Chapter 10
Automaton (semi)groups: Wang tilings and Schreier tries

Ines Klimann and Matthieu Picantin

Abstract Groups and semigroups generated by Mealy automata were formally introduced in the early sixties. They revealed their full potential over the years, by contributing to important conjectures in group theory. In the current chapter, we intend to provide various combinatorial and dynamical tools to tackle some decision problems all related to some extent to the growth of automaton (semi)groups. In the first part, we consider Wang tilings as a major tool in order to study and understand the behavior of automaton (semi)groups. There are various ways to associate a Wang tileset with a given complete and deterministic Mealy automaton, and various ways to interpret the induced Wang tilings. We describe some of these fruitful combinations, as well as some promising research opportunities. In the second part, we detail some toggle switch between a classical notion from group theory—Schreier graphs—and some properties of an automaton group about its growth or the growth of its monogenic subgroups. We focus on polynomial-activity automata and on reversible automata, which are somehow diametrically opposed families.

10.1 Introduction

Groups and semigroups generated by Mealy automata were formally introduced in the early sixties (for details, see [10, 32] and references therein). They revealed their full potential over the years, by contributing to important conjectures in group theory.

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We intend here to provide various combinatorial and dynamical tools to tackle some decision problems all related to some extent to the growth of automaton (semi)groups.

In Section 10.2, we consider Wang tilings as a major tool in order to study and understand the behavior of automaton (semi)groups. There are various ways to associate a Wang tileset with a given complete and deterministic Mealy automaton, and various ways to interpret the induced Wang tilings. We describe some of these fruitful combinations, as well as some promising research opportunities.

In Section 10.3, we detail some toggle switch between a classical notion from group theory—Schreier graphs—and some properties of an automaton group about its growth or the growth of its monogenic subgroups. We focus on polynomial-activity automata and on reversible automata, which are somehow diametrically opposed families.

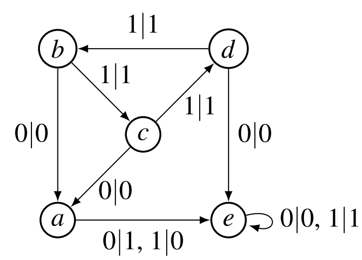


Fig. 10.1 The Grigorchuk automaton.

10.1.1 Mealy automata

We first recall the formal definition of an automaton. A *(finite, deterministic, and complete) automaton* is a triple $\big(Q,\Sigma,\delta=(\delta_i\colon Q\to Q)_{i\in\Sigma}\big)$, where the *state set* Q and the *alphabet* Σ are non-empty finite sets, and the δ_i are functions.

A *Mealy automaton* is a quadruple (Q,Σ,δ,ρ) such that (Q,Σ,δ) and (Σ,Q,ρ) are both automata. In other terms, a Mealy automaton is a complete, deterministic, letter-to-letter transducer with the same input and output alphabet. Its *size* is the cardinal of its state set.

The graphical representation of a Mealy automaton is standard, see Figures 10.2 and 10.1.

A Mealy automaton $\mathscr{A}=(Q,\Sigma,\delta,\rho)$ is *invertible* if each function ρ_x is a permutation of Σ and *reversible* if each function δ_i is a permutation of Q.

In the case where $\mathscr{A}$ is invertible, there is an explicit way to express the actions of the inverses functions by considering the *inverse automaton* $\mathscr{A}^{-1}$ having $Q^{-1}=\{x^{-1},x\in Q\}$ as state set, and a transition $x^{-1}\xrightarrow{j|i}y^{-1}$ whenever $x\xrightarrow{i|j}y$ is a transition in $\mathscr{A}$ (see Fig. 10.2). In the case where $\mathscr{A}$ is reversible, its connected components are strongly connected.

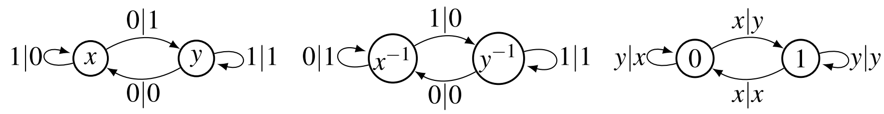


Fig. 10.2 The lamplighter automaton $\mathscr{L}$, its inverse automaton $\mathscr{L}^{-1}$, and its dual automaton $\mathfrak{d}\mathscr{L}$.

In any Mealy automaton $\mathscr{A} = (Q, \Sigma, \delta, \rho)$, the sets Q and Σ play dual roles. So we may consider the *dual (Mealy) automaton* defined by $\mathfrak{d}\mathscr{A} = (\Sigma, Q, \rho, \delta)$, where we have the transition $i \xrightarrow{x|y} j$ whenever $x \xrightarrow{i|j} y$ is a transition in $\mathscr{A}$ (see Fig. 10.2). Obviously, a Mealy automaton is reversible if and only if its dual is invertible.

An invertible Mealy automaton is *bireversible* if it is reversible (*i.e.* the input letters of the transitions act like permutations on the state set) and if so is its inverse (*i.e.* the output letters of the transitions act like permutations on the state set).

Whenever $\mathscr{A}$ is an invertible-reversible Mealy automaton, we can consider the letters and their inverses. By setting $\mathscr{A}' = \mathfrak{d}(\mathfrak{d}\mathscr{A} \sqcup (\mathfrak{d}\mathscr{A})^{-1})$, the (invertible-reversible) Mealy automaton $\widetilde{\mathscr{A}} = \mathscr{A}' \sqcup (\mathscr{A}')^{-1}$ is the extension of $\mathscr{A}$ with state set $Q \sqcup Q^{-1}$ and alphabet $\Sigma \sqcup \Sigma^{-1}$.

For any set Σ, we let Σ^+ denote the free semigroup over Σ (*resp.* Σ^* for the free monoid with unit $\mathtt{1}$) and call its elements *Σ-words*. We write $|w|$ for the length of a Σ-word w, and ww' for the product of two Σ-words w and w'.

A state of a Mealy automaton can be seen as acting on the set Σ^* of finite words (equivalently on a regular rooted tree of arity $|\Sigma|$) or on the set Σ^ω of infinite words (equivalently on the boundary of the former tree).

10.1.2 Minimization and Nerode classes

Let $\mathscr{A} = (Q, \Sigma, \delta, \rho)$ be a Mealy automaton.

The *Nerode equivalence* $\equiv$ *on* Q is the limit of the sequence of increasingly finer equivalences $(\equiv_k)_k$ recursively defined by:

$$\begin{aligned} \forall x, y \in Q, \qquad\qquad x \equiv_0 y &\iff \rho_x = \rho_y\,, \\ \forall k \geqslant 0,\ x \equiv_{k+1} y &\iff \big(x \equiv_k y \quad \wedge \quad \forall i \in \Sigma,\ \delta_i(x) \equiv_k \delta_i(y)\big)\,. \end{aligned}$$

Since the set Q is finite, this sequence is ultimately constant. For every element x in Q, we let $[x]$ denote the class of x *w.r.t.* the Nerode equivalence, called the *Nerode class* of x. Extending to the n-th power of $\mathscr{A}$, we let $[x]$ denote the Nerode class in Q^n of $x \in Q^n$.

Remark 10.1. The Nerode classes of a connected reversible Mealy automaton (*i.e.* a Mealy automaton with exactly one connected component) have the same cardinality.

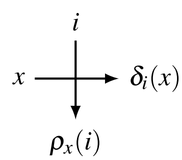
The *minimization* of $\mathscr{A}$ is the Mealy automaton $\mathfrak{m}(\mathscr{A}) = (Q/{\equiv}, \Sigma, \tilde{\delta}, \tilde{\rho})$, where for every (x,i) in $Q \times \Sigma$, $\tilde{\delta}_i([x]) = [\delta_i(x)]$ and $\tilde{\rho}_{[x]} = \rho_x$. This definition is consistent with the standard minimization of "deterministic finite automata" where instead of considering the mappings $(\rho_x : \Sigma \to \Sigma)_x$, the computation is initiated by the separation between terminal and non-terminal states.

Two Mealy automata are *equivalent* if their minimizations are isomorphic as labeled graphs. A Mealy automaton is *minimal* if it is equivalent to its minimization.

A pair of dual Mealy automata is *reduced* if both automata are minimal. The $\mathfrak{md}$-*reduction* of a Mealy automaton, introduced in [1], consists in minimizing the automaton or its dual until the resulting pair of dual Mealy automata is reduced. It is well-defined: if both a Mealy automaton and its dual automaton are non-minimal, the reduction is confluent.

10.1.3 Automaton (semi)groups

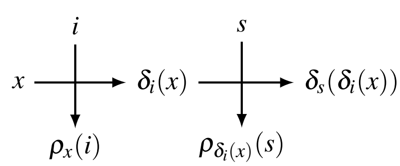
Let $\mathscr{A} = (Q, \Sigma, \delta, \rho)$ be a Mealy automaton. For each transition $x \xrightarrow{i|\rho_x(i)} \delta_i(x)$, we associate the *cross-transition* depicted in the following way:



Each state $x \in Q$ defines a mapping from Σ^* into itself recursively defined by:

$$\forall i \in \Sigma,\ \forall s \in \Sigma^*, \qquad \rho_x(is) = \rho_x(i)\rho_{\delta_i(x)}(s)\,.$$

that can also be depicted by a so-called *cross-diagram* obtained by gluing cross-transitions (see [1, 23]) representing the action of a word of states on a word of letters (or vice-versa):



In a dual way, each letter $i \in \Sigma$ defines also an action on Q^*.

Both actions naturally extend to words, respectively in Q^* and Σ^* with the following convention. The image of the empty word is itself. We define the function ρ_x *induced* by $x = x_1 \cdots x_n \in Q^n$, with $n > 0$, by setting, $\rho_x \colon \Sigma^* \to \Sigma^*, \rho_x =$

$\rho_{x_n} \circ \cdots \circ \rho_{x_1}$. We let $\delta_i \colon Q^* \to Q^*, i \in \Sigma$ denote dually the functions induced by the states of $\mathfrak{d}\mathscr{A}$. For $s = s_1 \cdots s_n \in \Sigma^n$ with $n > 0$, set $\delta_s \colon Q^* \to Q^*,\ \delta_s = \delta_{s_n} \circ \cdots \circ \delta_{s_1}$.

