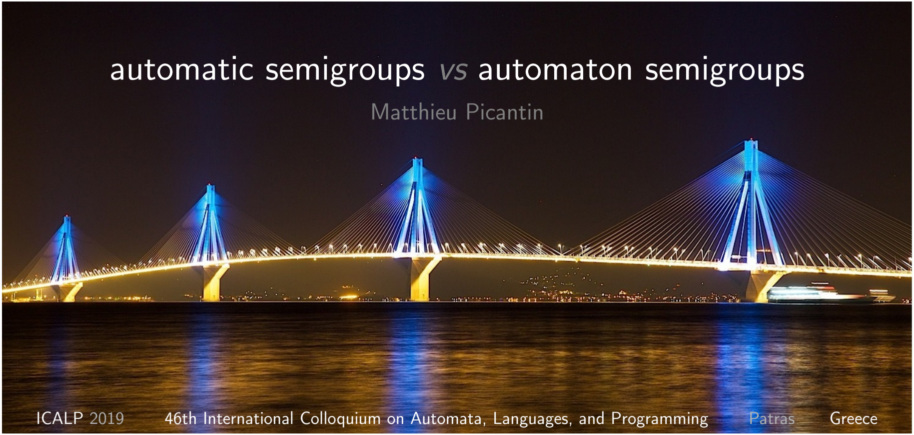
automatic semigroups *vs* automaton semigroups

Matthieu Picantin



ICALP 2019 46th International Colloquium on Automata, Languages, and Programming Patras Greece

INSTITUT DE RECHERCHE EN INFORMATIQUE FONDAMENTALE

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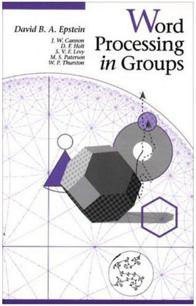
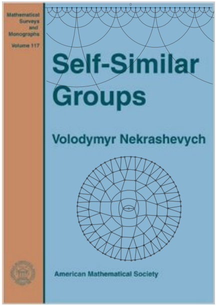
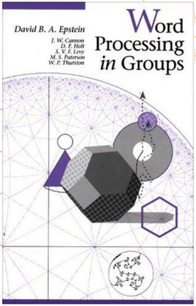
PROJET FINANCÉ PAR L'ANR - PROJECT FUNDED BY THE ANR

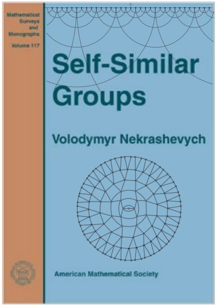
Automata theory & (semi)group theory

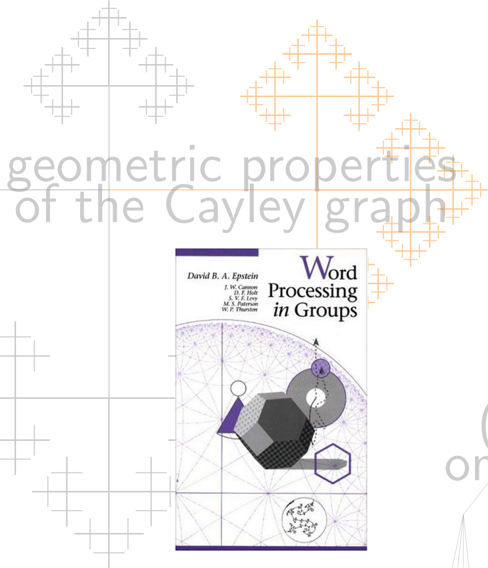


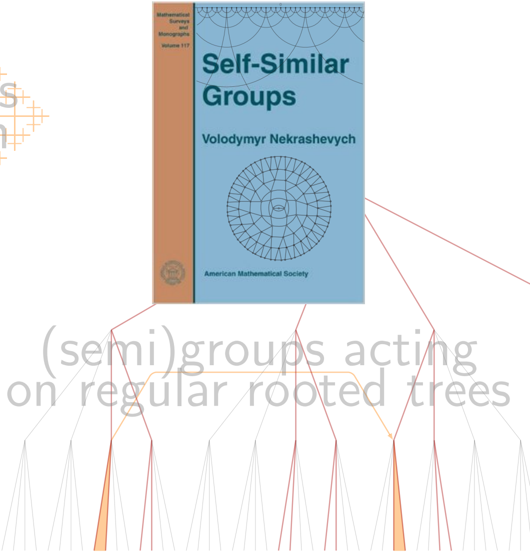


geometric properties of the Cayley graph



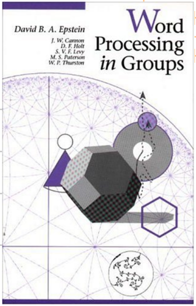


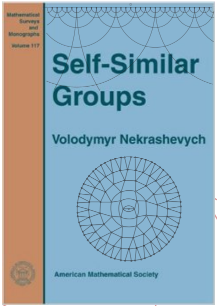

geometric properties
of the Cayley graph
David B. A. Epstein
J. W. Cannon
D. F. Holt
S. V. F. Levy
M. S. Paterson
W. P. Thurston
Word Processing in Groups

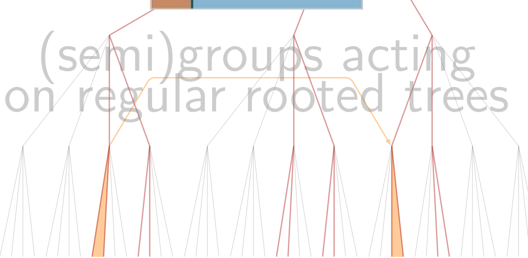

Mathematical Surveys and Monographs
Volume 117
Self-Similar Groups
Volodymyr Nekrashevych
American Mathematical Society
(semi)groups acting
on regular rooted trees

- ▷ recognize a language of normal forms
- ▷ execute the (semi)group operations

geometric properties of the Cayley graph

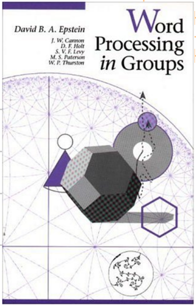


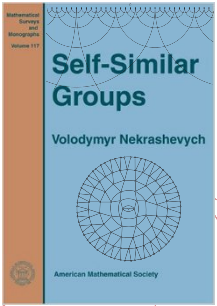




geometric properties of the Cayley graph

- ▷ recognize a language of normal forms
- ▷ execute the (semi)group operations



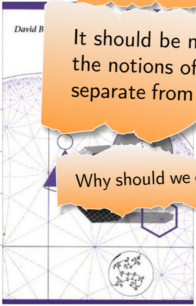


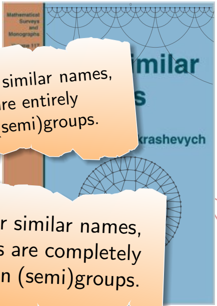
(semi)groups acting on regular rooted trees

- ▷ define sequential transformations
- ▷ represent the elements themselves

▷ recognize a languag

▷ execute the (semi)g





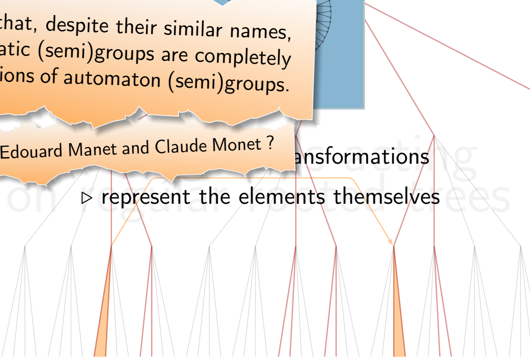
We should emphasize that, despite their similar names, the notions of automaton (semi)groups are entirely separate from the notions of automatic (semi)groups.

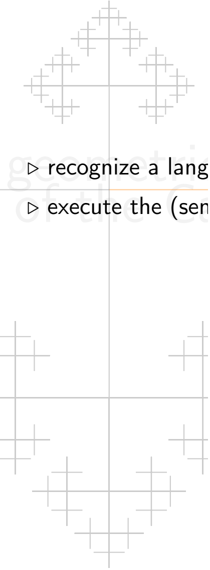
It should be noticed that, despite their similar names, the notions of automatic (semi)groups are completely separate from the notions of automaton (semi)groups.

Why should we compare Edouard Manet and Claude Monet ?

ansformations

▷ represent the elements themselves





▷ recognize a lang

▷ execute the (sen

Groups defined by automata

Laurent Bartholdi *Pedro V. Silva*

Automatha Handbook
Jean-Éric Pin Editor

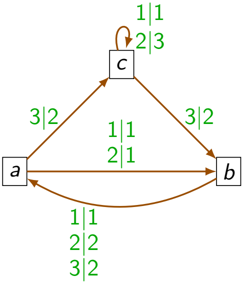
Contents

formations

themselves

$$\mathcal{M} = (Q, X, \tau, \sigma)$$

stateset alphabet transition output



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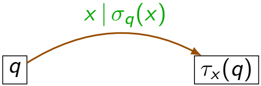

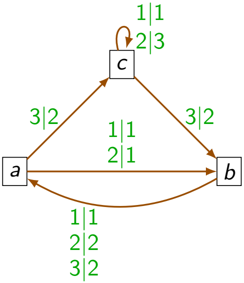

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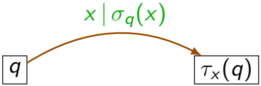

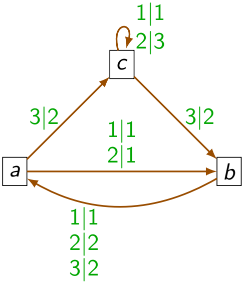

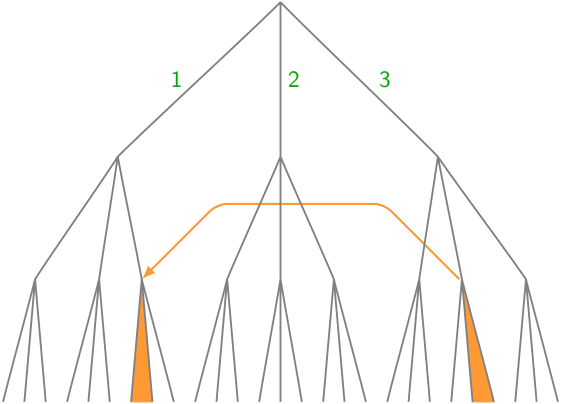

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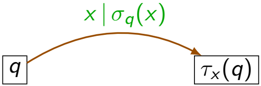

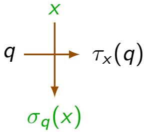

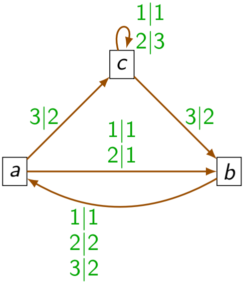

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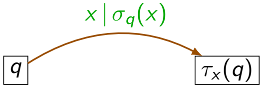

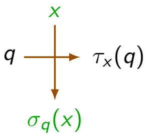

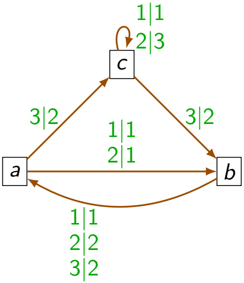

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stateset alphabet transition output

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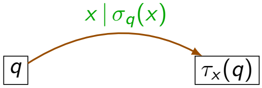
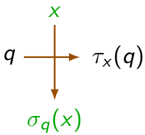
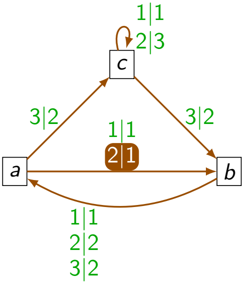


