcd} \tag{5.19}$$

By Proposition 5.3.14 we have that $\mathscr{P}\mathbb{X}$ has a nominal Boolean algebra with restriction structure. Moreover if $f : \mathbb{X} \to \mathbb{Y}$ is an equivariant function, $\mathscr{P}f : \mathscr{P}Y \to \mathscr{P}X$ is a morphism in $\mathsf{nBA}_{\text{И}}$. Hence we have a functor $\mathscr{P} : \mathsf{Nom} \to \mathsf{nBA}^{op}_{\text{И}}$.

Proposition 5.6.8. The functor $\mathscr{P} : \mathsf{Nom} \to \mathsf{nBA}^{op}_{\text{И}}$ has a right adjoint $\mathscr{S} : \mathsf{nBA}^{op}_{\text{И}} \to \mathsf{Nom}$.

Proof. On objects, $\mathscr{S}(\mathbb{B})$ is the nominal set of maximal n-filters of $\mathbb{B}$. Given $h : \mathbb{B} \to \mathbb{B}'$ a morphism in $\mathsf{nBA}_{\text{И}}$, $\mathscr{S}h : \mathscr{S}B' \to \mathscr{S}B$ is defined by $\mathscr{S}h(\mathfrak{f}') = h^{-1}(\mathfrak{f}')$.

For a nominal set $\mathbb{X}$, the unit $\eta_{\mathbb{X}} : \mathbb{X} \to \mathscr{S}\mathscr{P}\mathbb{X}$ is given by

$$x \mapsto \{Y \in \mathscr{P}\mathbb{X} \mid x \in Y\} \tag{5.20}$$

It is easy to check that $\eta(x)$ is an n-ultrafilter in $\mathscr{P}\mathbb{X}$.

For a nominal Boolean algebra $\mathbb{B}$, the counit $\varepsilon_{\mathbb{B}} : \mathscr{P}\mathscr{S}\mathbb{B} \to \mathbb{B}$ in $\mathsf{nBA}^{op}_{\text{И}}$ is given by the nominal Boolean algebra morphism ε^{op} that maps

$$b \mapsto \{\mathfrak{f} \in \mathscr{S}\mathbb{B} \mid b \in \mathfrak{f}\} \tag{5.21}$$

It is easy to check that for all $b \in \mathbb{B}$ we have that $\varepsilon^{op}(b)$ is supported by $\mathsf{supp}(b)$ and is a morphism in $\mathsf{nBA}_{\text{И}}$ by Lemma 5.5.5. Moreover, η and ε satisfy the usual triangular identities.

□

The adjunction $\mathscr{P} \dashv \mathscr{S}$ of Proposition 5.6.8 yields a monad (β, μ, η) on Nom where

1. $\beta\mathbb{X}$ is nominal set of n-ultrafilters in $\mathscr{P}\mathbb{X}$.

2. For $f:\mathbb{X}\to\mathbb{Y}$ equivariant $\beta f:\beta\mathbb{X}\to\beta\mathbb{Y}$ is defined by
$$\beta f(\mathfrak{f})=\{U\in\mathscr{P}\mathbb{Y}\mid f^{-1}(U)\in\mathfrak{f}\}. \tag{5.22}$$
3. The unit $\eta_{\mathbb{X}}:\mathbb{X}\to\mathscr{S}\mathscr{P}\mathbb{X}$ is given by
$$x\mapsto\{Y\in\mathscr{P}\mathbb{X}\mid x\in Y\}. \tag{5.23}$$
4. The multiplication $\mu_{\mathbb{X}}:\beta\beta\mathbb{X}\to\beta\mathbb{X}$, computed as $\mathscr{S}\varepsilon_{\mathscr{P}\mathbb{X}}$, is given by
$$\Phi\mapsto\{Y\in\mathscr{P}\mathbb{Y}\mid\{\mathscr{U}\in\mathscr{S}\mathscr{P}\mathbb{X}\mid Y\in\mathscr{U}\}\in\Phi\}. \tag{5.24}$$

Next we will generalise Manes' theorem to the nominal setting. In the classical case it is well known that a topological space (X,τ) is compact if and only if every filter on X has at least one limit. We say that a filter F has $x\in X$ as a limit when the filter $N(x)$ of neighbourhoods of x is included in F. A similar result can be proved for n-compact nominal topological spaces.

Let $(\mathbb{X},\tau)$ be a nominal topological space. First notice that for all $x\in\mathbb{X}$ we have that the neighbourhoods of x
$$\mathscr{N}(x)=\{U\in\mathscr{P}\mathbb{X}\mid\exists O\in\tau.x\in O,O\subseteq U\}$$
form a n-filter. Condition 3 of Definition 5.4.6 is satisfied using point 6 of Proposition 5.3.14.