Alternatively, we consider the powers of $\mathscr{A}$: for $n > 0$, its *n-th power* $\mathscr{A}^n$ is the Mealy automaton

$$\mathscr{A}^n = \left(Q^n, \Sigma, (\delta_i \colon Q^n \to Q^n)_{i\in\Sigma}, (\rho_x \colon \Sigma \to \Sigma)_{x\in Q^n} \right) .$$

By convention, $\mathscr{A}^0$ is the trivial automaton on the alphabet Σ. Note that all the powers of a reversible Mealy automaton are reversible as well.

The semigroup of mappings from Σ^* to Σ^* generated by $\{\rho_x, x \in Q\}$ is called the *semigroup generated by* $\mathscr{A}$ and is denoted by $\langle\ \mathscr{A}\ \rangle_+$. When $\mathscr{A}$ is invertible, the functions induced by its states are permutations on words of the same length and thus we may consider the group of mappings from Σ^* to Σ^* generated by $\{\rho_x, x \in Q\}$. This group is called the *group generated by* $\mathscr{A}$ and is denoted by $\langle\ \mathscr{A}\ \rangle$.

Two states of a Mealy automaton belong to the same Nerode class if and only if they represent the same element in the generated (semi)group, *i.e.* if and only if they induce the same action on Σ^*.

Remark 10.2. If two words of Q^* are equivalent, so are their images under the action of any element of $\langle\ \mathfrak{d}\mathscr{A}\ \rangle_+$.

Remark 10.3. If a state of a Mealy automaton induces the identity, so do all the states reachable from it. In particular, in a reversible connected component of a Mealy automaton, a state induces the identity if and only so do all of its states.

Remark 10.4. Let $\mathscr{A}$ and $\mathscr{B}$ be two reversible connected Mealy automata on the same alphabet Σ, and let x be some state of $\mathscr{A}$, and y be some state of $\mathscr{B}$. If x and y have the same action on Σ^*, then $\mathfrak{m}(\mathscr{A})$ and $\mathfrak{m}(\mathscr{B})$ are isomorphic; in particular they have the same size. Indeed the image of x in $\mathscr{A}$ by some word $s \in \Sigma^*$ and the image of y in $\mathscr{B}$ by this same word s have necessarily the same action on Σ^*, and $\mathscr{A}$ and $\mathscr{B}$ being strongly connected (because they are connected and reversible), for every state of $\mathscr{A}$ there is a state of $\mathscr{B}$ which acts similarly on Σ^*, and vice-versa.

Let us recall some known results from [1] and [39] (see also [14, 46, 49]) that will be used in our proofs.

Proposition 10.5. *An invertible-reversible Mealy automaton $\mathscr{A}$ and its extension $\widetilde{\mathscr{A}}$ generate isomorphic groups.*

Proposition 10.6. *An invertible Mealy automaton generates a finite group if and only if it generates a finite semigroup.*

Theorem 10.7. *A Mealy automaton generates a finite (semi)group if and only if so does its dual.*

Corollary 10.8. *A Mealy automaton generates a finite (semi)group if and only if so does its $\mathfrak{md}$-reduction.*

The trivial Mealy automaton generates the trivial (semi)group. If the $\mathfrak{md}$-reduction of a Mealy automaton $\mathscr{A}$ leads to the trivial Mealy automaton, $\mathscr{A}$ is said to be $\mathfrak{md}$-*trivial*. It is decidable whether a Mealy automaton is $\mathfrak{md}$-trivial. An $\mathfrak{md}$-trivial Mealy automaton generates a finite semigroup, but in general the converse is false [1].

The (bi)reversible Mealy automata seem to be especially sensitive to $\mathfrak{md}$-reduction. The four minimal examples of non-$\mathfrak{md}$-trivial bireversible Mealy automata generating finite groups are displayed on Figure 10.3 (see [1] and [48] for other examples).

Theorem 10.9. *Any 2-letter and/or 2-state bireversible Mealy automaton generates a finite group if and only if it is* $\mathfrak{md}$*-trivial.*

We shall see in Subsection 10.2.3 why and how we intend to generalize this fundamental result.

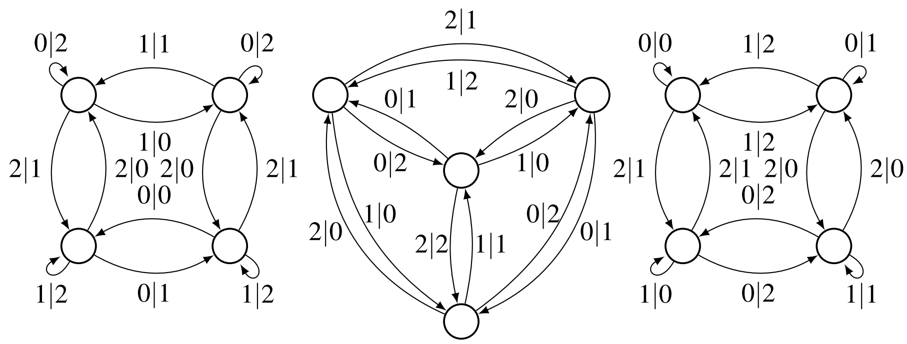


Fig. 10.3 Three non-$\mathfrak{md}$-trivial bireversible automata that generate finite groups. The inverse of the leftmost one provides a fourth minimal example.

10.2 A matter of tilings

Wang tilings are a major tool in order to study and understand the behavior of automaton (semi)groups. Without even thought for the potential generated algebraic structures, Mealy automata have early been associated with Wang tilings by K. Culik [11], by J. Kari [37], and ultimately by E. Jeandel and M. Rao [35] in their overwhelming and successful pursuit of small aperiodic Wang tilesets. Such Mealy automata need not to be either complete or deterministic, preventing to easily define automaton (semi)groups as in the current framework. Now there are various ways to associate a Wang tileset with a given complete and deterministic Mealy automaton, and various ways to interpret the induced Wang tilings. We aim to describe some of these fruitful combinations, as well as some promising research opportunities.

Subsection 10.2.1 recalls basic definitions and undecidability results about Wang tilings. Subsection 10.2.2 is devoted to the undecidability result by P. Gillibert of the finiteness problem for automaton semigroups and then sketches a bright connection between reset Mealy automata and some one-way cellular automata. Subsection 10.2.3 focuses on so-called helix graphs, a crucial notion capturing the whole dynamics by placing on a same footing the symmetric roles of the state set and the alphabet. Subsection 10.2.4 outlines an effective and natural approach to interpret any semigroup admitting a special language of greedy normal forms—based on rewriting systems and Wang tilings—as an automaton semigroup, namely the semigroup generated by a Mealy automaton encoding the behavior of such a language of greedy normal forms under one-sided multiplication. In each of all these cases, the key notion is duality.

10.2.1 Background on tilings

Named after H. Wang [55], a *Wang tile* is a unit square tile with a color on each edge: it is a quadruple $t = (t_w, t_s, t_e, t_n) \in C^4$ where C is a finite set of colors, as typically depicted in Figure 10.4. A *Wang tileset* is a finite set $\mathscr{T}$ of Wang tiles, and for each $t \in \mathscr{T}$ and $d \in \{n, s, e, w\}$, we put t_d for the color of the edge in the d-side. Given a Wang tileset $\mathscr{T}$, a *Wang tiling* of a subset P of $\mathbb{Z}^2$ is a map $f \colon P \to \mathscr{T}$. We say that such a Wang tiling f is *valid* whenever, with each point $(x,y) \in P$, f associates a tile $f(x,y)$ such that adjacent tiles share the same color on their common edge:

$$f(x,y)_n = f(x,y+1)_s, \qquad \text{for } (x,y) \in P \text{ and } (x,y+1) \in P,$$
$$f(x,y)_e = f(x+1,y)_w, \qquad \text{for } (x,y) \in P \text{ and } (x+1,y) \in P.$$

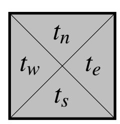


Fig. 10.4 A Wang tile.

A simple compactness argument gives the following classical result.

Theorem 10.10. *For any Wang tileset $\mathscr{T}$, $\mathbb{Z}^2$ admits a valid Wang tiling for $\mathscr{T}$ if and only if so does each finite subset of $\mathbb{Z}^2$.*

In particular, if $\mathbb{Z}^2$ admits no valid Wang tiling, then there is a least integer $n \in \mathbb{N}$ such that the square $\{0, 1, \dots, n\}^2$ admits no valid Wang tiling.

Following K. Culik [11] and J. Kari [37], any Wang tileset may be interpreted as a letter-to-letter transducer with the same input and output alphabet, according to the correspondence in Figure 10.5. Note that such a transducer may be neither complete nor deterministic (and has neither initial nor final states). Follow-

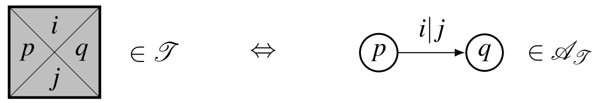


Fig. 10.5 The transducer $\mathscr{A}_{\mathscr{T}}$ associated with a Wang tileset $\mathscr{T}$ according to Culik-Kari.

ing [43] for instance, we say that a Wang tileset $\mathscr{T}$ is *cd-deterministic* with $(c,d) \in \mathscr{I} = \{(n,e),(s,e),(n,w),(s,w)\}$ if each tile $t \in \mathscr{T}$ is uniquely determined by its pair (t_c,t_d) of colors. Whenever $\mathscr{T}$ is *cd*-deterministic for each $(c,d) \in \mathscr{I}$, we say that $\mathscr{T}$ is *4-way deterministic*. The following lemma links properties of the Wang tileset $\mathscr{T}$ with properties of the associated transducer $\mathscr{A}_{\mathscr{T}}$.

Lemma 10.11. *Let $\mathscr{T}$ be a Wang tileset and $\mathscr{A}_{\mathscr{T}}$ be the associated transducer according to Culik-Kari. A necessary condition for $\mathscr{A}_{\mathscr{T}}$ to be a Mealy automaton is that $\mathscr{T}$ is nw-deterministic. In such a case, we have the following.*

- *$\mathscr{T}$ is ne-deterministic if and only if $\mathscr{A}_{\mathscr{T}}$ is reversible;*
- *$\mathscr{T}$ is sw-deterministic if and only if $\mathscr{A}_{\mathscr{T}}$ is invertible;*
- *$\mathscr{T}$ is 4-way deterministic if and only if $\mathscr{A}_{\mathscr{T}}$ is bireversible.*

This original Wang tiling viewpoint could provide a special insight to the dynamics of the Mealy automaton (see for instance [13]).