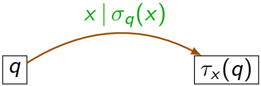


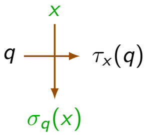


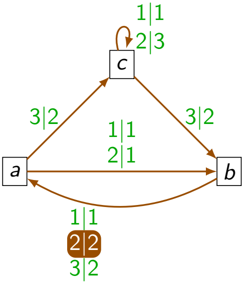
Automaton (semi)groups

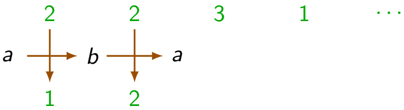
stateset, alphabet, transition, output

$$\mathcal{M} = (Q, X, \tau, \sigma)$$









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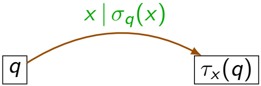

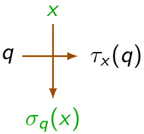

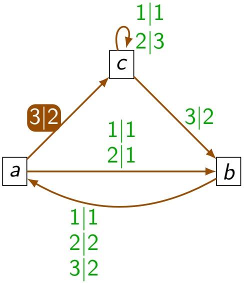

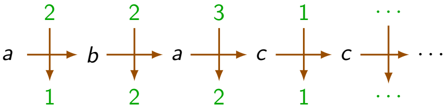

stateset alphabet transition output

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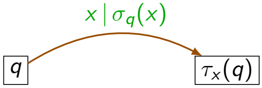

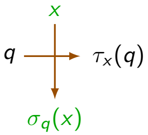

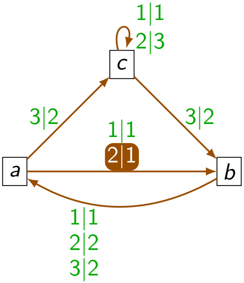

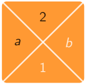

Automaton (semi)groups

stateset, alphabet, transition, output

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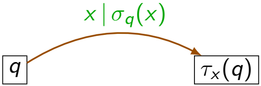
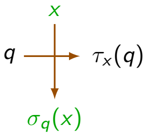
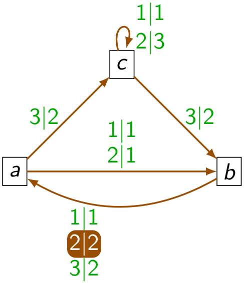
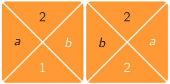


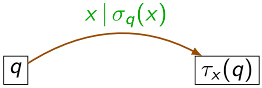


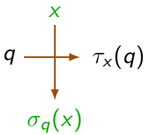


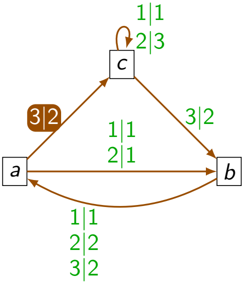


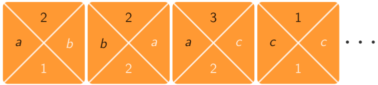
stateset alphabet transition output

$$\mathcal{M} = (Q, X, \tau, \sigma)$$









stateset alphabet transition output

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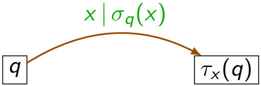
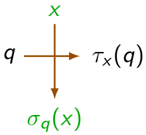
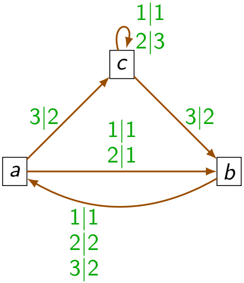




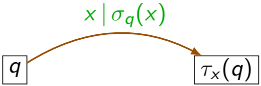


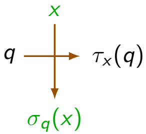
$$\mathcal{M} = (Q, X, \tau, \sigma)$$

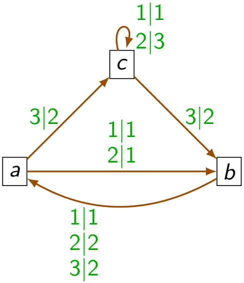
(stateset, alphabet, transition, output)

$$\langle\, \mathcal{M} \,\rangle_+ = \langle\, \sigma_q, q \in Q \,\rangle_+$$

($\sigma_q : X^* \to X^*$)







stateset alphabet transition output

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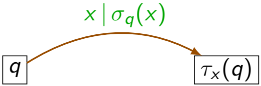

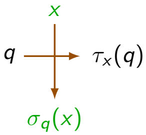

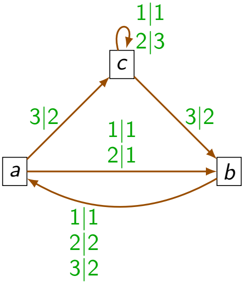

Automaton (semi)groups

stateset, alphabet, transition, output

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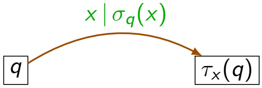
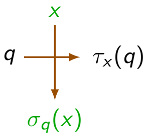
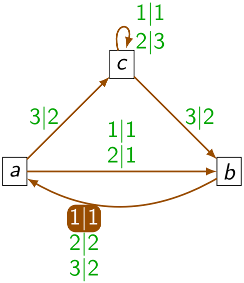
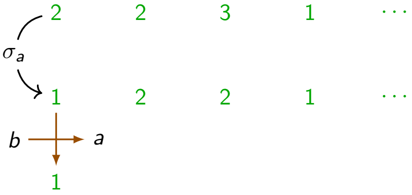


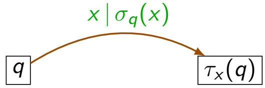


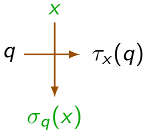


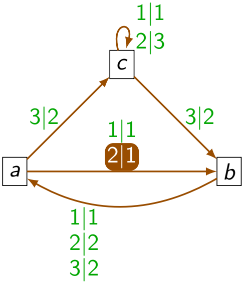


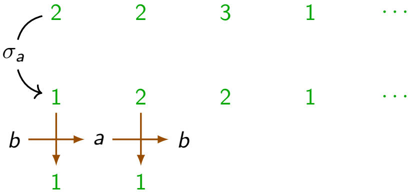
$$\mathcal{M} = (\underset{\text{stateset}}{Q}, \underset{\text{alphabet}}{X}, \underset{\text{transition}}{\tau}, \underset{\text{output}}{\sigma})$$

$$\langle\ \mathcal{M}\ \rangle_+ = \langle\ \underset{X^* \to X^*}{\sigma_q}, q \in Q\ \rangle_+$$







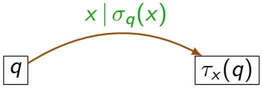


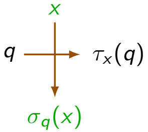
Automaton (semi)groups

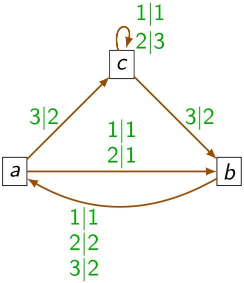
stateset, alphabet, transition, output

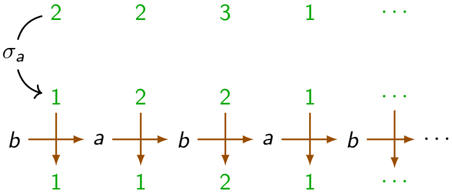
$$\mathcal{M} = (Q, X, \tau, \sigma)$$

$$\langle\ \mathcal{M}\ \rangle_+ = \langle\ \underbrace{\sigma_q}_{X^* \to X^*}, q \in Q\ \rangle_+$$









Automaton (semi)groups

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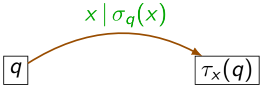
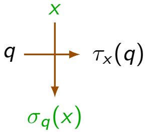
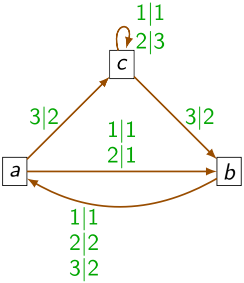
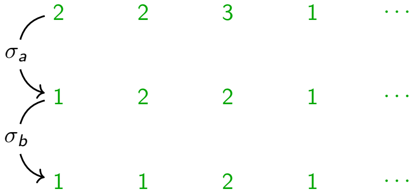








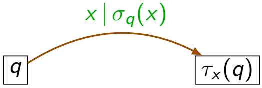
Automaton (semi)groups

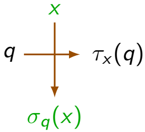
$$\mathcal{M} = (Q, X, \tau, \sigma)$$

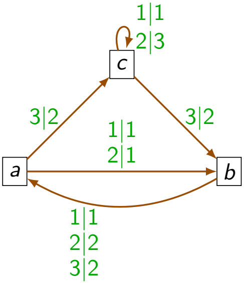
stateset, alphabet, transition, output

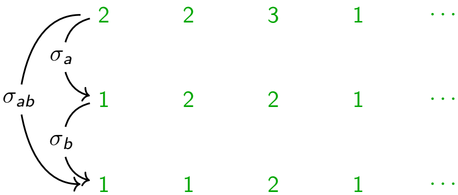
$$\langle\ \mathcal{M}\ \rangle_+ = \langle\ \sigma_q, q \in Q\ \rangle_+$$

$\sigma_q : X^* \to X^*$









stateset alphabet transition output

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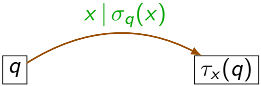
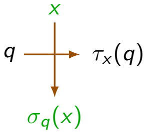
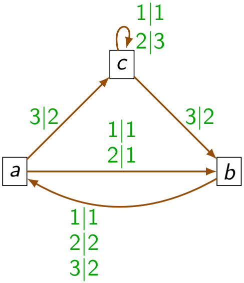
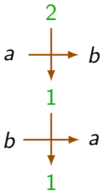
$X^* \to X^*$









stateset, alphabet, transition, output

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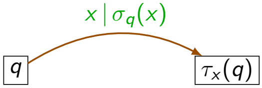
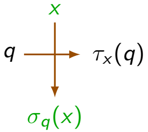
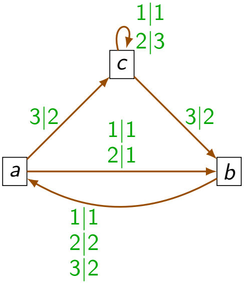
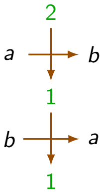
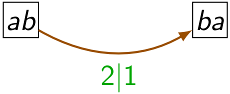