Definition 5.6.9. Let $(\mathbb{X},\tau)$ be a nominal topological space. A maximal n-filter $\mathfrak{f}$ in $\mathscr{P}\mathbb{X}$ converges to $x\in\mathbb{X}$, or has x as a limit, when $\mathscr{N}(x)\subseteq\mathfrak{f}$.

Lemma 5.6.10. A nominal topological space $(\mathbb{X},\tau)$ is n-compact if and only if every maximal n-filter $\mathfrak{f}$ in $\mathscr{P}\mathbb{X}$ has at least one limit.

Proof. Assume $(\mathbb{X},\tau)$ and $\mathfrak{f}$ is a maximal n-filter. The set $\{V\mid V\in\mathfrak{f},\mathbb{X}\setminus V\in\tau\}$ is n-stable (because $\mathfrak{f}$ is an n-filter and by Lemma 5.5.3) and has the finite intersection property, thus has a non-empty intersection. Let
$$x\in\bigcap\{V\mid V\in\mathfrak{f},\mathbb{X}\setminus V\in\tau\}.$$
If U is open such that $x\in U$ then $U\in\mathfrak{f}$. Hence $\mathscr{N}(x)\subseteq\mathfrak{f}$.

Conversely, if every n-ultrafilter has a limit, let $(U_i)_{i\in I}$ be an n-stable finitely-supported cover of $\mathbb{X}$ with no finite subcover. Then the set $\{\mathbb{X}\setminus U_i\mid i\in I\}$ is n-stable and has the finite intersection property. By Lemma 5.4.11 it is included in a maximal n-filter $\mathfrak{f}$. Let x be a limit of $\mathfrak{f}$. It follows that for all i we have $x\notin\bigcap U_i$ because $X\setminus U_i\in\mathfrak{f}$. But this contradicts the fact that $(U_i)_{i\in I}$ covers $\mathbb{X}$. □

Lemma 5.6.11. A nominal topological space is Hausdorff if and only if every maximal n-filter in $\mathscr{P}\mathbb{X}$ has at most a limit in $\mathbb{X}$.

Proof. Assume $(\mathbb{X},\tau)$ is Hausdorff and $\mathfrak{f}$ is a maximal n-filter in $\mathscr{P}\mathbb{X}$ such that $\mathscr{N}(x)\subseteq\mathfrak{f}$ and $\mathscr{N}(y)\subseteq\mathfrak{f}$. If x and y were distinct we would find $U,V\in\tau$ such that $U\cap V=\emptyset$ and $x\in U$, $y\in V$. But then $\emptyset\in\mathfrak{f}$, contradiction. Conversely, consider $x,y\in\mathbb{X}$ such that $x\neq y$. Assume by contradiction that for all $U\in\tau$ with $x\in U$ and for all $V\in\tau$ with $y\in V$ we have $U\cap V\neq\emptyset$. Then the finitely-supported set

$$\{U\cap V\mid U\in\mathscr{N}(x),V\in\mathscr{N}(y)\}$$

has the finite intersection property and is n-stable, thus, by Lemma 5.4.11, it is contained in a maximal n-filter $\mathfrak{f}$. Notice that $\mathscr{N}(x)\subseteq\mathfrak{f}$ and $\mathscr{N}(y)\subseteq\mathfrak{f}$, contradicting the fact that $\mathfrak{f}$ has at most one limit. Hence x and y can be separated by disjoint opens, so $(\mathbb{X},\tau)$ is Hausdorff. □

Let nCHaus denote the category of n-compact nominal topological spaces and continuous equivariant maps. We define a functor $G:\mathsf{nCHaus}\to\mathsf{Nom}^{\beta}$. Consider a nominal topological space $(\mathbb{X},\tau)$ that is n-compact and Hausdorff. By Lemma 5.6.11 and Lemma 5.6.10 each maximal n-filter $\mathfrak{f}$ has a unique limit $\lim(\mathfrak{f})$. From the proof of Lemma 5.6.10 we obtain that

$$\{\lim(\mathfrak{f})\}=\bigcap\{V\mid V\in\mathfrak{f},\mathbb{X}\setminus V\in\tau\}. \tag{5.25}$$

This shows that $\lim:\beta\mathbb{X}\to\mathbb{X}$ is equivariant.