10.2.2 Finiteness and order problems

A second correspondence between Mealy automata and Wang tilings has allowed P. Gillibert [21] to prove that the finiteness and the order problems for automaton semigroups are undecidable (these decision problems are still open for automaton groups, see [32, Problems 7.2.1(a) and 7.2.1(b)]). P. Gillibert's proof relies on the local correspondence from *nw*-deterministic Wang tilesets to Mealy automata displayed in Figure 10.6, inspired by J. Kari's proof of the undecidability of the nilpotency problem for cellular automata [36].

The following results of R. Berger in [4] and of J. Kari in [36] illustrate how the existence of valid Wang tilings is hard to determine.

Theorem 10.12. *It is undecidable whether or not a Wang tileset admits a valid Wang tiling for the discrete plane $\mathbb{Z}^2$.*

Theorem 10.13. *It is undecidable whether or not a nw-deterministic Wang tileset admits a valid Wang tiling for the discrete plane $\mathbb{Z}^2$.*

Enhancing a result by K. Culik, J. K. Pachl, and S. Yu in [12], J. Kari proved the following in [36].

Theorem 10.14. *It is undecidable whether or not a one-dimensional cellular automaton is nilpotent.*

P. Gillibert adapted J. Kari's argument to address the finiteness and order problems for automaton semigroups. We have to be careful about a side effect: a cellular automaton acts on words indexed by $\mathbb{Z}$, while an automaton semigroup acts on words indexed by $\mathbb{N}$. According to P. Gillibert [21], we first define a Mealy automaton from a Wang tileset (Kari uses a similar construction to obtain a cellular automaton).

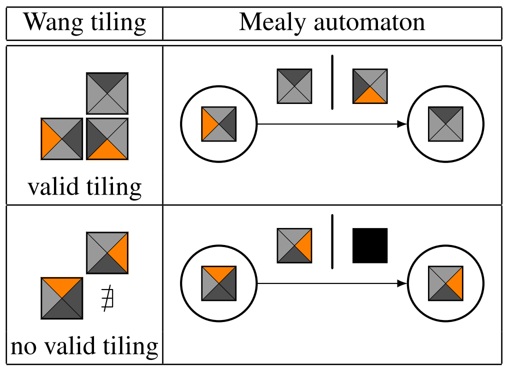


Fig. 10.6 The Mealy automaton $\mathscr{W}_{\mathscr{T}}$ associated with a *nw*-deterministic Wang tileset $\mathscr{T}$ according to Gillibert-Kari (see Definition 10.15).

Definition 10.15. With any *nw*-deterministic Wang tileset $\mathscr{T}$, as illustrated in Figure 10.6, we associate the Mealy automaton $\mathscr{W}_{\mathscr{T}} = (Q,\Sigma,\delta,\rho)$ with $Q = \Sigma = \mathscr{T} \sqcup \{\blacksquare\}$, $\delta_b(a) = b$ for $(a,b) \in Q^2$, and

$$\rho_a(b) = \begin{cases} c & \text{for } a,b,c \in \mathscr{T} \text{ with } a_e = c_w \text{ and } c_n = b_s, \\ \blacksquare & \text{otherwise.} \end{cases}$$

The Mealy automaton $\mathscr{W}_{\mathscr{T}}$ associated with a *nw*-deterministic Wang tileset $\mathscr{T}$ should be understood in the following way. Any word over the state set can be seen as a word written over the tileset along some diagonal, the Mealy automaton transforms this word to the word written on the tiles along the diagonal right below. If it is impossible to put a tile at some place, then the *mistake* tile $\blacksquare$ is placed instead.

For the next three statements, we consider a *nw*-deterministic Wang tileset $\mathscr{T}$, with its associated Mealy automaton $\mathscr{W}_{\mathscr{T}} = (Q,\Sigma,\delta,\rho)$, as in Definition 10.15. The following is straightforward.

Lemma 10.16. *For any state $a \in Q$ and any infinite word $u = (u_k)_{k\in\mathbb{N}} \in \Sigma^\omega$, we have*

$$\rho_a(u) = \rho_a(u_0)(\rho_{u_k}(u_{k+1}))_{k\in\mathbb{N}}. \tag{10.1}$$

Lemma 10.17. *If $\mathbb{Z}^2$ admits some valid Wang tiling for $\mathscr{T}$, then $\langle\, \mathscr{W}_{\mathscr{T}} \,\rangle_+$ is infinite.*

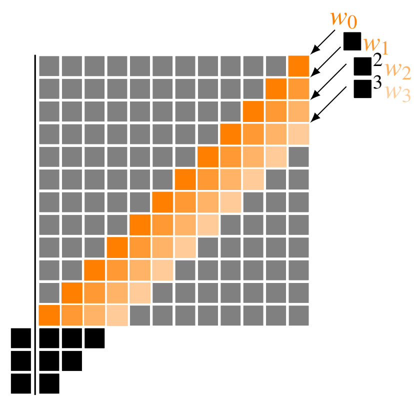


Fig. 10.7 The action of the tile ■ corresponds to an element of infinite order (proof of Lemma 10.17).

Proof. Let $t\colon \mathbb{Z}^2 \to \mathscr{T}$ be some valid Wang tiling. Then we claim that the tile ■ induces an element of infinite order. The point is to show that, for any $n \in \mathbb{N}$, the word $w_n = (t(k+n,k))_{k\in\mathbb{N}}$ satisfies $\rho_{■}^m(w_n) = ■^m w_{m+n}$ for every $m \in \mathbb{N}$, as illustrated in Figure 10.7. By very definition, for $(i,j) \in \mathbb{N}^2$, we have

$$\rho_{t(i,j)}(t(i+1,j+1)) = t(i+1,j)\,. \tag{10.2}$$

Given $n \in \mathbb{N}$, we find:

$$\begin{aligned}\rho_{■}(w_n) &\overset{(10.1)}{=\!=} \rho_{■}(t(n,0))(\rho_{t(n+k,k)}(t(n+k+1,k+1)))_{k\in\mathbb{N}}\,,\\ &\overset{(10.2)}{=\!=} ■(t(n+k+1,k))_{k\in\mathbb{N}}\,,\\ &= ■w_{n+1}\,.\end{aligned}$$

The result follows by induction. In particular, we deduce that the maps $\rho_{■}^m$ are pairwise different, which means that the element $\rho_{■}$ has infinite order and implies that the semigroup $\langle\, \mathscr{W}_{\mathscr{T}} \,\rangle_+$ is infinite. □

Proposition 10.18. *If $\mathbb{Z}^2$ admits no valid Wang tiling for $\mathscr{T}$, then $\langle\, \mathscr{W}_{\mathscr{T}} \,\rangle_+$ is finite.*

Proof. According to Theorem 10.10, there exists $n \in \mathbb{N}$ such that the set $\{0,1,\ldots,n\}^2$ admits no valid Wang tiling for $\mathscr{T}$. Let fix $(p,q) \in \Sigma^n \times \Sigma^\omega$. As illustrated in Figure 10.8, we want to prove that any word $u \in Q^{2n}$ satisfies

$$\rho_u(pq) = \rho_u(p)■^\omega\,. \tag{10.3}$$

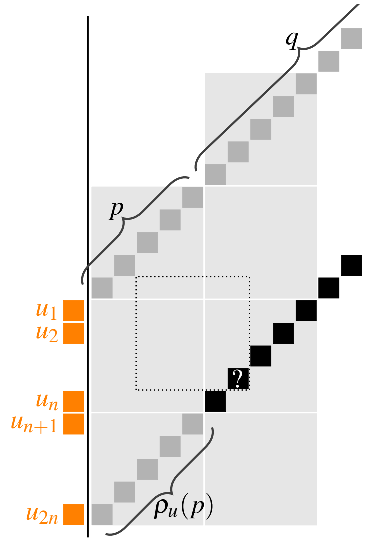


Fig. 10.8 If some tile (along the black diagonal) was not ■, the $n \times n$ square (with dotted line) could be tiled, inside the $2n$-width corridor (proof of Proposition 10.18).

Write $u = u_1 \cdots u_{2n}$ (those orange tiles on Figure 10.8), and set $\tau_0 = \mathrm{id}$ and

$$\tau_k = \rho_{u_1 u_2 \cdots u_k} = \rho_{u_k} \circ \rho_{u_{k-1}} \circ \cdots \circ \rho_{u_1}, \quad \text{for } 1 \le k \le 2n.$$

We have:

$$\rho_{u_{k+1}} \circ \tau_k = \tau_{k+1}, \quad \text{for } 0 \le k \le 2n-1. \tag{10.4}$$

Denoting by $f(i,j)$ the j-th letter of $\tau_i(pq)$ for $(i,j) \in \mathbb{N}^2$ with $0 \le i \le 2n$, we have:

$$\tau_i(pq) = (f(i,j))_{j \in \mathbb{N}}, \quad \text{for } 0 \le i \le 2n. \tag{10.5}$$

For $0 \le i < 2n$, we find:

$$\begin{aligned} f(i+1,j)_{j\in\mathbb{N}} &\overset{(10.5)}{=\!=} \tau_{i+1}(pq), \\ &\overset{(10.4)}{=\!=} \rho_{u_{i+1}}(\tau_i(pq)), \\ &\overset{(10.5)}{=\!=} \rho_{u_{i+1}}(f(i,j)_{j\in\mathbb{N}}), \\ &\overset{(10.1)}{=\!=} \rho_{u_{i+1}}(f(i,0))(\rho_{f(i,j)}(f(i,j+1)))_{j\in\mathbb{N}}. \end{aligned}$$

We deduce

$$\rho_{f(i,j)}(f(i,j+1)) = f(i+1,j+1) \tag{10.6}$$

for $(i,j) \in \mathbb{N}^2$ with $0 \le i < 2n$.

Now assume $f(2n, n+k) \neq \blacksquare$ for some $k \in \mathbb{N}$ (among the black diagonal on Figure 10.8). Applying inductively (10.6), we obtain $f(i+j, i+k) \neq \blacksquare$ for $0 \leq i, j \leq n$, which yields in particular a valid Wang tiling for some $n \times n$ square (with dotted line on Figure 10.8): this is a contradiction that allows to prove (10.3) as well.