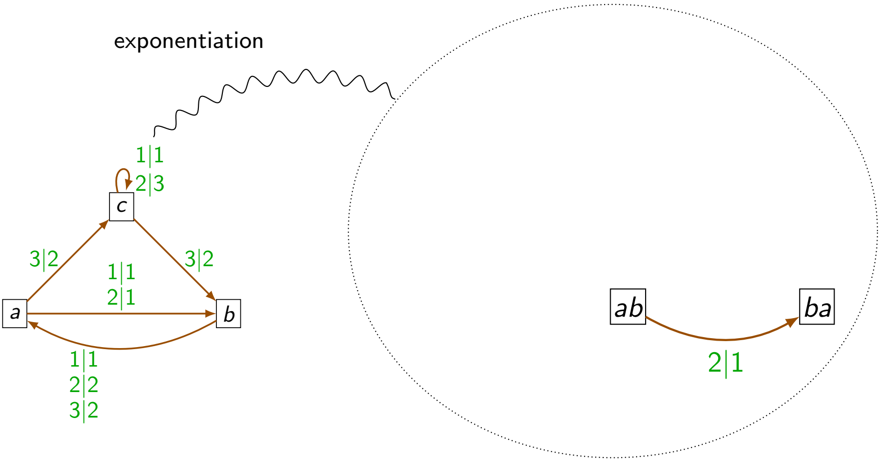


$\mathcal{M} = (Q, X, \tau, \sigma)$

stateset alphabet transition output

$\mathcal{M}^2$

exponentiation



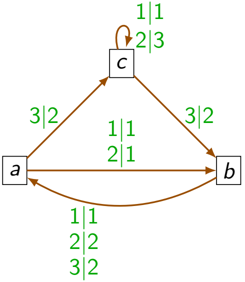
stateset alphabet transition output

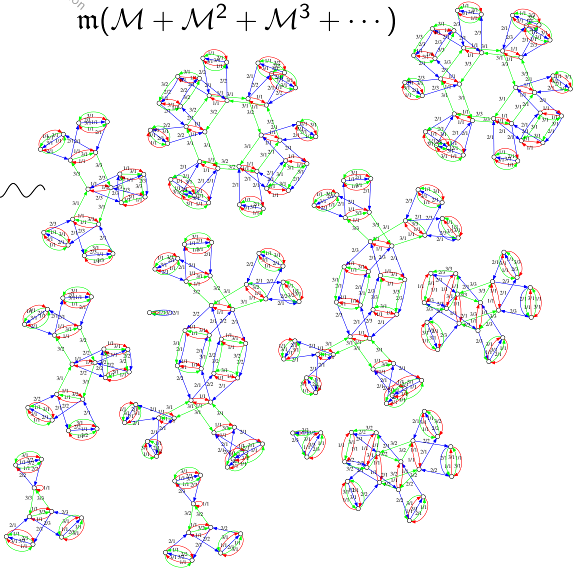
$$\mathcal{M} = (Q, X, \tau, \sigma)$$

minimisation

$$\mathfrak{m}(\mathcal{M} + \mathcal{M}^2 + \mathcal{M}^3 + \cdots)$$

completion attempt





stateset alphabet transition output

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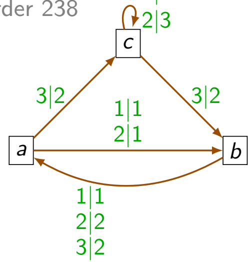
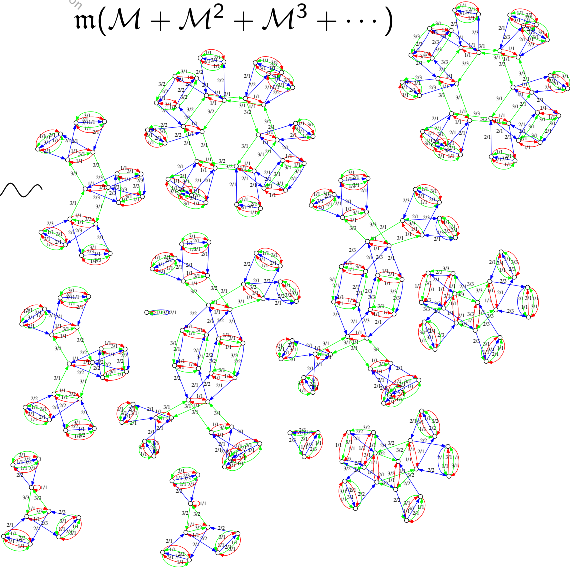
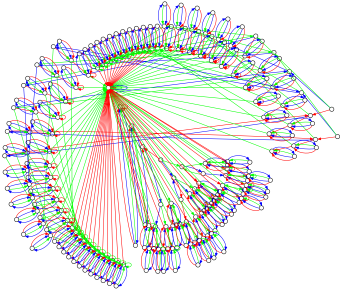
completion attempt

order 238





Automaton (semi)groups



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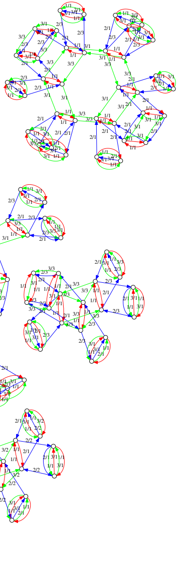
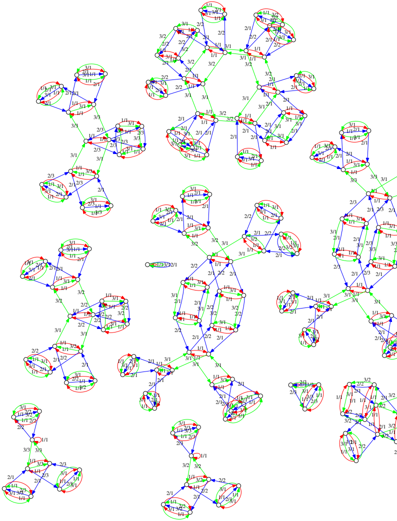
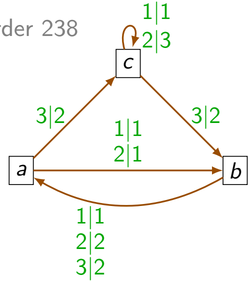


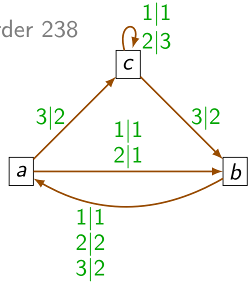


order 238

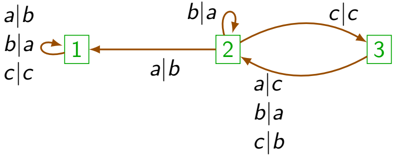


$$\mathcal{M} = (Q, X, \tau, \sigma) \quad \overset{\mathfrak{d}}{\longleftrightarrow} \quad \mathfrak{d}\mathcal{M} = (X, Q, \sigma, \tau)$$

order 238



$\overset{\mathfrak{d}}{\longleftrightarrow}$



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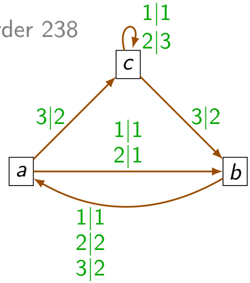
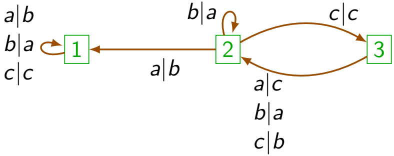
order 238



$\stackrel{\mathfrak{d}}{\longleftrightarrow}$

order ?



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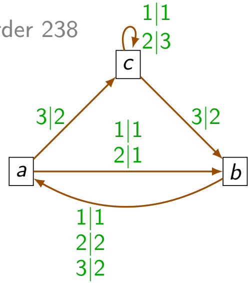
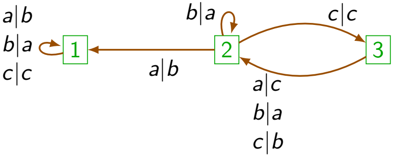
order 238



$\xleftrightarrow{\ \mathfrak{d}\ }$

order 1 494 186 269 970 473 680 896



Let S be a (semi)group with a finite generating subfamily $\mathcal{Q}$:

$$\mathsf{EV} : \mathcal{Q}^* \longrightarrow S$$

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$$\mathsf{EV} : \mathcal{Q}^* \longrightarrow S \quad (\mathsf{NF} : S \to \mathcal{Q}^*)$$

A normal form for $(S, \mathcal{Q})$ is a map NF that assigns to each element of S a distinguished representative $\mathcal{Q}$-word.

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for the Adjan-Garside-Thurston normal form, $\mathcal{Q}$ is chosen to be {positive braids in which any two strands cross at most once}

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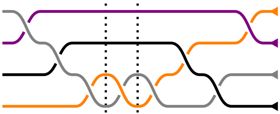

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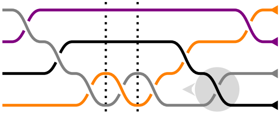

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Automatic (semi)groups

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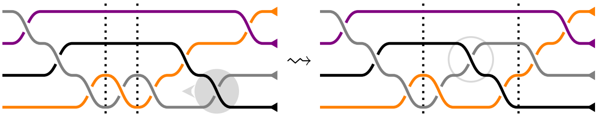

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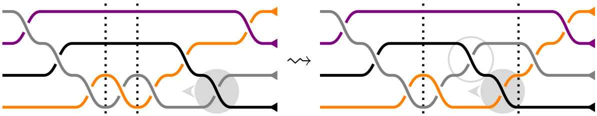

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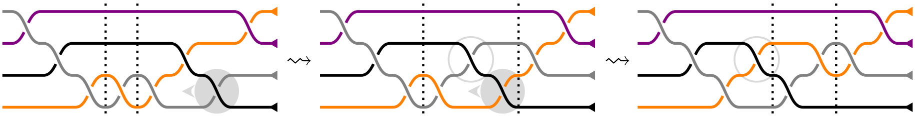

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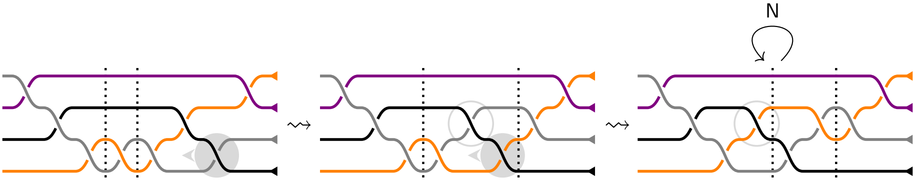

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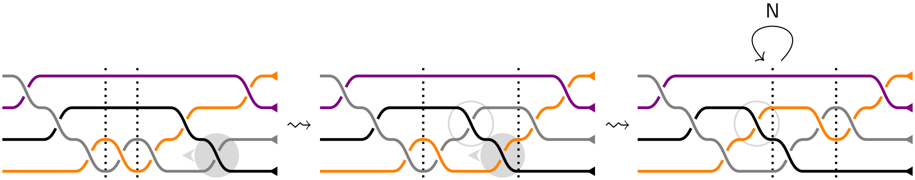

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Whenever NF(S) is regular, it provides a so-called right-automatic structure for S if the language $\bigcup_{q\in\mathcal{Q}}\{\ (\mathsf{NF}(a)\#^{|\mathsf{NF}(aq)|}, \mathsf{NF}(aq)\#^{|\mathsf{NF}(a)|}) : a \in S\ \}$ is regular.

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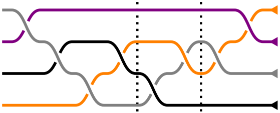

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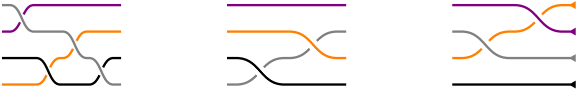

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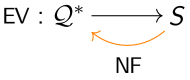

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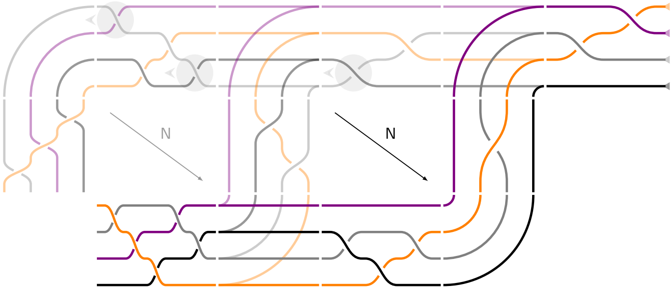

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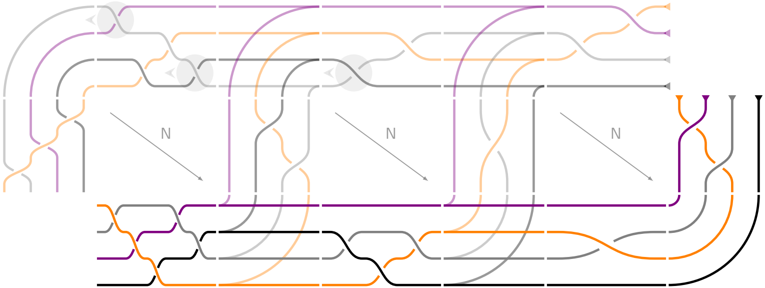

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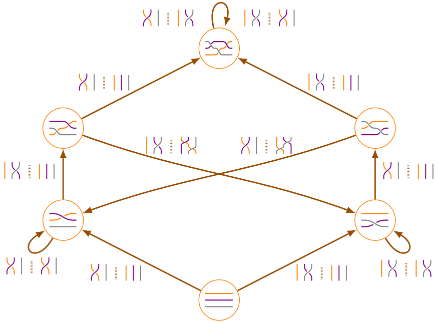