We prove next that $\lim:\beta\mathbb{X}\to\mathbb{X}$ is an Eilenberg-Moore algebra for β. By (5.25) it follows easily that for all $x\in\mathbb{X}$ we have $\lim(\eta(x))=x$. To check that the diagram

$$\begin{tikzcd}
\beta\beta\mathbb{X} \arrow[r, "\mu_{\mathbb{X}}"] \arrow[d, "\beta\lim"'] & \beta\mathbb{X} \arrow[d, "\lim"] \\
\beta\mathbb{X} \arrow[r, "\lim"] & \mathbb{X}
\end{tikzcd} \tag{5.26}$$

commutes, consider $\Phi\in\beta\beta\mathbb{X}$. Using (5.22) and (5.24) It is enough to check that the n-filters

$$\mu_{\mathbb{X}}(\Phi)=\{U\in\mathscr{P}\mathbb{X}\mid\{\mathscr{U}\in\beta\mathbb{X}\mid U\in\mathscr{U}\}\in\Phi\}$$

and

$$\beta\lim(\Phi)=\{U\in\mathscr{P}\mathbb{X}\mid\lim^{-1}(U)\in\Phi\}$$

converge to the same limit. If x is the limit of $\beta\lim(\Phi)$, then for all open $V\in\mathscr{N}(x)$ we have $\lim^{-1}(V)\in\Phi$. But $\lim^{-1}(V)\subseteq\{\mathscr{U}\in\beta\mathbb{X}\mid V\in\mathscr{U}\}$, hence $V\in\mu_{\mathbb{X}}(\Phi)$. It follows that x is also a limit of $\mu_{\mathbb{X}}(\Phi)$.

We put $G(\mathbb{X},\tau) = \lim : \beta\mathbb{X} \to \mathbb{X}$. Let $(\mathbb{X}_1,\tau_1)$ and $(\mathbb{X}_2,\tau_2)$ be two nominal n-compact Hausdorff topological spaces. Given $f : \mathbb{X}_1 \to \mathbb{X}_2$ a continuous map between n-compact Hausdorff spaces, we have to check that $f \circ \lim_1 = \lim_2 \circ \beta f$. This is exactly as the proof in the classical case, see [Joh82, Lemma 2.3].

Lemma 5.6.12. The functor $G : \mathsf{nCHaus} \to \mathsf{Nom}^{\beta}$ defined above is full.

Proof. Let $(\mathbb{X}_1,\tau_1)$ and $(\mathbb{X}_2,\tau_2)$ be two nominal n-compact Hausdorff topological spaces and $(\mathbb{X}_1,\lim_1)$, $(\mathbb{X}_2,\lim_2)$ the corresponding Eilenberg-Moore algerbras. If $f : (\mathbb{X}_1,\lim_1) \to (\mathbb{X}_2,\lim_2)$ is a morphism in Nom^{β}, we show that $f : \mathbb{X}_1 \to \mathbb{X}_2$ is continuous. Let V be an open in $\mathbb{X}_2$ and assume to the contrary that $f^{-1}(V)$ is not equal to the finitely-supported union

$$\mathit{int}(f^{-1}(V)) = \bigcup\{U \in \tau_1 \mid U \subseteq f^{-1}(V)\}.$$

Let $x \in f^{-1}(V) \setminus \mathit{int}(f^{-1}(V))$. Then the set $\{U \setminus f^{-1}(V) \mid U \in \mathcal{N}(x)\}$ is n-stable and has the finite intersection property, thus, by Lemma 5.4.11, it is contained in a n-ultrafilter $\mathfrak{f} \in \beta\mathbb{X}_1$. It follows that $\mathfrak{f}$ converges to x. Since $f \circ \lim_1 = \lim_2 \circ \beta f$, we have that $\beta f(\mathfrak{f})$ converges to $f(x)$. This leads to a contradiction, because $V \in \mathcal{N}(f(x))$ but $V \notin \beta f(\mathfrak{f})$. ☐

Theorem 5.6.13. The category of Eilenberg-Moore algebras for the monad β is isomorphic to the category nCHaus of n-compact Hausdorff nominal topological spaces.

Proof. We have seen that the functor G is well-defined, faithful and full. It remains to show that each Eilenberg-Moore algebra is of the form $G(\mathbb{X},\tau)$ for a unique n-compact Hausdorff space $(\mathbb{X},\tau)$. Given $\psi : \beta\mathbb{X} \to \mathbb{X}$ in Nom^{β} we define a map cl on $\mathscr{P}\mathbb{X}$ by

$$U \mapsto \{\psi(\mathfrak{f}) \mid U \in \mathfrak{f}\}.$$

The map $\mathsf{cl} : \mathscr{P}\mathbb{X} \to \mathscr{P}\mathbb{X}$ is equivariant and moreover satisfies the next properties; the first two can be proved exactly as in the classical case, see [Joh82, Theorem 2.4]:

1. $U \subseteq \mathsf{cl}(U)$, since for all $x \in U$ we have $\psi(\eta(x)) = x$ and $U \in \eta(x)$.
2. $\mathsf{cl}(U \cup V) = \mathsf{cl}(U) \cup \mathsf{cl}(V)$, since $U \cup V \in \mathfrak{f}$ iff $U \in \mathfrak{f}$ or $V \in \mathfrak{f}$.
3. $\mathsf{cl}(\mathsf{cl}(U)) = \mathsf{cl}(U)$ follows using the following claim:

Claim 5.6.14. If $\mathfrak{f}$ is a n-ultrafilter and $\mathsf{cl}(U) \in \mathfrak{f}$ there exists a n-ultrafilter $\mathfrak{g}$ with $U \in \mathfrak{g}$ and $\psi(\mathfrak{f}) = \psi(\mathfrak{g})$.