Let $u = vw$ with $v \in Q^{2n}$ and $w \in Q^*$. We have

$$\rho_u(pq) = \rho_{vw}(pq) = \rho_w(\rho_v(pq)) \overset{(10.3)}{=\!=} \rho_w(\rho_v(p)\blacksquare^{\omega}) = \rho_{vw}(p)\blacksquare^{\omega} = \rho_u(p)\blacksquare^{\omega}.$$

The cardinality of $\{\rho_u : u \in Q^{2n}Q^*\}$ is bounded by $|(\Sigma^n)^{(\Sigma^n)}|$. From $\langle\, \mathscr{W}_{\mathscr{T}} \,\rangle_+ = \{\rho_u : u \in Q^{<2n}\} \cup \{\rho_u : u \in Q^{2n}Q^*\}$, we deduce

$$|\langle\, \mathscr{W}_{\mathscr{T}} \,\rangle_+| \leq 1 + |Q| + |Q|^2 + \cdots + |Q|^{2n-1} + |(\Sigma^n)^{(\Sigma^n)}|.$$

Hence $\langle\, \mathscr{W}_{\mathscr{T}} \,\rangle_+$ is finite. □

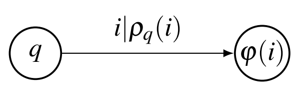
Gathering Lemma 10.17 and Proposition 10.18, we deduce:

Theorem 10.19. *The semigroup $\langle\, \mathscr{W}_{\mathscr{T}} \,\rangle_+$ is infinite if and only if the discrete plane $\mathbb{Z}^2$ admits some valid Wang tiling for $\mathscr{T}$.*

Gathering Theorems 10.13 and 10.19, we finally obtain:

Corollary 10.20. *The finiteness problem for automaton semigroups is undecidable.*

It is worthwhile noting that the Mealy automaton $\mathscr{W}_{\mathscr{T}}$ associated with any *nw*-deterministic tileset $\mathscr{T}$ is reset: a Mealy automaton $\mathscr{A} = (Q, \Sigma, \delta, \rho)$ is said to be a *reset automaton* if, for any state $q \in Q$, the state $\delta_i(q)$ does not depend on q, that is, if there is a map $\varphi : \Sigma \to Q$ such that any transition of $\mathscr{A}$ has the form



The very definition of $\mathscr{W}_{\mathscr{T}}$ induces also that its state set and its alphabet coincide. However this feature is not specific in the framework of reset automata. Indeed, any reset automaton is equivalent to a reset automaton with $Q = \Sigma$ (and $\varphi = \mathrm{id}$).

The point is now that, by construction (with the mandatory adjunction of a tile $\blacksquare$), such reset Mealy automata $\mathscr{W}_{\mathscr{T}}$ are highly non-invertible, which seems prevent to adapt the previous approach for automaton groups. Silva and Steinberg have studied groups generated by invertible reset automata [52]. They proved in particular that such a group is infinite if and only if any generator is of infinite order. However, the following problem remains open.

Problem 10.21. Is the finiteness problem for reset automaton groups decidable?

To conclude this subsection, we describe a link between the finiteness of reset automaton groups and the periodicity of one-way cellular automata. A *one-way cellular automaton* is a triple (Q, r, f) where Q is the finite state set, $r \in \mathbb{N}$ is the radius, and $f : Q^{r+1} \to Q$ is the local transition rule. A *configuration* of such an automaton is an element in $Q^{\mathbb{N}}$. The whole dynamics is described by the *global transition function* F defined by the local transition function f as

$$F(c)(k) = f(c(k), c(k+1), \ldots, c(k+r))$$

for every configuration $c \in Q^{\mathbb{N}}$ and every $k \in \mathbb{N}$. Such a cellular automaton is said to be *periodic* if $F^p = \mathrm{id}$ holds for some integer $p > 0$.

Problem 10.22. Is the periodicity problem for one-way cellular automata decidable?

It must be recalled that J. Kari and N. Ollinger have shown that the periodicity problem for (reversible) cellular automata is undecidable [38]. One can restrict the study of the periodicity to those one-way cellular automata with radius 1 without loss of generality. Let (Q, f) denote the one-way cellular automaton $(Q, 1, f)$ with $f : Q^2 \to Q$.

Any such periodic cellular automaton (Q, f) has to preserve the value of the cell $c(0)$, hence, for any state $b \in Q$, the map $\rho_b : a \mapsto f(a, b)$ has to be a permutation. Such a cellular automaton is said to be *center-permutive*. The latter being a purely syntactic property, Problem 10.22 is equivalent to the following.

Problem 10.23. Is the periodicity problem for one-way center-permutive radius 1 cellular automata decidable?

As illustrated on Figure 10.9 and stated in [18], Problems 10.21 and 10.22 turn out to be a single one open problem.

Proposition 10.24. *For any family $\rho = (\rho_b)_{b\in\Sigma}$ of permutations of the alphabet Σ, the group generated by the Mealy automaton $(\Sigma, \Sigma, \mathrm{id}, \rho)$ is finite if and only if the cellular automaton $(\Sigma, (a,b) \mapsto \rho_b(a))$ is periodic.*

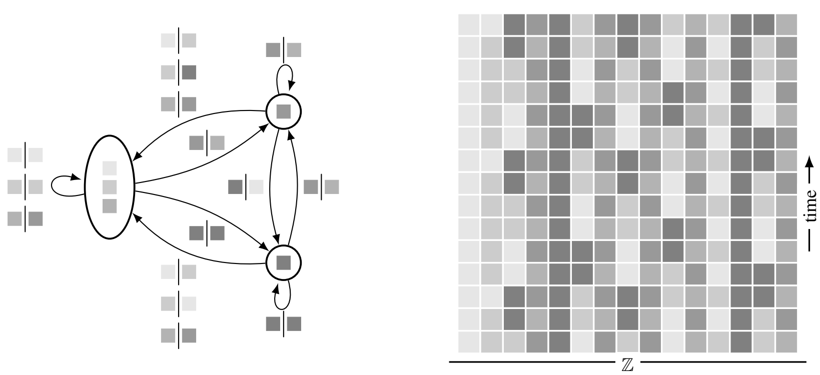


Fig. 10.9 Any reset Mealy automaton $(\Sigma, \Sigma, \mathrm{id}, \rho)$ (or its minimization, on the left) corresponds to a cellular automaton $(\Sigma, (a,b) \mapsto \rho_b(a))$ (with a fragment of a space-time diagram on the right), according to Proposition 10.24.

On the one hand, systematic experimentations on small reset Mealy automata (as well as randomly chosen large ones) seems to indicate that whenever a reset automaton group is finite, the semigroup generated by the dual automaton is very small. On the other hand, Delacourt and Ollinger have managed to inject some computations in one-way center-permutive cellular automata [18]. Tilting in opposite directions, these cooperating two points of view remains for now this crucial open question in some swing state.

10.2.3 Helix graphs and rigidity

The notion of a helix graph is a dynamical tool introduced in [1] and can be thought as some one-dimensional tiling.

Definition 10.25. The *helix graph* $\mathscr{H}_{n,k}$ of a Mealy automaton $\mathscr{A} = (Q, \Sigma, \delta, \rho)$ is defined to be the directed graph with nodes $Q^n \times \Sigma^k$ and arrows

$$\boxed{u, v} \longrightarrow \boxed{\delta_v(u), \rho_u(v)}$$

for all $(u, v) \in Q^n \times \Sigma^k$ (see Figure 10.10).

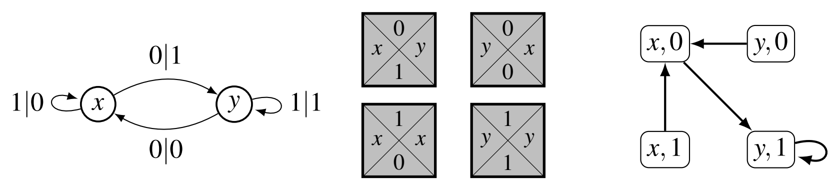


Fig. 10.10 The Mealy automaton $\mathscr{L}$ generating the lamplighter group, its associated Wang tileset $\mathscr{T}(\mathscr{L})$, and its helix graph $\mathscr{H}_{1,1}(\mathscr{L})$.

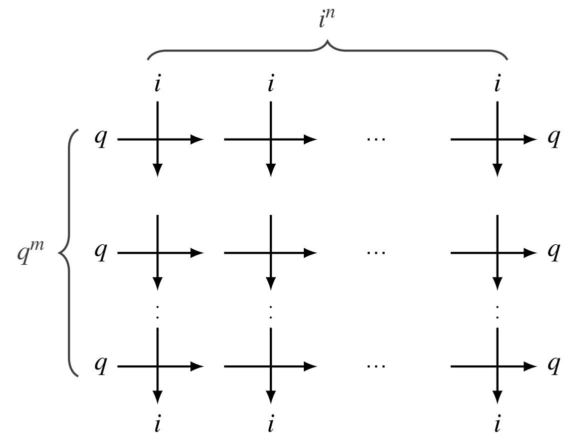
Merging together a Mealy automaton $\mathscr{A}$ and its dual $\mathfrak{d}\mathscr{A}$, their helix graphs allow to capture the whole dynamics by placing on a same footing the symmetric roles of the state set and the alphabet. This way, the two faces of the coin are coalesced into one.

Let us mention that a helix graph could be defined for any letter-to-letter transducer where input and output alphabets coincide. Such a transducer is a Mealy automaton if and only if from any vertex starts a unique edge.

Proposition 10.26. *If the group generated by an invertible-reversible Mealy automaton is finite, then any of its helix graphs is is a union of disjoint cycles.*

Proof. Let $\mathscr{A} = (Q, \Sigma, \delta, \rho)$ be an invertible-reversible Mealy automaton that generates a finite group. Theorem 10.7 implies that $\mathfrak{d}\mathscr{A}$ generates a finite group as well. Let $\mathscr{H}$ be the helix graph of $\mathscr{A}$ and s be the map from the finite set of vertices of $\mathscr{H}$ into itself that maps any vertex to its (unique) successor. The helix graph $\mathscr{H}$ is a union of disjoint cycles if and only if the map s is bijective, that is, if and only if the map s is surjective.

Let $x \in Q$ and $i \in \Sigma$. We have to show that the vertex (q,i) admits a unique predecessor in $\mathscr{H}$. There exist integers $m,n > 0$ satisfying $\rho_q^m = \rho_{q^m} = \mathrm{id}_{\langle \mathscr{A} \rangle}$ and $\delta_i^n = \delta_{i^n} = \mathrm{id}_{\langle \mathfrak{d}\mathscr{A} \rangle}$. This means that there is a transition $q^m \xrightarrow{i^n|i^n} q^m$ in the associated automaton of order (m,n). The corresponding cross-diagram is:



The most south-east cross gives a predecessor to the vertex (q,i). □

The condition of Proposition 10.26 is not sufficient: there are invertible-reversible Mealy automata whose helix graph is a union of disjoint cycles and that generate an infinite group, as for instance the Alešin automaton displayed on Figure 10.11. Proposition 10.27 characterizes the class of those invertible-reversible automata whose helix graph is a union of disjoint cycles.