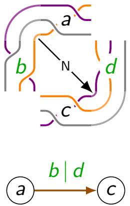

Thurston's transducer

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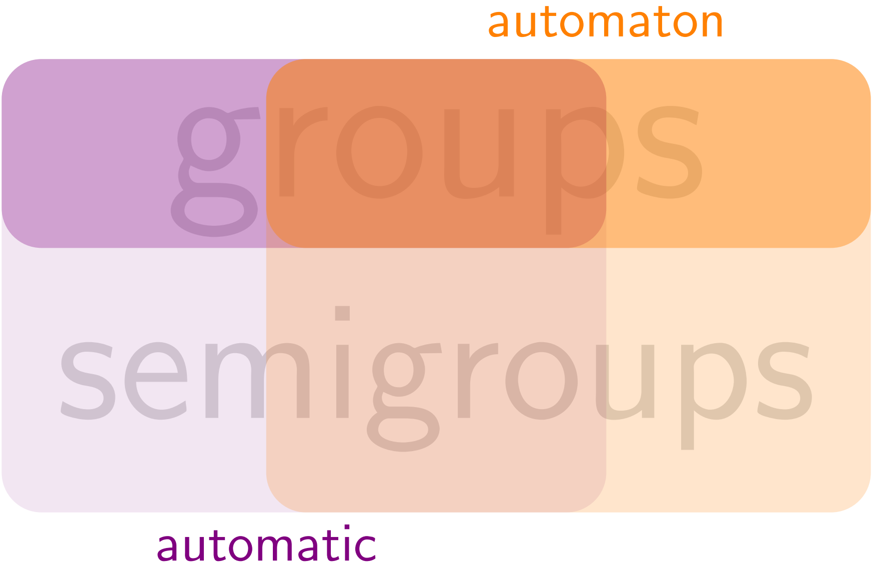
automaton
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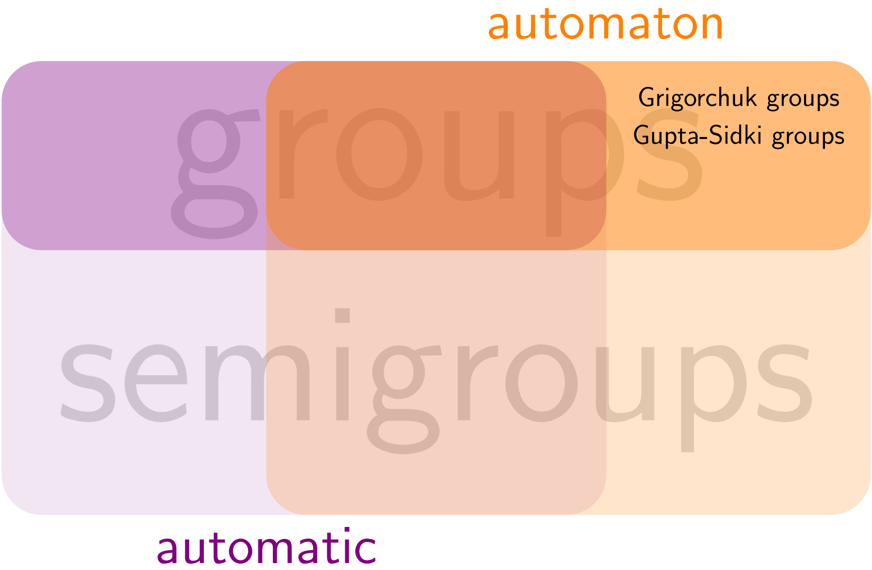
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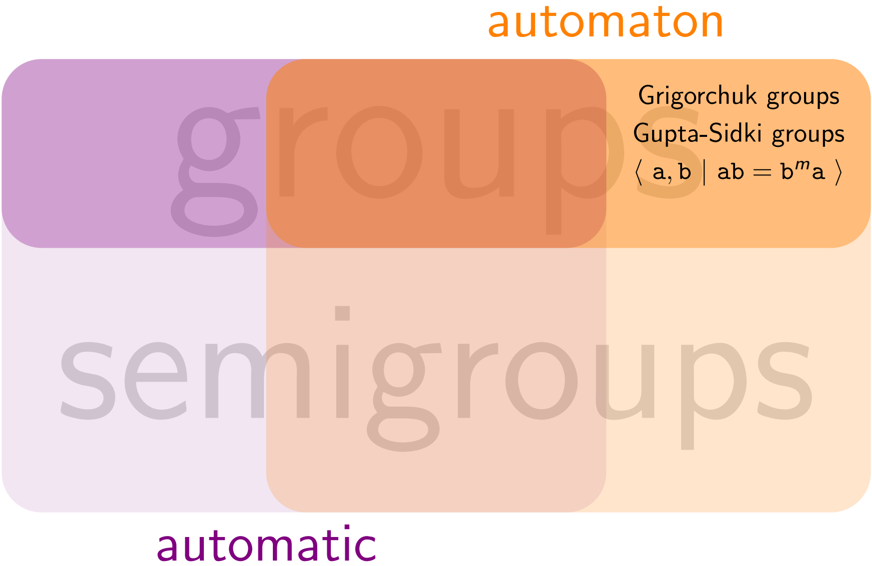
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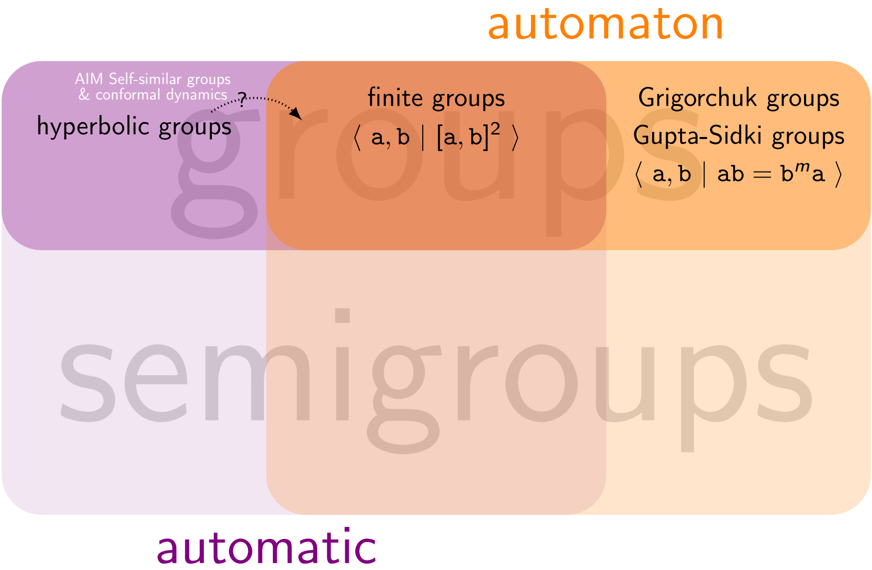
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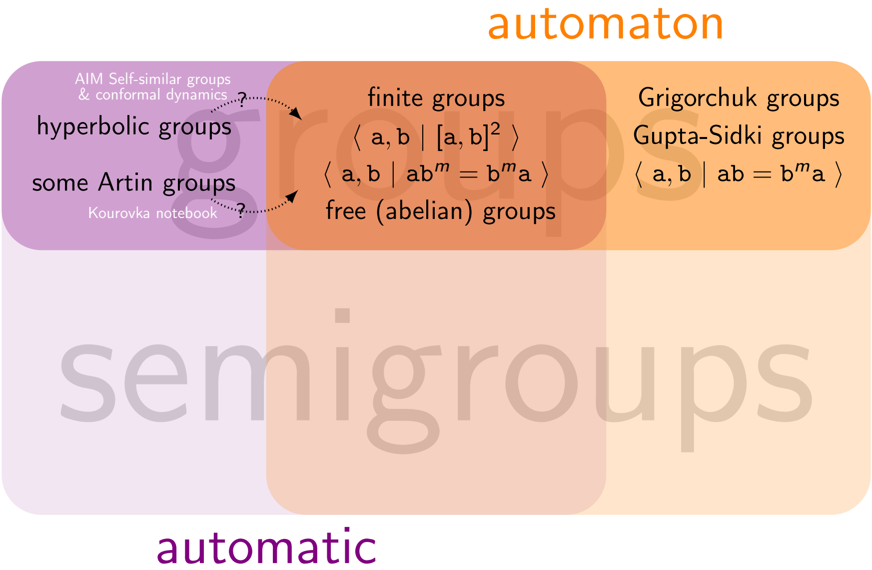
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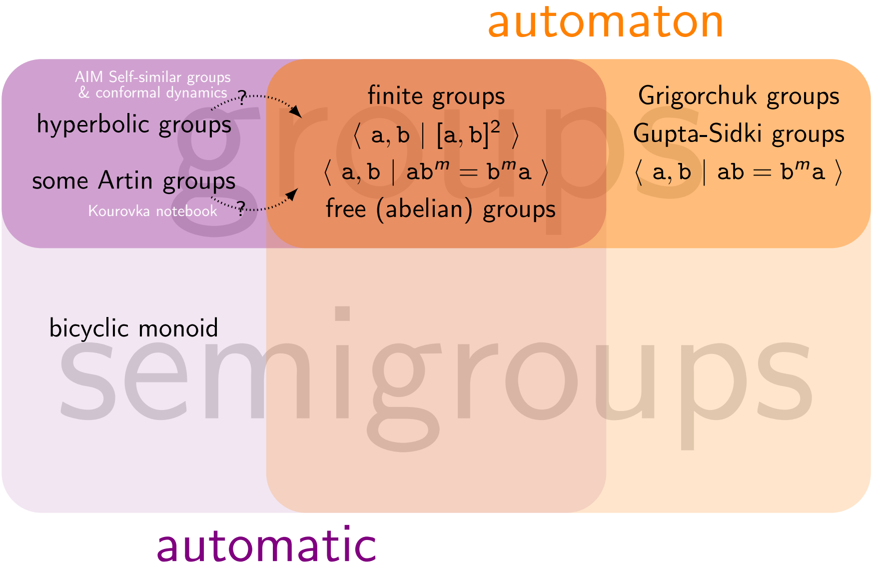
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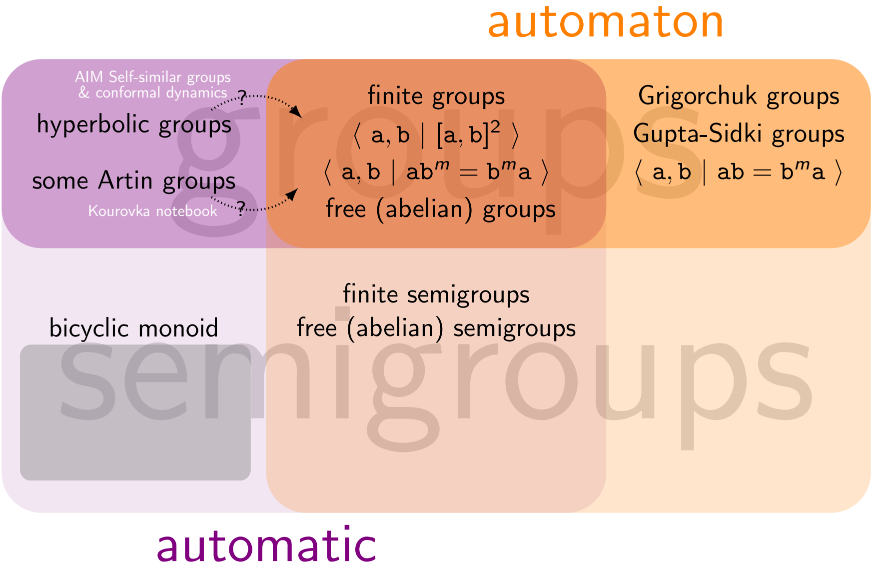
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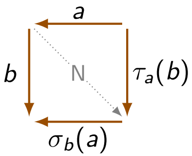

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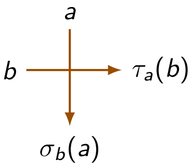