For $V \in \mathfrak{f}$ the set $\psi^{-1}(V) \cap \{\mathscr{U} \in \beta\mathbb{X} \mid U \in \mathscr{U}\}$ is non-empty because $V \cap \mathsf{cl}(U)$ is non-empty. Then the set $\{\psi^{-1}(V) \cap \{\mathscr{U} \in \beta\mathbb{X} \mid U \in \mathscr{U}\} \mid V \in \mathfrak{f}\}$ is finitely-supported subset of $\mathscr{P}\beta\mathbb{X}$, n-stable and has the finite intersection property. Thus, by Lemma 5.4.11, it can be extended to a maximal n-filter $\Phi \in \beta\beta\mathbb{X}$. Put $\mathfrak{g} = \mu_{\mathbb{X}}(\Phi)$. Since $\beta\psi(\Phi) = \mathfrak{f}$ we have that $\psi(\mathfrak{f}) = \psi(\mathfrak{g})$.

We consider a nominal topology on $\mathbb{X}$ given by

$$\tau = \{U \in \mathscr{P}\mathbb{X} \mid \mathbb{X} \setminus U = \mathsf{cl}(\mathbb{X} \setminus U)\}.$$

It is easy to check that τ is a nominal subset of $\mathscr{P}\mathbb{X}$ and is indeed a nominal topology. Next we show that

$$\mathfrak{f} \text{ converges to } x \iff \psi(\mathfrak{f}) = x. \tag{5.27}$$

Suppose $\lim \mathfrak{f} = x$. Then for every $V \in \mathfrak{f}$, we have $x \in \mathsf{cl}(V)$. By Claim 5.6.14 there exists an ultrafilter $\mathfrak{g}_V$ that converges to x and contains V. Hence $\psi^{-1}(x) \cap \{\mathscr{U} \in \beta\mathbb{X} \mid V \in \mathscr{U}\} \neq \emptyset$. Then the set $\{\psi^{-1}(x) \cap \{\mathscr{U} \in \beta\mathbb{X} \mid V \in \mathscr{U}\} \mid V \in \mathfrak{f}\}$ is finitely-supported subset of $\mathscr{P}\beta\mathbb{X}$, n-stable and has the finite intersection property. Hence it is contained in a maximal n-filter in $\Psi \in \beta\beta\mathbb{X}$.

For the right-to-left implication, notice that for all $V \in \mathfrak{f}$ we have $x \in \mathsf{cl}(V)$. Using the argument in Lemma 5.6.10 it follows that $x = \lim \mathfrak{f}$.

On one hand, (5.27) implies that each n-ultrafilter has a unique limit in $\mathbb{X}$, hence by Lemmas 5.6.10 and 5.6.11 it follows that $(\mathbb{X}, \tau)$ is n-compact and Hausdorff. On the other hand, using (5.27) we immediately get that $G(\mathbb{X}, \tau) = (\mathbb{X}, \psi)$. □

5.7 Conclusions and further work

In this chapter we proved that the ultrafilter theorem cannot be internalised in nominal sets. We can still obtain Stone type dualities in nominal setting by exploiting the rich structure of the power object of a nominal set. In particular, the power object functor $\mathscr{P}$ can be restricted to a functor from Nom to the opposite category of nominal Boolean algebras with a name restriction operation.

Then, all the apparent ad-hoc constructions from [GLP11] can be easily explained. The n-ultrafilters of a such a nominal Boolean algebra with restriction are needed to find the right adjoint for $\mathscr{P}$. The key for proving the duality lies in the fact that this adjunction is of descent type. The notion of n-compactness can be explained by understanding the Eilenberg-Moore algebras for the induced monad on Nom.

We have to mention a different approach to the dualities presented in this chapter. Staton observed that the category of nominal restriction sets is equivalent to $\mathsf{Set}^{p\mathbb{I}}$, where $p\mathbb{I}$ is the category of finite sets and partial injective maps. Since $p\mathbb{I}$ is self dual, an equivalent description of our nominal Stone spaces is $\mathsf{Stone}^{p\mathbb{I}}$. However, notice that in our setting the topological spaces are internal in nominal sets, rather than in nominal restriction sets.

Future work includes restricting the dualities for nominal distributive lattices with restriction to categories that might be of interest for domain theory. Another potential application would be a duality-based approach to developing coalgebraic logic with names.

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