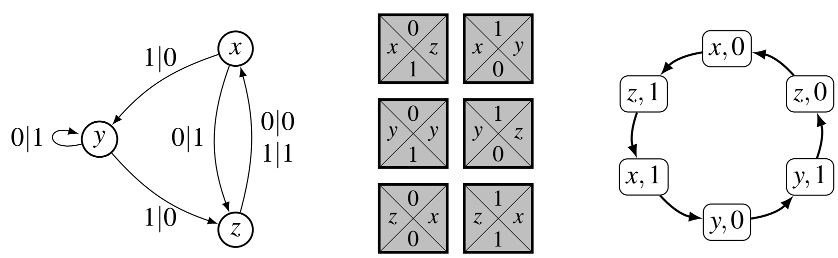


Fig. 10.11 Alešin automaton generates an infinite group, namely the rank 3 free group. Its helix graph is a cycle.

Proposition 10.27. *Let $\mathscr{A}$ be an invertible-reversible Mealy automaton. The following are equivalent.*

1. $\mathscr{A}$ is bireversible;
2. $\mathfrak{dida}\mathscr{A}$ is a Mealy automaton;
3. the helix graph of $\mathscr{A}$ is a union of disjoint cycles.

Proof. $(1 \Rightarrow 2)$ By hypothesis of bireversibility, $\mathscr{A}$ is invertible and $\mathfrak{i}\mathscr{A}$ is reversible. The latter means that $\mathfrak{d}\mathfrak{i}\mathscr{A}$ is invertible. Therefore $\mathfrak{i}\mathfrak{d}\mathfrak{i}\mathscr{A}$ is a Mealy automaton, and so is its dual $\mathfrak{d}\mathfrak{i}\mathfrak{d}\mathfrak{i}\mathscr{A}$.

$(2 \Rightarrow 1)$ First, by hypothesis of invertibility, $\mathfrak{i}\mathscr{A}$ is a Mealy automaton, and so is its dual $\mathfrak{d}\mathfrak{i}\mathscr{A}$. Next, since $\mathfrak{d}\mathfrak{i}\mathfrak{d}\mathfrak{i}\mathscr{A}$ is assumed to be a Mealy automaton, so is its dual $\mathfrak{i}\mathfrak{d}\mathfrak{i}\mathscr{A}$. This means that $\mathfrak{d}\mathfrak{i}\mathscr{A}$ is invertible, that is, $\mathfrak{i}\mathscr{A}$ reversible. We deduce that $\mathscr{A}$ is bireversible.

$(2 \Leftrightarrow 3)$ In any helix graph (of a Mealy automaton), each vertex admits a unique successor. Such a helix graph is a union of disjoint cycles if and only each vertex admits a unique predecessor. Let $\mathscr{G}$ be the graph with set of vertices $Q^{-1} \times \Sigma^{-1}$ and with an edge $(y^{-1}, j^{-1}) \to (x^{-1}, i^{-1})$ whenever $(x,i) \to (y,j)$ is an edge in the helix graph $\mathscr{H}$ of $\mathscr{A}$: the graph $\mathscr{G}$ is the helix graph of $\mathfrak{d}\mathfrak{i}\mathfrak{d}\mathfrak{i}\mathscr{A}$:

- $(\Rightarrow)$ if $\mathfrak{d}\mathfrak{i}\mathfrak{d}\mathfrak{i}\mathscr{A}$ is a Mealy automaton, each vertex of $\mathscr{G}$ admits a unique successor, hence each vertex of $\mathscr{H}$ admits a unique predecessor, and $\mathscr{H}$ is a union of disjoint cycles;
- $(\Leftarrow)$ if $\mathscr{H}$ is a union of disjoint cycles, so is $\mathscr{G}$. This implies that $\mathfrak{d}\mathfrak{i}\mathfrak{d}\mathfrak{i}\mathscr{A}$ is a Mealy automaton. □

We deduce a simple infiniteness criterion, which is very easy to check.

Corollary 10.28. *Any invertible-reversible Mealy automaton which is not bireversible generates an infinite group.*

We can also state the following characterization:

Theorem 10.29. *Let $\mathscr{A}$ be an invertible-reversible Mealy automaton. The group $\langle\, \mathscr{A} \,\rangle$ is finite if and only if there exists an integer C such that, for all k,ℓ, the helix graphs $\widetilde{\mathscr{H}_{k,\ell}}$ of $\widetilde{\mathscr{A}}$ are unions of disjoint cycles of lengths bounded by C.*

Note that such a characterization is not effective and does not directly lead to a decision procedure of finiteness.

Recall that, for any invertible-reversible automaton $\mathscr{A}$ with state set Q and alphabet Σ, we let $\widetilde{\mathscr{A}}$ denote the extension with state set $Q \sqcup Q^{-1}$ and alphabet $\Sigma \sqcup \Sigma^{-1}$.

Proof. Assume first that $\langle\, \mathscr{A} \,\rangle$ is finite: so is $\langle\, \widetilde{\mathscr{A}} \,\rangle$ by Proposition 10.5. Proposition 10.26 shows that the helix graphs of any level are unions of disjoint cycles. It remains to prove that the lengths of these cycles are uniformly bounded. By Theorem 10.7, the group $\langle\, \mathfrak{d}\widetilde{\mathscr{A}} \,\rangle$ is finite as well. Let $\mathscr{C}$ be a cycle in a helix graph of $\widetilde{\mathscr{A}}$ and let $(u,v) \in (Q \sqcup Q^{-1})^* \times (\Sigma \sqcup \Sigma^{-1})^*$ be a node of this cycle. Each node of $\mathscr{C}$ is of the form $(h(u), g(v))$, where g (*resp.* h) is an element of $\langle\, \widetilde{\mathscr{A}} \,\rangle$ (*resp.* $\langle\, \mathfrak{d}\widetilde{\mathscr{A}} \,\rangle$). Since the nodes are pairwise distinct, the length of the cycle $\mathscr{C}$ is at most $|\langle\, \widetilde{\mathscr{A}} \,\rangle| \times |\langle\, \mathfrak{d}\widetilde{\mathscr{A}} \,\rangle|$.

Let us prove the converse and assume that the group $\langle\, \mathscr{A}\, \rangle$ is infinite: so is $\langle\, \widetilde{\mathscr{A}}\, \rangle$ by Proposition 10.5. First we argue that the orders of the elements of $\langle\, \widetilde{\mathscr{A}}\, \rangle$ are unbounded. Indeed, automata groups are residually finite by construction since they act faithfully on rooted locally finite trees. Moreover it follows from Zelmanov's solution of the restricted Burnside problem [54, 56, 57] that any residually finite group with bounded torsion is finite. Since $\langle\, \widetilde{\mathscr{A}}\, \rangle$ is infinite, the orders of its elements are unbounded.

There exists either $x \in (Q \sqcup Q^{-1})^*$ such that the order of ρ_x is infinite, or a sequence $(x_n)_{n\in\mathbb{N}} \subseteq (Q \sqcup Q^{-1})^*$ such that the sequence $(k_n)_n$ of orders of the elements ρ_{x_n} goes to infinity. We carry out the proof in the second case, the first one can be treated similarly. Let us concentrate on ρ_{x_n}, element of order k_n of $\langle\, \widetilde{\mathscr{A}}\, \rangle$. For all $1 \leq k < k_n$, there exists a word $u_k \in (\Sigma \sqcup \Sigma^{-1})^*$ satisfying $\rho^k_{x_n}(u_k) = u'_k \neq u_k$.

Say that a word $v \in (\Sigma \sqcup \Sigma^{-1})^*$ is *unital* if δ_v is the identity of $\langle\, \mathfrak{d}\widetilde{\mathscr{A}}\, \rangle$. Since $\langle\, \mathfrak{d}\widetilde{\mathscr{A}}\, \rangle$ is a group, the word u_k can be extended into a unital word $u_k v_k$. Set $w_n = u_1 v_1 \cdots u_{k_n-1} v_{k_n-1}$. By construction, we have $\rho_{x_n}(w_n) = u'_1 \cdots \neq w_n$. Since $u_1 v_1$ is unital, we also have:

$$\begin{aligned}\rho^2_{x_n}(w_n) &= \rho^2_{x_n}(u_1 v_1)\rho^2_{x_n}(u_2 v_2 \cdots u_{k_n-1} v_{k_n-1}) \\ &= \rho^2_{x_n}(u_1 v_1) u'_2 \cdots \neq w_n\,.\end{aligned}$$

In the same way, we prove that, for all $k < k_n$, we have $\rho^k_{x_n}(w_n) \neq w_n$.

In the helix graph of $\widetilde{\mathscr{A}}$ of level $(|x_n|, |w_n|)$, consider the cycle containing the node (x_n, w_n). Since w_n is unital, the successors of (x_n, w_n) on the cycle are: $(x_n, \rho_{x_n}(w_n))$, $(x_n, \rho^2_{x_n}(w_n))$, ... Therefore the cycle is of length k_n. Since k_n goes to infinity, the lengths of the cycles of the helix graphs of $\widetilde{\mathscr{A}}$ are not uniformly bounded. □

We finally mention a relevant perpective based on helix graphs. The original observation is that the cyclic helix graph of Alešin on Figure 10.11 happens to be what we call *rigid*: it admits a trivial symmetry group. We can simply illustrate the phenomenon on Figure 10.12 by comparing the latter (at the center) with the other four cyclic helix graphs with the same size.

This helix rigidity notion can be easily translated in term of size of equivalence class to formulate the following claimed criterion:

Conjecture 10.30. Let $\mathscr{A}$ be a bireversible automaton. If the (non-trivial) $\mathfrak{md}$-reduction of $\mathscr{A}$ admits an equivalence class of maximal size, then the group $\langle\, \mathscr{A}\, \rangle$ is infinite.

Note that Conjecture 10.30 is trivially true for 2-letter and/or 2-state bireversible automata by Theorem 10.9. Beyond, it would apply to more and more automata as suggesting by some experimentations:

states	letters	$\mathfrak{md}$-reduced	rigid	%
2	5	190	154	81%
3	3	148	140	95%
4	3	6293	6117	97%

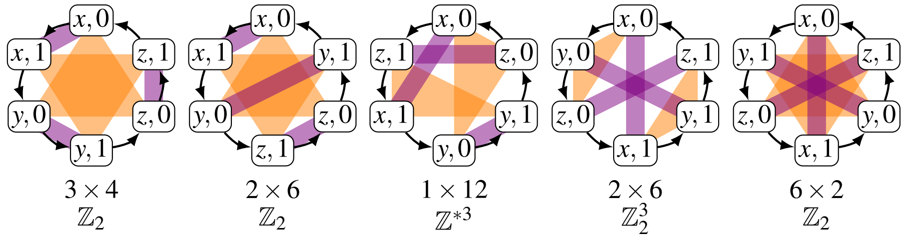


Fig. 10.12 The five helix graphs made of a unique cycle of size $2 \cdot 3 = 6$. Additional colors (orange for the letters *vs* violet for the states) emphasize the symmetries (or the absence of such ones). We give first the size s of the symmetry group of the helix graph together with the size e of the equivalence class of the corresponding automaton (with $s \times e = 2!\,3!$), and then the generated group (only its finite or infinite nature matters here).