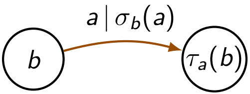

Automaticon semigroups

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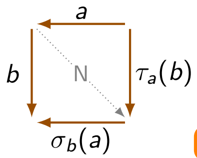

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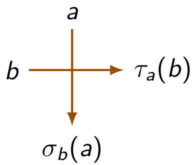

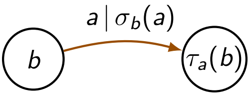

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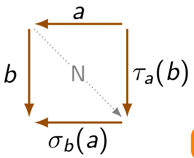

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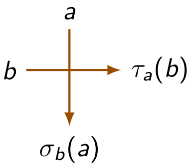

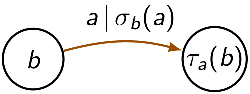

Lemma P 2019 — top-approximation

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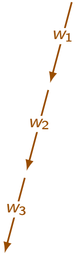

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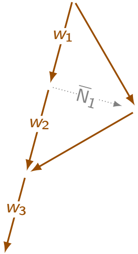

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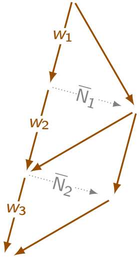

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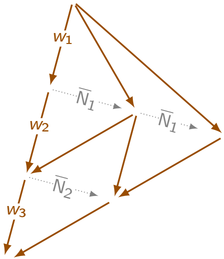

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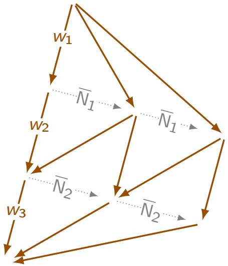

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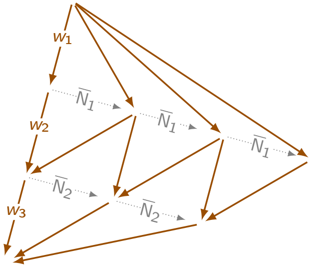

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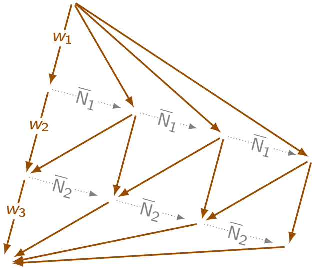

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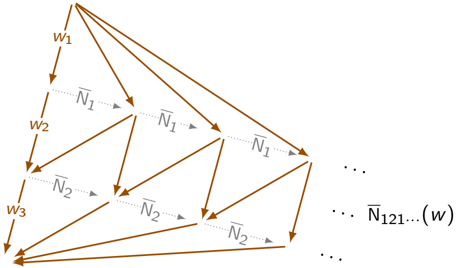

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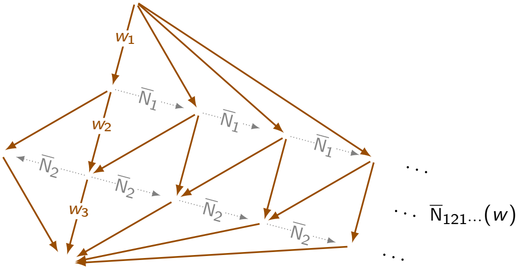

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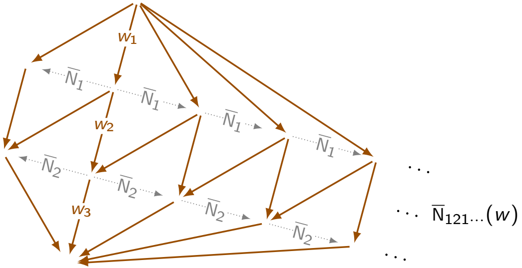

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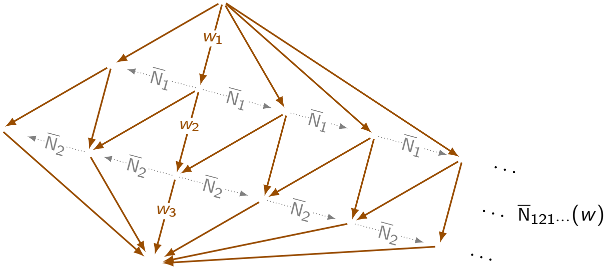

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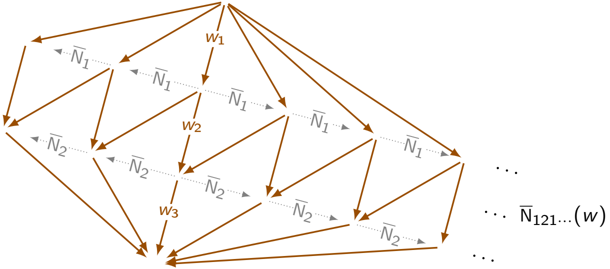

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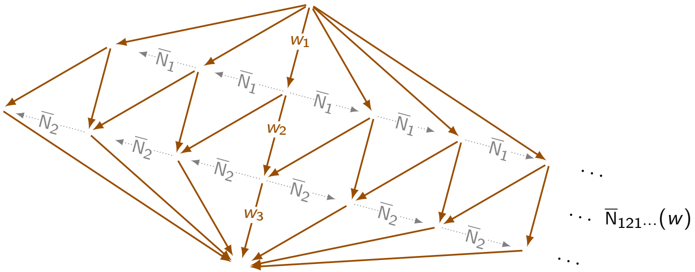

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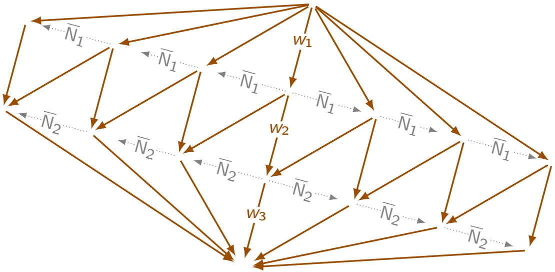

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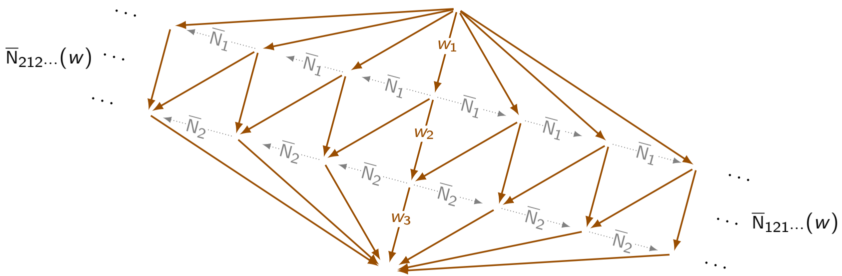

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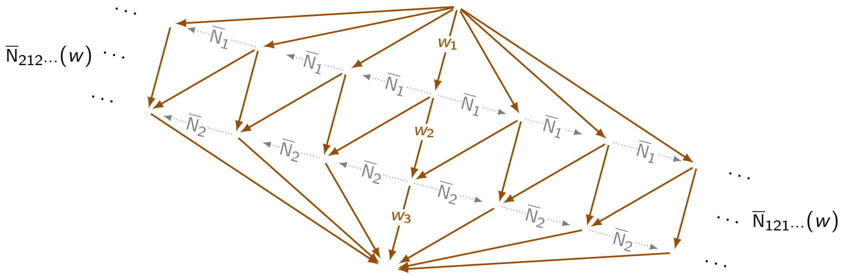

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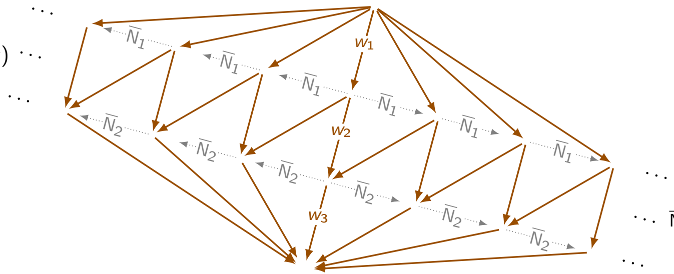

$$\left(\max_{w\in\mathcal{Q}^3} \min\{\ \ell : \mathsf{N}(w) = \overline{\mathsf{N}}_{\underbrace{212\cdots}_{\text{length } \ell}}(w)\},\ \max_{w\in\mathcal{Q}^3} \min\{\ \ell : \mathsf{N}(w) = \overline{\mathsf{N}}_{\underbrace{121\cdots}_{\text{length } \ell}}(w)\} \right)$$

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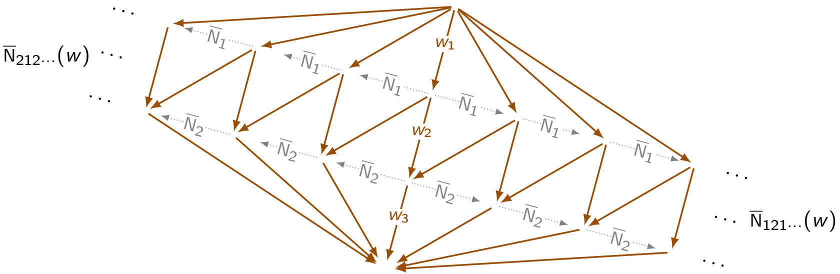

The breadth of $(\mathcal{Q}, \mathsf{N})$ is:

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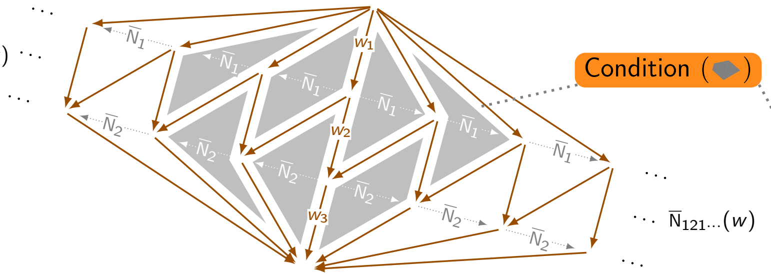

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Proposition P 2019 — bottom-approximation

If N has breadth at most $(4, 3)$, then $\langle\ \mathcal{M}_{\mathcal{Q},\mathsf{N}}\ \rangle_+$ is some quotient of S.

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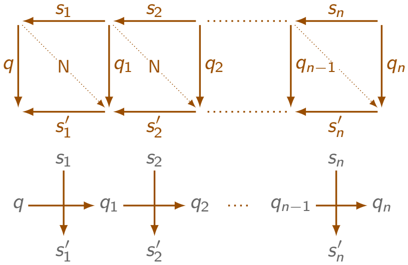
For $\mathsf{N}(s) = s_n \cdots s_1$ and $\mathsf{N}(sq) = q_n s'_n \cdots s'_1$, we obtain diagrammatically:



Condition ( )

We deduce $\sigma_q(s_1 \cdots s_n) = s'_1 \cdots s'_n$ for any $q \in \mathcal{Q}$.

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Automaticon semigroups: a trivial example

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- let $(\mathcal{J}, \mathtt{N})$ verify $\mathtt{N}(xy) = \mathtt{1}(xy)$ for every $(x, y) \in \mathcal{J}^2$;

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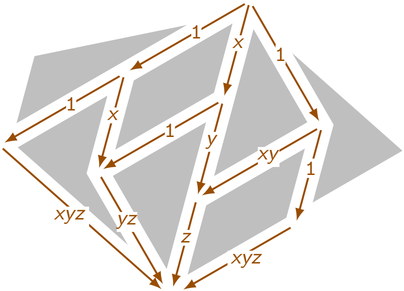
- its breadth is $(3, 2)$, implying Condition (◆).