Furthermore, it seems that the lack of helix rigidity yields a conjugator such that the $\mathfrak{md}$-reduction of the conjugated is smaller, and so on. Based on the helix rigidity, we claim the following generalization of Corollary 10.8 and Theorem 10.9.

Conjecture 10.31. Let $\mathscr{A}$ be a bireversible automaton. The group $\langle\, \mathscr{A} \,\rangle$ is finite if and only if some conjugate of $\mathscr{A}$ is $\mathfrak{md}$-trivial.

The latter would validate in particular the idea that $\mathfrak{md}$-reduction allows to solve the finiteness problem for prime bireversible automata, that is, for those bireversible that admit no non-trivial decomposition. This notion of primality turns out to be recurrent in various contexts and seems to be especially relevant for the Burnside problem in Section 10.3.

Let us come back to the specimens from Figure 10.3. We have seen that they are $\mathfrak{md}$-reduced and generate finite groups. Each of their helix graphs admits a non-trivial symmetry, hence it is not rigid.

For instance, the leftmost one (on Figure 10.3) admits the symmetry $0 \leftrightarrow 1, a \leftrightarrow c, b \leftrightarrow d$. As shown on Figure 10.13, it admits a decomposition as a product of two components $\mathscr{C}_1$ and $\mathscr{C}_2$. The conjugate $\mathscr{C}_2 \times \mathscr{C}_1$ happens to be $\mathfrak{md}$-trivial. The statement of Conjecture 10.31 might be stronger, and we could use some notion of $\mathfrak{mdc}$-reduction that would alternate $\mathfrak{md}$-reduction and conjugacy.

10.2.4 Automat-ic-on semigroups

Considering tilings as computations, they can be used to encode some rewriting systems, that, according to [15, 16], we call *quadratic normalizations*. This allow to develop an effective and natural approach to interpret any semigroup admitting a special language of greedy normal forms as an automaton semigroup, namely the semigroup generated by a Mealy automaton encoding the behavior of such a language of greedy normal forms under one-sided multiplication [47].

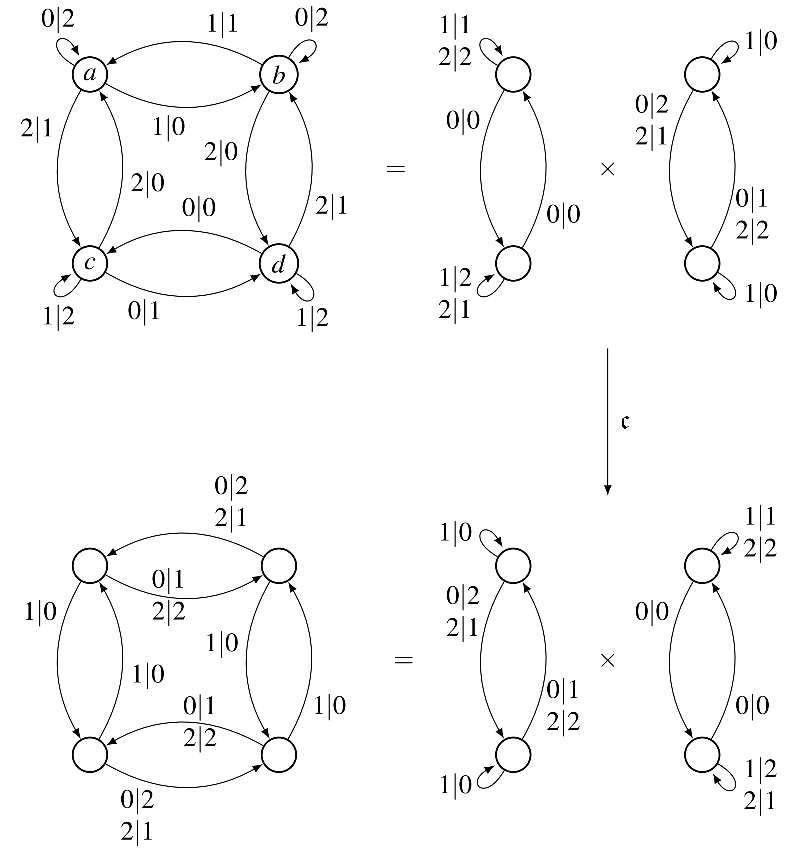


Fig. 10.13 An $\mathfrak{m}\mathfrak{d}$-reduced bireversible automaton, that admits an $\mathfrak{m}\mathfrak{d}$-trivial conjugate.

The framework embraces many of the well-known classes of (automatic) semigroups: finite monoids, free semigroups, free commutative monoids, trace or divisibility monoids, braid or Artin-Tits or Krammer or Garside monoids, Baumslag-Solitar semigroups, etc. Like plactic monoids or Chinese monoids, some neither left- nor right-cancellative automatic semigroups are also investigated, as well as some residually finite variations of the bicyclic monoid.

It is worthwhile noting that, in all these cases, ”being an automatic semigroup” and ”being an automaton semigroup” become dual properties in a very automata-theoretical sense.

Definition 10.32. Assume that S is a semigroup with a generating subfamily Q.

$$\mathrm{EV} : Q^+ \underset{\mathrm{NF}}{\overset{}{\rightleftarrows}} S$$

A *normal form* for (S,Q) is a (set-theoretic) section of the canonical projection EV from the language of Q-words onto S, that is, a map NF that assigns to each element of S a distinguished representative Q-word.

Whenever $\mathrm{NF}(S)$ is regular, it provides a *right-automatic structure* for S if the language $\mathscr{L}_q = \{\ (\mathrm{NF}(a),\mathrm{NF}(aq)) : a \in S\ \}$ is regular for each $q \in Q$. The semigroup S then can be called a *(right-)automatic semigroup*.

We mention here the thorough and precious study in [34] of the different notions (right- or left-reading *vs* right- or left-multiplication) of automaticity for semigroups.

Remark 10.33. In his seminal work [19, Chapter 9], Thurston shows how the set of these different automata recognizing the multiplication—that is, recognizing the languages of those pairs of normal forms of elements differing by a right factor $q \in Q$—and the one recognizing the equality in Definition 10.32 can be replaced with advantage by a single letter-to-letter transducer (Definition 10.44) that computes the normal forms via iterated runs, each run both providing one brick of the final normal from and outputting a word still to be normalized.

One will often consider the associated normalization $\mathrm{N} = \mathrm{NF} \circ \mathrm{EV}$.

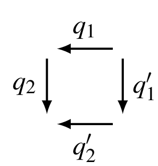
Definition 10.34. A *normalization* is a pair (Q,N), where Q is a set and N is a map from Q^+ to itself satisfying, for all Q-words u,v,w:

- $|\mathrm{N}(w)| = |w|$,
- $|w| = 1 \Rightarrow \mathrm{N}(w) = w$,
- $\mathrm{N}(u\,\mathrm{N}(w)\,v) = \mathrm{N}(u\,w\,v)$.

A Q-word w satisfying $\mathrm{N}(w) = w$ is called N-*normal*. If S is a semigroup (*resp.* a monoid), we say that (Q,N) is a *normalization for S* if S admits the presentation

$$\langle\ Q : \{w = \mathrm{N}(w) \mid w \in Q^+\}\ \rangle_+ \quad (\textit{resp.}\ \langle\ Q : \{w = \mathrm{N}(w) \mid w \in Q^*\}\ \rangle_+^1\ ).$$

Following [17], we associate with every element $q \in Q$ a q-labeled edge and with a product the concatenation of the corresponding edges, and represent equalities in the ambient semigroup using commutative diagrams, what we call here *square-diagram*: for instance, the following square illustrates an equality $q_1q_2 = q_1'q_2'$.



For a normalization (Q,N), we let $\overline{\mathrm{N}}$ denote the restriction of N to Q^2 and, for $i \geq 1$, by $\overline{\mathrm{N}}_i$ the (partial) map from Q^+ to itself that consists in applying $\overline{\mathrm{N}}$ to the entries in positions i and $i+1$. For any finite sequence $\boldsymbol{i} = i_1 \cdots i_n$ of positive integers, we write $\overline{\mathrm{N}}_{\boldsymbol{i}}$ for the composite map $\overline{\mathrm{N}}_{i_n} \circ \cdots \circ \overline{\mathrm{N}}_{i_1}$ (so $\overline{\mathrm{N}}_{i_1}$ is applied first).

Definition 10.35. A normalization (Q, N) is *quadratic* if both following conditions hold:

- a Q-word w is N-normal if, and only if, so is every length-two factor of w;
- for every Q-word w, there exists a finite sequence i of positions, depending on w, such that $\mathrm{N}(w)$ is equal to $\overline{\mathrm{N}}_i(w)$.

Definition 10.36. As illustrated on Figure 10.14, with any quadratic normalization (Q, N) is associated its *breadth* (d, p) (called minimal left and right classes in [15, 16]) defined as:

$$d = \max_{(q_1,q_2,q_3)\in Q^3} \min\{\,\ell : \mathrm{N}(q_1q_2q_3) = \overline{\mathrm{N}}_{\underbrace{212\ldots}_{\text{length}\,\ell}}(q_1q_2q_3)\},$$

and

$$p = \max_{(q_1,q_2,q_3)\in Q^3} \min\{\,\ell : \mathrm{N}(q_1q_2q_3) = \overline{\mathrm{N}}_{\underbrace{121\ldots}_{\text{length}\,\ell}}(q_1q_2q_3)\}.$$

Such a breadth is ensured to be finite provided that Q is finite, and then satisfies $|d - p| \leq 1$. For $d \leq 4$ and $p \leq 3$, (Q, N) is said to satisfy Condition (⬟), corresponding to the so-called *domino rule* in [15, 16, 17] but with a different reading direction.

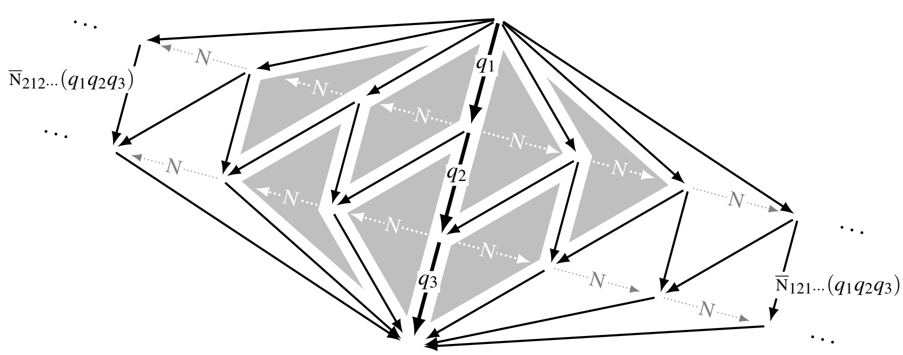


Fig. 10.14 From an initial Q-word $q_1q_2q_3$, one applies normalizations on the first and the second 2-factors alternatively up to stabilization, beginning either on the first 2-factor q_1q_2 (on the right-hand side here) or on the second q_2q_3. The gray zone corresponds to Condition (⬟) as defined in Definition 10.36.

Remark 10.37. One can build quadratic normalizations with a (finite) breadth arbitrarily high (see [16]). A natural question would be to know what is the maximal breadth for a fixed size of Q. For instance, the semigroup

$$\mathbf{W} = \langle\, \mathtt{a}, \mathtt{b}, \mathtt{c} : \mathtt{aa} = \mathtt{cc} = \mathtt{bc}, \mathtt{ba} = \mathtt{cb} = \mathtt{ab}, \mathtt{bb} = \mathtt{ca} = \mathtt{ac} \,\rangle_+$$

admits a quadratic normalization with breadth $(11, 10)$, corresponding to the maximal breadth for $|Q| = 3$. Such a large breadth corresponds with a great height of the associated $\overline{\text{N}}$-graph as displayed on Figure 10.15. An easy general observation is the following: the larger the breadth, the higher the $\overline{\text{N}}$-graph, the most the associated semigroup approximates the rank 1 free semigroup. Here, as an ultimate example, **W** is precisely isomorphic to $\langle\, \texttt{a} :\ \rangle_+$. Conversely, any quadratic normalization (Q, N) with a zero breadth corresponds to the rank $|Q|$ free semigroup $\langle\, Q :\ \rangle_+$ (except for $|Q| = 1$).

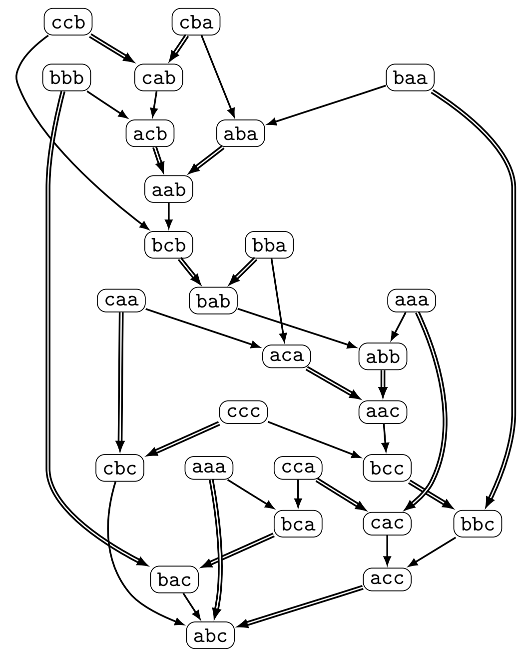


Fig. 10.15 The $\overline{\text{N}}_{1,2}$-graph for the quadratic normalization associated with **W**: simple arrows correspond to $\overline{\text{N}}_1$ and double arrows to $\overline{\text{N}}_2$, while loops (on the sinks and on some sources) are simply omitted for better readability.

The first main result of [16] is an axiomatization of these quadratic normalizations satisfying Condition (⬟) in terms of their restrictions to length-two words: any idempotent map $\overline{\text{N}}$ on Q^2 that satisfies $\overline{\text{N}}_{2121} = \overline{\text{N}}_{121} = \overline{\text{N}}_{1212}$ extends into a quadratic normalization (Q, N) satisfying Condition (⬟). For larger breadths, a map on length-two words normalizing length-three words need not normalize words of greater length.

The second main result involves termination. Every quadratic normalization (Q, N) gives rise to a quadratic rewriting system, namely the one with rules $w \longrightarrow \overline{\text{N}}(w)$ for $w \in Q^2$. By Definition 10.35, such a rewriting system is confluent and normalizing, meaning that, for every initial word, there exists a finite sequence of rewriting steps leading to a unique N-normal word, but its convergence, meaning that *any* sequence of rewriting steps is finite, is a quite different problem.

Theorem 10.38. [16] *If* (Q, N) *is a quadratic normalization satisfying Condition* (⬟), *then the associated rewriting system is convergent.*

More precisely, every rewriting sequence starting from a word of Q^p has length at most $\frac{p(p-1)}{2}$ (*resp.* $2^p - p - 1$) in the case of a breadth $(3,3)$ (*resp.* either $(3,4)$ or $(4,3)$). Theorem 10.38 is essentially optimal since there exist non-convergent rewriting systems with breadth $(4,4)$.

The results of the current subsection rely on the special Condition (⬟) outlined by Dehornoy and Guiraud (see [16]). However, none of their results (in particular Theorem 10.38 mentioned here for completeness) is neither applied nor required here. We want to emphasize that Condition (⬟) appears as a common denominator for the different approaches.

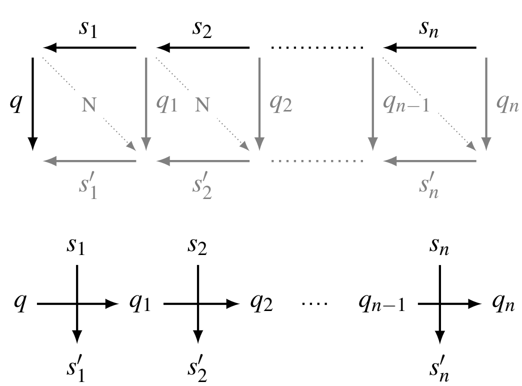
All the ingredients are now in place to effectively and naturally interpret as an automaton monoid any automatic monoid admitting a special language of normal forms—namely, a quadratic normalization satisfying Condition (⬟). The point is to construct a Mealy automaton encoding the behavior of its language of normal forms under one-sided multiplication.

Definition 10.39. Assume that S is a semigroup with a quadratic normalization (Q, N). We define the Mealy automaton $\mathscr{A}_{S,Q,\mathrm{N}} = (Q, Q, \delta, \rho)$ such that, for every $(a,b) \in Q^2$, $\rho_b(a)$ is the rightmost element of Q in the normal form $\mathrm{N}(ab)$ of ab and $\delta_a(b)$ is the left one:

$$\mathrm{N}(ab) = \delta_a(b)\rho_b(a).$$

The latter correspondence can be simply interpreted via square-diagram *vs* cross-diagram:

Then, for $\mathrm{N}(s) = s_n \cdots s_1$ and $\mathrm{N}(sq) = q_n s'_n \cdots s'_1$, we obtain diagrammatically:



We choose on purpose to always draw a normalization square-diagram backward, such that it coincides with the associated cross-diagram. The function ρ_q induced by the state q maps any normal form (read backward) to the normal form of the right-product by q (read backward).

We now aim to strike reasonable (most often optimal) hypotheses for a quadratic normalization (Q,N), associated with an original semigroup S to generate a semigroup $\langle\ \mathscr{A}_{S,Q,\mathrm{N}}\ \rangle_+$ that approximates S as sharply as possible. Since the generating sets coincide by Definition 10.39, we shall first focus on the case where S should be a quotient of $\langle\ \mathscr{A}_{S,Q,\mathrm{N}}\ \rangle_+$ (top-approximation), and next, on the case where $\langle\ \mathscr{A}_{S,Q,\mathrm{N}}\ \rangle_+$ should be a quotient of S (bottom-approximation).

Before establishing the top-approximation statement, we just recall that semigroups could appear much more difficult to handle, especially when it comes to automaticity (see [34]) or selfsimilarity (see [7, 8]). To any (not monoid) semigroup S with a quadratic normalization (Q,N), one obtains a monoid S^1 with a quadratic normalization (Q^1,N^1) by adjoining a unit $\mathtt{1}$ and then by setting $Q^1 = Q \sqcup \{\mathtt{1}\}$ and defining N^1 by $\mathrm{N}^1(w) = \mathrm{N}(w)$ and

$$\mathrm{N}^1(\mathtt{1}w) = \mathrm{N}^1(w\mathtt{1}) = \mathtt{1}\mathrm{N}(w) \qquad (\boxed{\scriptstyle\mathrm{N}^1})$$

for $w \in Q^+$. The choice made for Condition ($\boxed{\scriptstyle\mathrm{N}^1}$) becomes natural whenever we think of the (adjoined or not) unit $\mathtt{1}$ as some *dummy* element that simply ensures the length-preserving property for N^1 (see Definition 10.34 and also [16, Section 2.2]).

Lemma 10.40. *Assume that S is a monoid with a quadratic normalization (Q,N) satisfying Condition ($\boxed{\scriptstyle\mathrm{N}^1}$). Then the Mealy automaton $\mathscr{A}_{S,Q,\mathrm{N}}$ generates a monoid of which S is a quotient.*

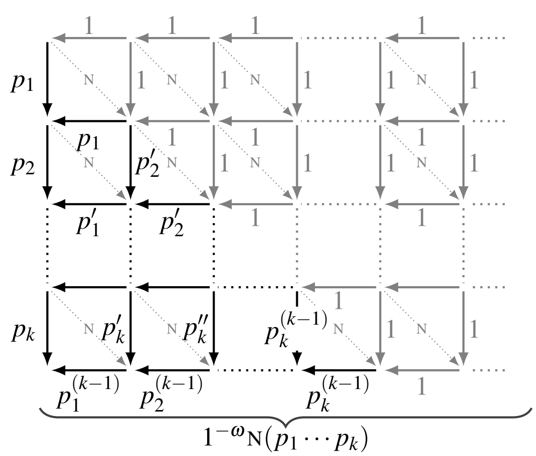


Fig. 10.16 Proof of Lemma 10.40: any Q-words inducing a same action have normal forms that coincide.