Every finite monoid $\mathcal{J}$ is an automaticon monoid:

- ▷ let $(\mathcal{J}, \mathtt{N})$ verify $\mathtt{N}(xy) = 1(xy)$ for every $(x, y) \in \mathcal{J}^2$;
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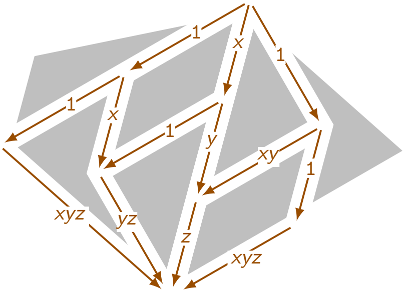

The bicyclic monoid $\mathbf{B} = \langle\ \mathtt{a}, \mathtt{b} : \mathtt{ab} = 1\ \rangle_{+}^{1}$ is not an automaton monoid:

- ▷ let $(\{\mathtt{a}, \mathtt{b}, 1\}, \mathtt{N})$ with $\mathtt{N}(\mathtt{ab}) = 11$, $\mathtt{N}(x1) = 1x$, and $\mathtt{N}(xy) = xy$;

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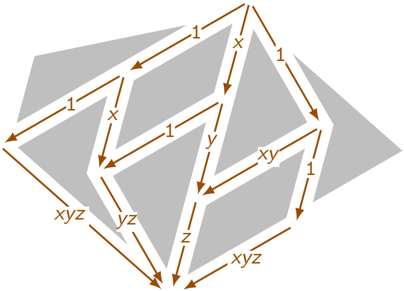

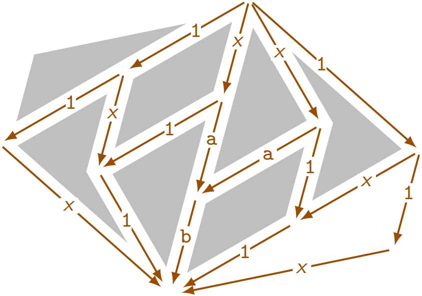

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- ▷ let $(\{\mathtt{a}, \mathtt{b}, \mathtt{1}\}, \mathtt{N})$ with $\mathtt{N}(\mathtt{ab}) = \mathtt{11}$, $\mathtt{N}(x\mathtt{1}) = \mathtt{1}x$, and $\mathtt{N}(xy) = xy$;
- ▷ its breadth is $(3, 4)$, contradicting Condition ( ).

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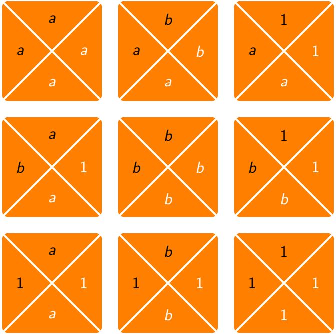

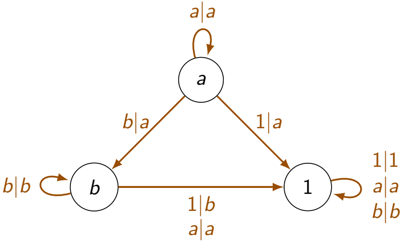

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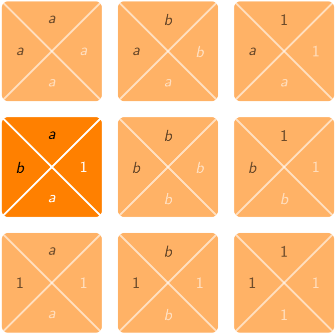

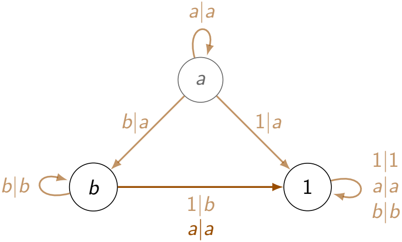

Baumslag-Solitar Artin-Krammer

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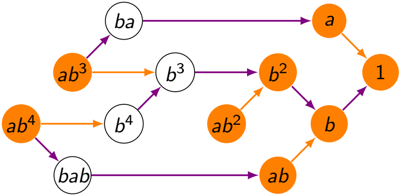

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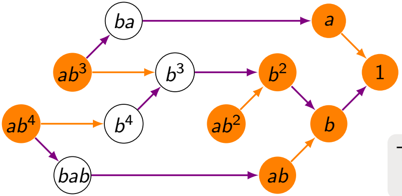

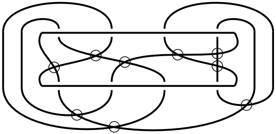

There exists a group-embeddable automaticon monoid whose enveloping group is not an automaticon group

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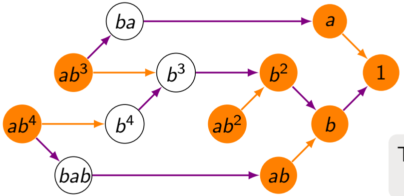

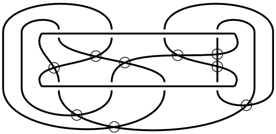

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Hoffmann 2001   P 2019

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Baumslag-Solitar — Artin-Krammer

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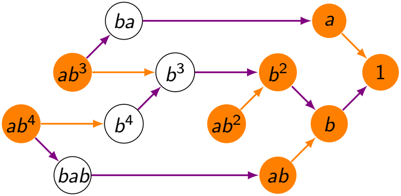

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Baumslag-Solitar | Artin-Krammer

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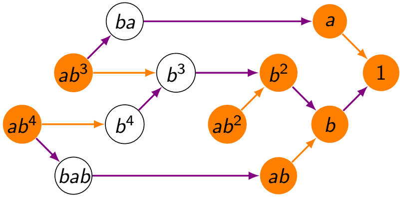

Hoffmann 2001 **P 2019**

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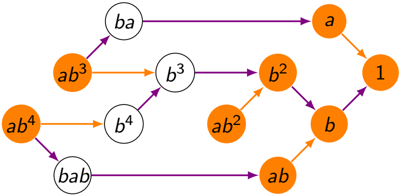

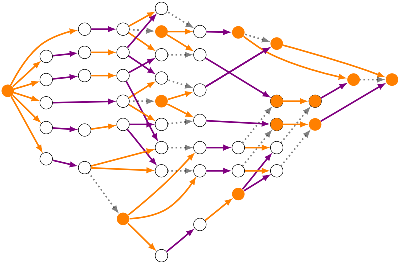

Hoffmann 2001 P 2019

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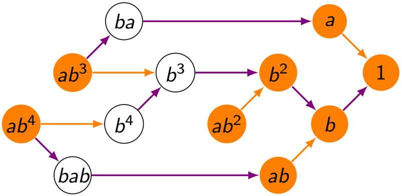

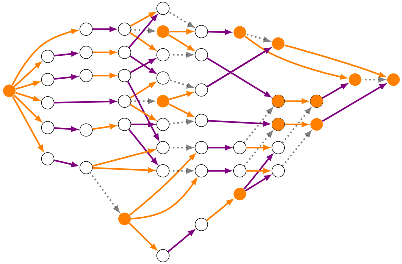

Hoffmann 2001 P 2019

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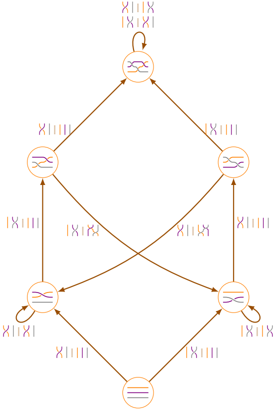
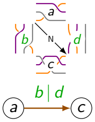
Dehornoy Guiraud P 2019 P 2019

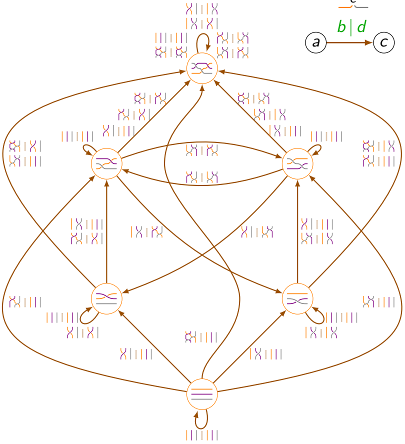
$\mathrm{AK}_+^1(\Gamma)$ is an automaticon monoid

Automaticon semigroups: the paradigmatic braid monoids



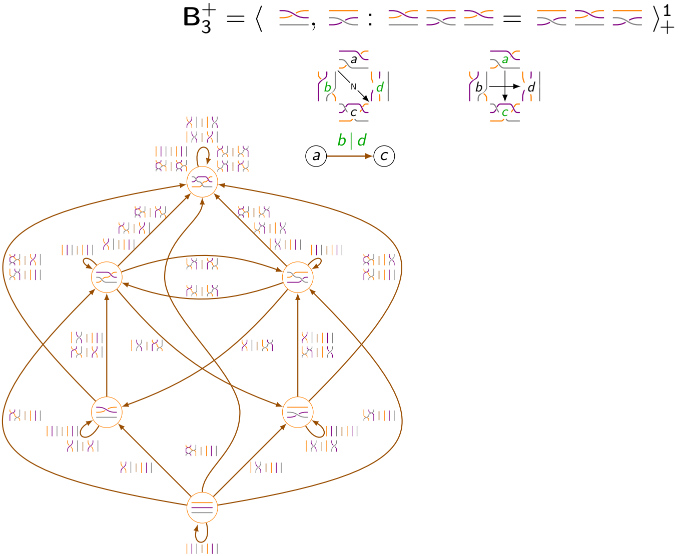
Thurston transducer





(completed)
Thurston transducer

Automaticon semigroups: the paradigmatic braid monoids



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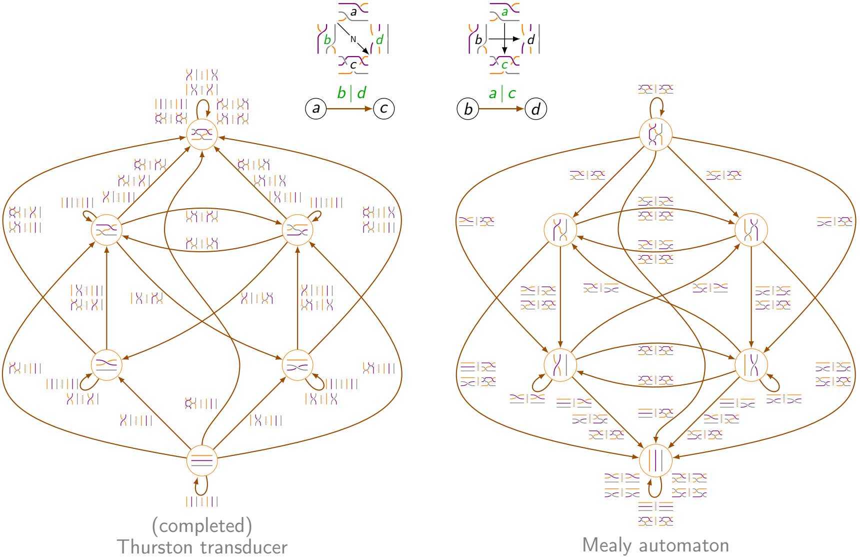
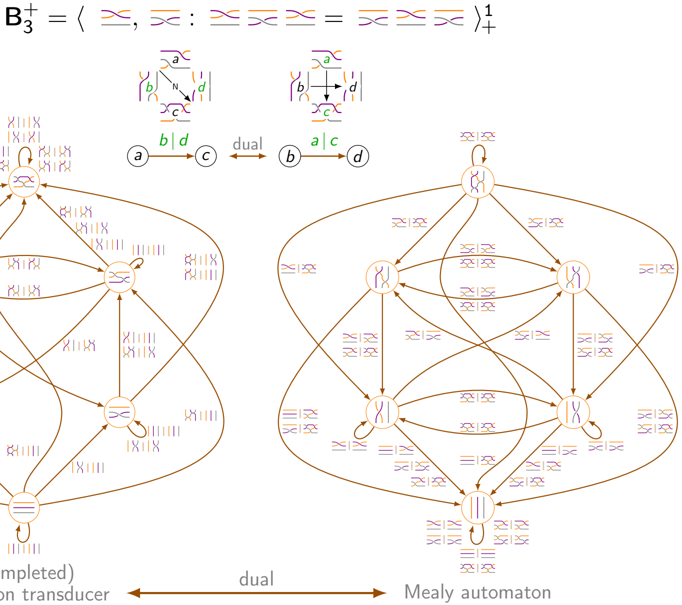
Thurston transducer

Automaticon semigroups: the paradigmatic braid monoids




B_3^+
a
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d
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$b \mid d$
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dual
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$a \mid c$
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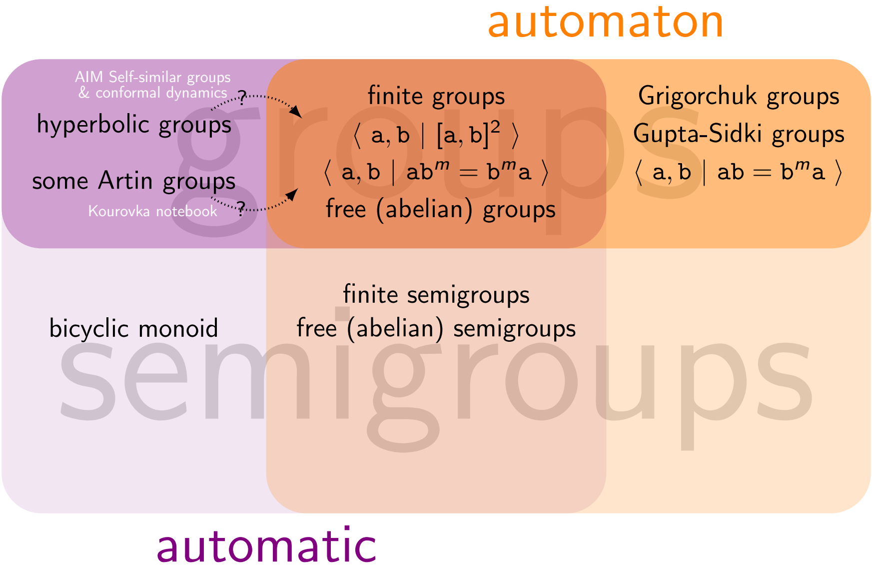
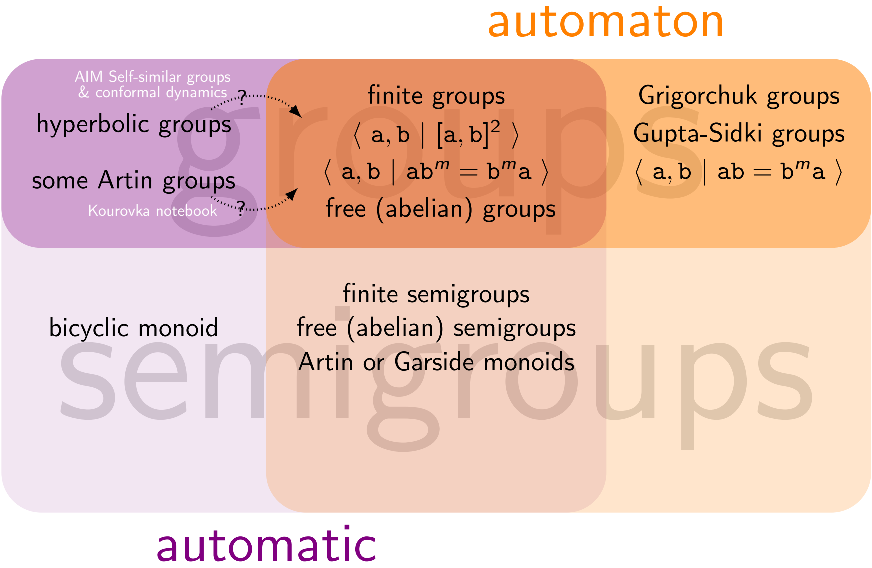
(completed)
Thurston transducer
dual
Mealy automaton


automaton
groups
semigroups
AIM Self-similar groups & conformal dynamics
?
hyperbolic groups
some Artin groups
Kourovka notebook
?
finite groups
$\langle\, a, b \mid [a, b]^2 \,\rangle$
$\langle\, a, b \mid ab^m = b^m a \,\rangle$
free (abelian) groups
Grigorchuk groups
Gupta-Sidki groups
$\langle\, a, b \mid ab = b^m a \,\rangle$
bicyclic monoid
finite semigroups
free (abelian) semigroups
automatic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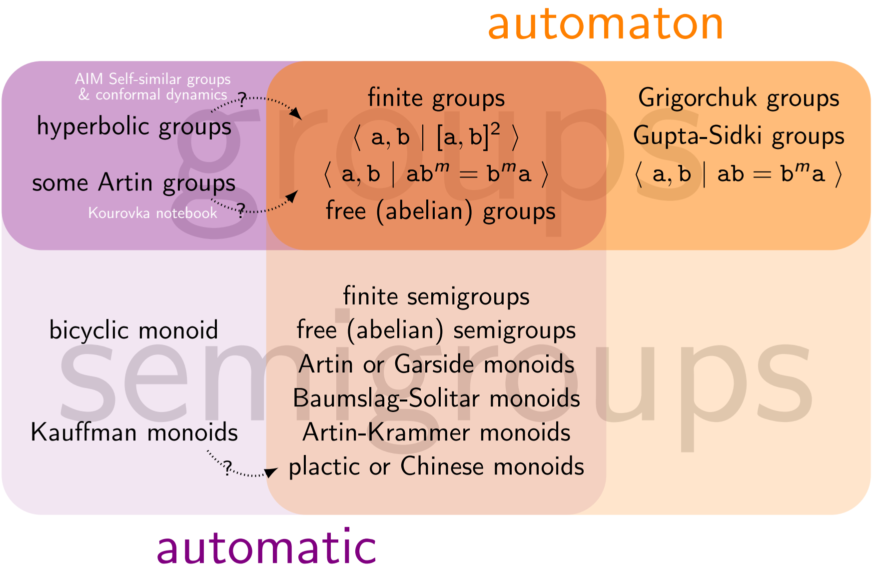
plactic or Chinese monoids

automatic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The plactic monoid of rank n is

$$\mathbf{P}_n = \left\langle 1 < \cdots < n \;\middle|\; \begin{array}{ll} \mathrm{zxy} = \mathrm{xzy} & \text{for } \mathrm{x} \leq \mathrm{y} < \mathrm{z} \\ \mathrm{yxz} = \mathrm{yzx} & \text{for } \mathrm{x} < \mathrm{y} \leq \mathrm{z} \end{array} \right\rangle^{+}.$$

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Cain Gray Malheiro 2014

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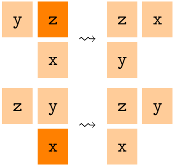

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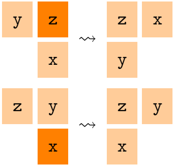

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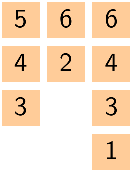

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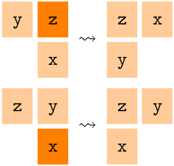

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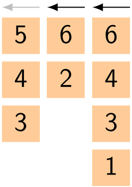

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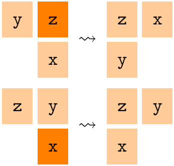

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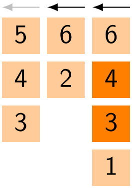

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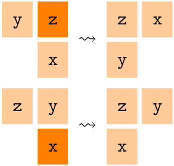

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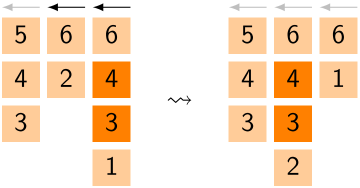

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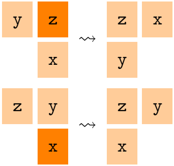

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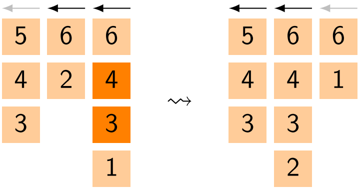

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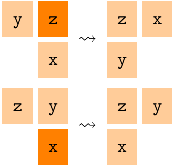

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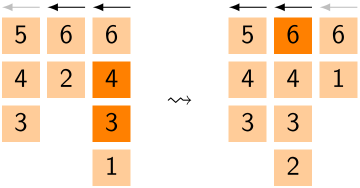

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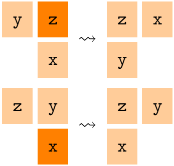

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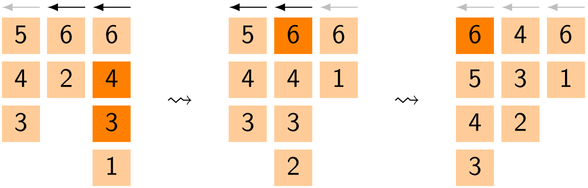

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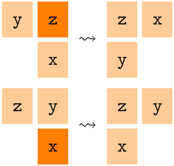

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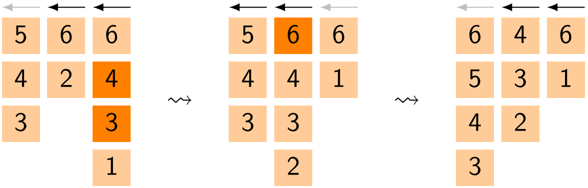

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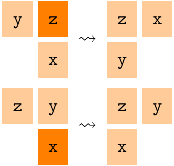

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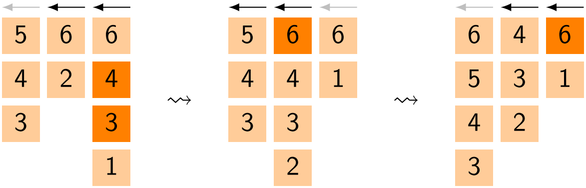

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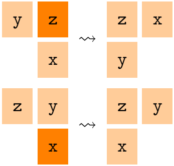

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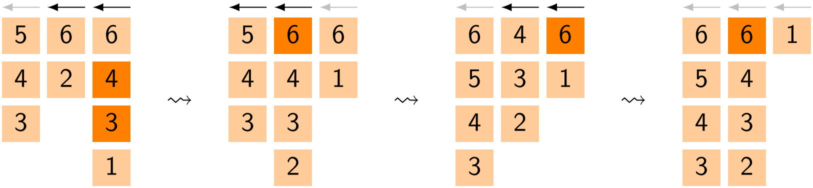

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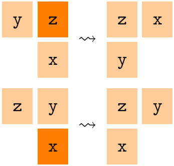

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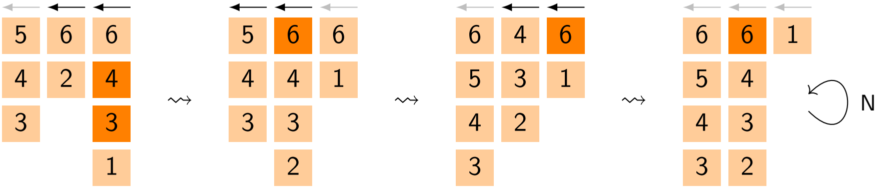

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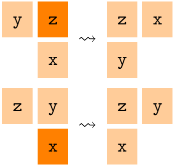

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P 2019

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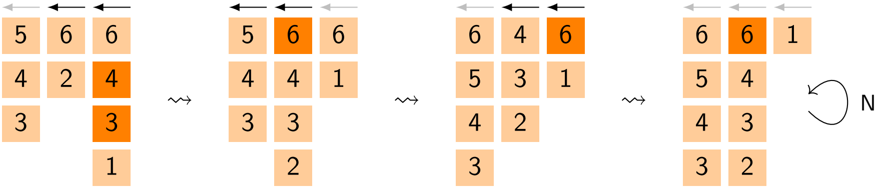

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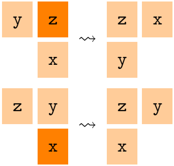

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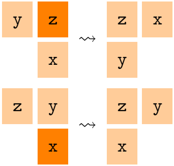

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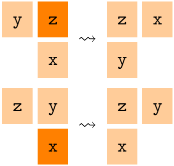
Then $\mathbf{C}_n$ is generated by $\mathcal{Q} = \{\mathrm{x} : n \geq \mathrm{x} \geq 1\} \cup \{\mathrm{yx} : \mathrm{y} > \mathrm{x}\}$.

Cain Gray Malheiro 2016

$\mathbf{C}_n$ is an automatic monoid

The plactic monoid of rank n is

$$\mathbf{P}_n = \left\langle 1 < \cdots < n \;\middle|\; \begin{array}{ll} \mathrm{zxy} = \mathrm{xzy} & \text{for } \mathrm{x} \leq \mathrm{y} < \mathrm{z} \\ \mathrm{yxz} = \mathrm{yzx} & \text{for } \mathrm{x} < \mathrm{y} \leq \mathrm{z} \end{array} \right\rangle^{+}.$$



Then $\mathbf{P}_n$ is also generated by the family $\mathcal{Q}$ of columns,
defined to be strictly decreasing products of elements of $\{1, \ldots, n\}$.

Cain Gray Malheiro 2014

$\mathbf{P}_n$ is an automatic monoid

P 2019

$\mathbf{P}_n$ is an automaton monoid

The Chinese monoid of rank n is

$$\mathbf{C}_n = \langle\, 1 < \cdots < n \mid \mathrm{zyx} = \mathrm{zxy} = \mathrm{yzx} \quad \text{for } \mathrm{x} \leq \mathrm{y} \leq \mathrm{z} \,\rangle^{+}.$$

Then $\mathbf{C}_n$ is generated by $\mathcal{Q} = \{\mathrm{x} : n \geq \mathrm{x} \geq 1\} \cup \{\mathrm{yx} : \mathrm{y} > \mathrm{x}\} \cup \{\mathrm{x}^2 : n > \mathrm{x} > 1\}$.

Cain Gray Malheiro 2016

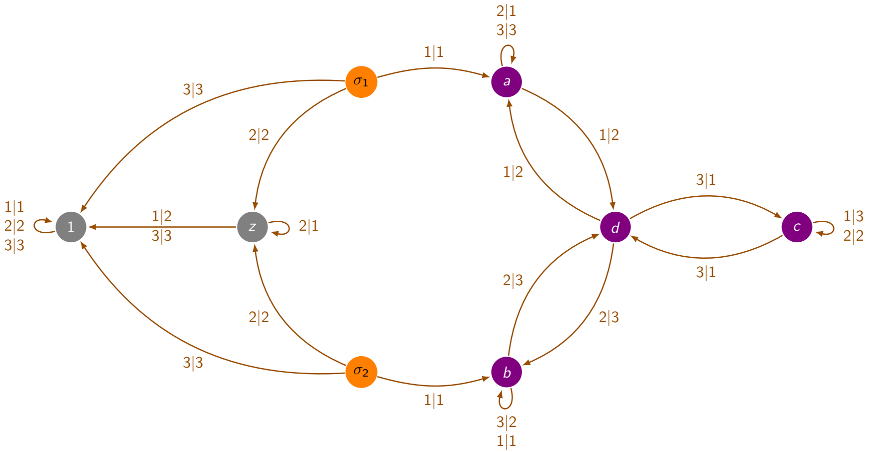
$\mathbf{C}_n$ is an automatic monoid

P 2019

$\mathbf{C}_n$ is an automaton monoid

the 3-strand braid group as an automaton **sub**group

$$\mathbf{B}_3 = \langle\, \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \,\rangle_+^1$$



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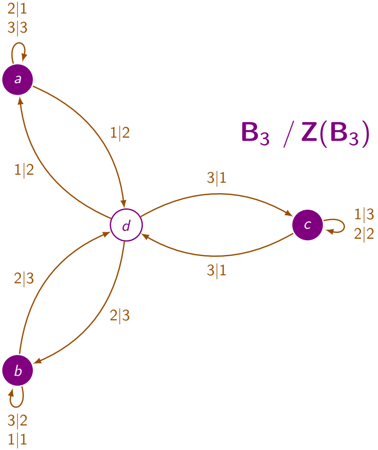

$\mathbf{B}_3\ /\ \mathbf{Z}(\mathbf{B}_3)$

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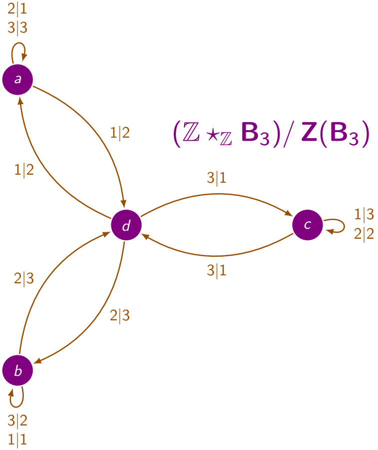

$(\mathbb{Z} \star_{\mathbb{Z}} \mathbf{B}_3) / \mathbf{Z}(\mathbf{B}_3)$

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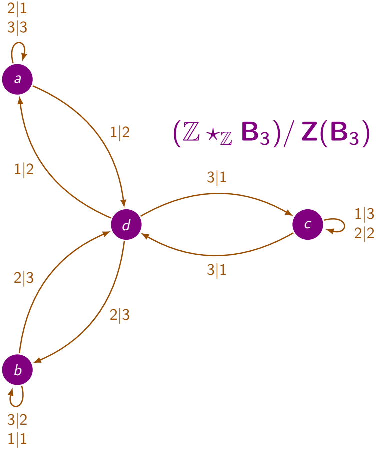
$\mathbb{Z}$



$(\mathbb{Z} \star_{\mathbb{Z}} \mathbf{B}_3) / \mathbf{Z}(\mathbf{B}_3)$

the 3-strand braid group as an automaton **sub**group

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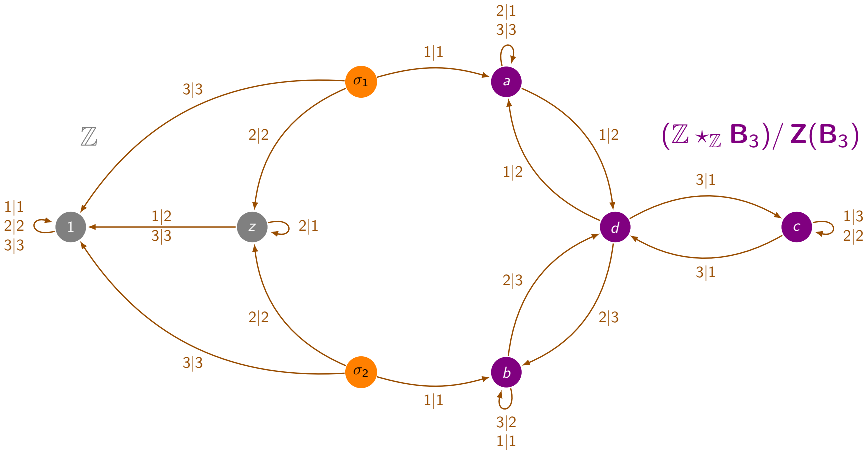

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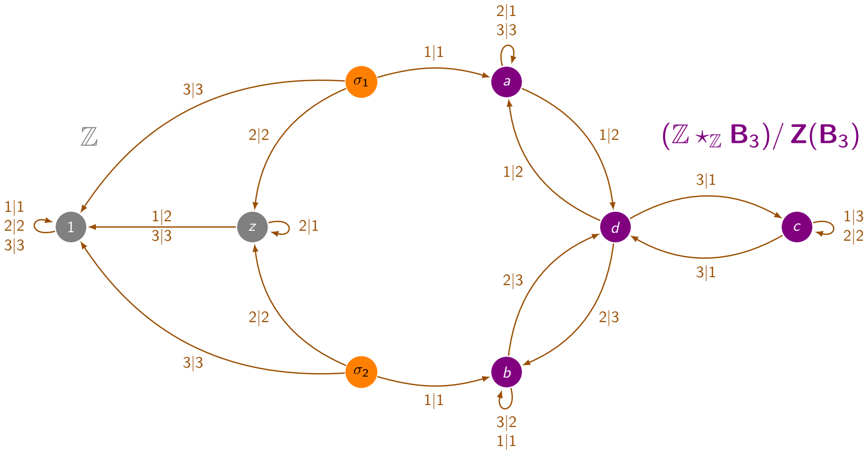
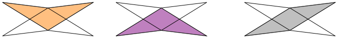


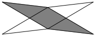
Independently, Lavrenyuk, Mazorchuk, Oliynyk, and Sushchansky (2005) gave 3-letter Mealy automata for σ_1 (14 states) and σ_2 (13 states).

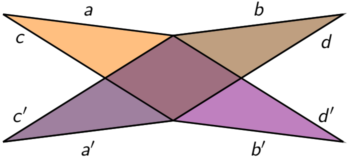
The semigroup



$$\mathbf{T} = \langle\, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{a}', \mathtt{b}', \mathtt{c}', \mathtt{d}' : \mathtt{ab} = \mathtt{cd}, \mathtt{a}'\mathtt{b}' = \mathtt{c}'\mathtt{d}', \mathtt{a}'\mathtt{d} = \mathtt{c}'\mathtt{b} \,\rangle_+$$

is known by Malcev work to be cancellative but not group-embeddable:

from these three relations, 
we cannot deduce the relation $\mathtt{ad}' = \mathtt{cb}'$
that holds in the enveloping group.

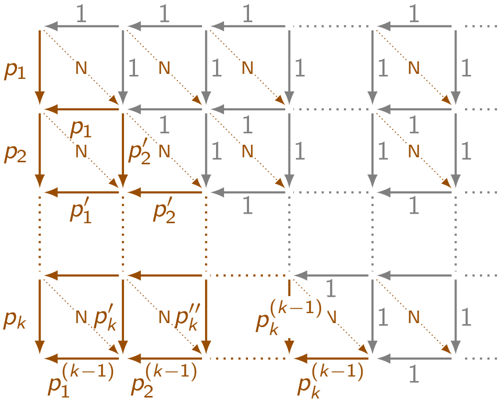


For instance, the quadratic normalisation $(\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{a}', \mathtt{b}', \mathtt{c}', \mathtt{d}'\}, \mathtt{N})$ defined by

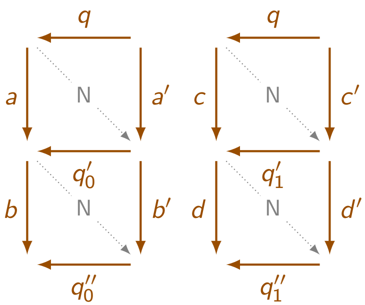
$$\mathtt{N}(\mathtt{ab}) = \mathtt{cd}, \quad \mathtt{N}(\mathtt{a}'\mathtt{b}') = \mathtt{c}'\mathtt{d}', \quad \text{and} \quad \mathtt{N}(\mathtt{a}'\mathtt{d}) = \mathtt{c}'\mathtt{b},$$

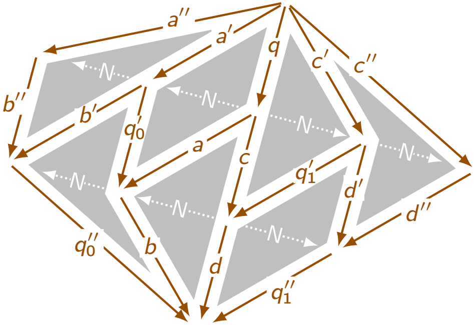
has breadth $(2, 2)$, hence satisfies Condition ().
Moreover, **T** admits several elements that escape the normalisation $\mathtt{N}$.
Therefore Main Theorem applies: **T** is an automaticon semigroup.

This answers in particular a question by Alan J. Cain.



Any $\mathcal{Q}$-words inducing a same action (on 1^ω for instance) are N-equivalent.





Assume $\mathsf{N}(ab) = \mathsf{N}(cd) = ab$. Let $\mathbf{u} = q\mathbf{v} \in \mathcal{Q}^n$ for some $n > 0$ and $q \in \mathcal{Q}$. We prove both $\sigma_{ab}(\mathbf{u}) = \sigma_{cd}(\mathbf{u})$ (coordinatewise) and $\tau_{\mathbf{u}}(ab) \equiv_{\mathsf{N}} \tau_{\mathbf{u}}(cd)$ by induction on $n > 0$.