Proof. Let $S^1 = Q^*/\equiv_{\mathrm{N}^1}$ and $\mathscr{A}_{S^1,Q^1,\mathrm{N}^1} = (Q^1, Q^1, \delta, \rho)$ as in Definition 10.39. We have to prove that any relation in $\langle\, \mathscr{A}_{S^1,Q^1,\mathrm{N}^1} \,\rangle_+^1$ is a relation in S^1, thereby implying for all $u, v \in Q^*$:

$$\rho_u = \rho_v \implies u \equiv_{\mathrm{N}^1} v.$$

Let $\rho_{p_1} \cdots \rho_{p_k} = \rho_{q_1} \cdots \rho_{q_\ell}$ be some relation in $\langle\, \mathscr{A}_{S,Q,\mathrm{N}} \,\rangle_+^1$ with $p_i \in Q$ for $0 \le i \le k$ and $q_j \in Q$ for $0 \le j \le \ell$. Any word w over Q admits hence the same image under $\rho_{p_1} \cdots \rho_{p_k}$ and under $\rho_{q_1} \cdots \rho_{q_\ell}$. By taking $w = \mathtt{1}^\omega$ (or any sufficiently long power of $\mathtt{1}$, precisely any word from $\mathtt{1}^{\max(k,\ell)}\mathtt{1}^*$), such a common image corresponds to their normal forms by very definition of $\mathscr{A}_{S^1,Q^1,\mathrm{N}^1}$ (see Figure 10.16). Therefore the resulting letterwise equality $\mathtt{1}^{-\omega}\mathrm{N}(p_1 \cdots p_k) = \mathtt{1}^{-\omega}\mathrm{N}(q_1 \cdots q_\ell)$ (where $\mathtt{1}^{-\omega}$ denotes the left-infinite word $\cdots \mathtt{111}$) implies that the two corresponding Q-words $p_1 \cdots p_k$ and $q_1 \cdots q_\ell$ represent a same element in S^1 by definition of N^1. $\square$

Although specific to a monoidal framework and then requiring the innocuous Condition ($\boxed{\scriptstyle \mathrm{N}^1}$), the previous straightforward proof relies only on the definition of a quadratic normalization and on the well-fitted associated Mealy automaton (Definition 10.39). For the bottom-approximation statement, we consider an extra assumption, which happens to be necessary and sufficient.

Proposition 10.41. *Assume that S is a semigroup with a quadratic normalization (Q, N). If Condition (⬬) is satisfied, then the Mealy automaton $\mathscr{A}_{S,Q,\mathrm{N}}$ generates a semigroup quotient of S. The converse holds provided that Condition ($\boxed{\scriptstyle \mathrm{N}^1}$) is satisfied.*

Proof. Let $S = Q^+/\equiv_{\mathrm{N}}$ and $\mathscr{A}_{S,Q,\mathrm{N}} = (Q, Q, \delta, \rho)$ as in Definition 10.39.

($\Leftarrow$) Assume that Condition (⬬) is satisfied and that there exists $(a,b,c,d) \in Q^4$ with $ab \equiv_{\mathrm{N}} cd$. We have to prove $\rho_{ab} = \rho_{cd}$. Without lost of generality, the word ab can be supposed to be N-normal, that is, $\mathrm{N}(ab) = \mathrm{N}(cd) = ab$ holds.

Let $u = qv \in Q^n$ for some $n > 0$ and $q \in Q$. We shall prove both $\rho_{ab}(u) = \rho_{cd}(v)$ (coordinatewise) and $\delta_u(ab) \equiv_{\mathrm{N}} \delta_v(cd)$ by induction on $n > 0$. For $n = 1$, we obtain the two square-diagrams on Figure 10.17 (left). With these notations, we have to prove $q_0'' = q_1''$ and $a'b' \equiv_{\mathrm{N}} c'd'$, the latter meaning $\mathrm{N}(a'b') = \mathrm{N}(c'd')$, that is, with the notations from Figure 10.17, the conjunction of $a'' = c''$ and $b'' = d''$. Now these three equalities hold whenever (Q, N) satisfies Condition (⬬), as shown on Figure 10.17 (right).

This allows to proceed the induction and to prove the implication ($\Leftarrow$).

($\Rightarrow$) Consider an arbitrary length-three word over Q, say qcd. Let a, b denote the elements in Q satisfying $\mathrm{N}(cd) = ab$. By definition, we deduce $ab \equiv_{\mathrm{N}} cd$. This implies $\rho_{ab} = \rho_{cd}$ by hypothesis. In particular, the images of any word qv under ρ_{ab} and ρ_{cd} coincide: $\rho_{ab}(qv) = \rho_{cd}(qv)$, hence

$$\rho_{ab}(q) = q_0'' = q_1'' = \rho_{cd}(q)$$

and

$$\rho_{\delta_q(ab)}(v) = \rho_{a'b'}(v) = \rho_{c'd'}(v) = \rho_{\delta_q(cd)}(v)$$

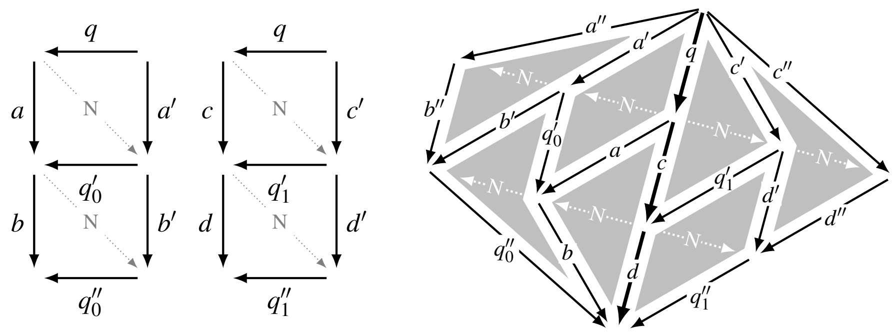


Fig. 10.17 Proof of Proposition 10.41: initial data (left) can be pasted into Condition (⬟) (right).

(with notations of Figure 10.17). The last equality holds for any original word $v \in Q^*$ and implies $\rho_{a'b'} = \rho_{c'd'}$. Whenever, Condition (◻) is satisfied, we deduce $\mathrm{N}(a'b') = \mathrm{N}(c'd')$ according to Lemma 10.40. We obtain

$$\overline{\mathrm{N}}_{121}(qcd) = \overline{\mathrm{N}}_{2121}(qcd).$$

Therefore (Q, N) satisfies Condition (⬟). □

Gathering Lemma 10.40 and Proposition 10.41, we obtain the following result.

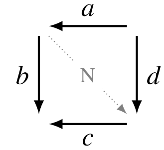
Theorem 10.42. *Assume that S is a monoid with a quadratic normalization (Q, N) satisfying Conditions (◻) and (⬟). Then the Mealy automaton $\mathscr{A}_{S,Q,\mathrm{N}}$ generates a monoid isomorphic to S.*

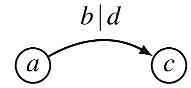
Proof. By construction, S and $\langle\, \mathscr{A}_{S,Q,\mathrm{N}} \,\rangle_+^1$ share a same generating subset Q. Now, any defining relation for S maps to a defining relation for $\langle\, \mathscr{A}_{S,Q,\mathrm{N}} \,\rangle_+^1$ by Proposition 10.41, and conversely by Lemma 10.40. □

Corollary 10.43. *Any monoid with a quadratic normalization satisfying Conditions (◻) and (⬟) is residually finite.*

To conclude this subsection, we come back to Remark 10.33 about the original transducer approach by Thurston.

Definition 10.44. With any quadratic normalization (Q, N) is associated its *Thurston transducer* defined as the Mealy automaton $\mathscr{T}_{Q,\mathrm{N}}$ with state set Q, alphabet Q, and transitions as follows:





Corollary 10.45. *Assume that S is a monoid with a quadratic normalization* (Q,N) *satisfying Conditions* (▣) *and* (⬟). *The Thurston transducer* $\mathscr{T}_{Q,\mathrm{N}}$ *and the Mealy automaton* $\mathscr{A}_{S,Q,\mathrm{N}}$ *being dual automaton, S possesses both the explicitly dual properties of automaticity and selfsimilarity.*

One of the simplest non-trivial examples is the following. Many others can be found in [47, 48]. The automatic monoid $\mathbf{M} = \langle\ \mathtt{a},\mathtt{b} : \mathtt{ab} = \mathtt{a}\ \rangle^1_+$ admits a quadratic normalization (Q,N) with $Q = \{\mathtt{1},\mathtt{a},\mathtt{b}\}$, $\mathrm{N}(\mathtt{ab}) = \mathtt{1a}$, and $\mathrm{N}(xy) = xy$ for $(x,y) \in Q^2 \smallsetminus \{(\mathtt{a},\mathtt{b})\}$. The latter has width $(3,3)$, hence satisfies Condition (⬟). According to Theorem 10.42, $\mathbf{M}$ is therefore an automaton monoid. The corresponding Wang tileset and the Mealy automaton are displayed on Figure 10.18.

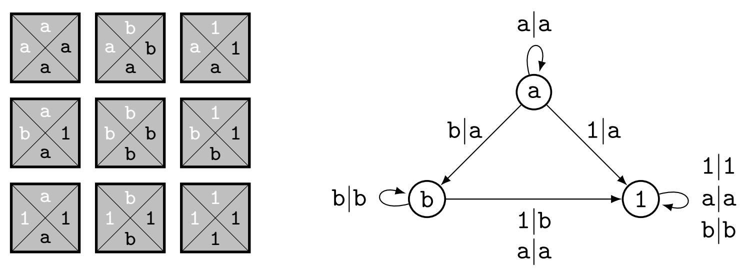


Fig. 10.18 The Wang tileset and the Mealy automaton associated with the monoid $\langle\ \mathtt{a},\mathtt{b} : \mathtt{ab} = \mathtt{a}\ \rangle^1_+$.

10.3 A matter of orbits

In this section, we detail some toggle switch between a classical notion from group theory—Schreier graphs—and some properties of an automaton group about its growth or the growth of its monogenic subgroups. In the first part (Subsection 10.3.1) we see how automaton groups can provide examples for questions on Schreier graphs, and in the second part (Subsection 10.3.2) we see how Schreier graphs viewed in a meta structure called an orbit tree or a Schreier trie[1] can give answers to questions about these groups generated by invertible-reversible Mealy automata.

The first part considers essentially polynomial-activity automata which were introduced by S. Sidki [51], whereas the second part considers reversible automata. As we will see, these families of Mealy automata are somehow diametrically opposed.

[1] This latter denomination is introduced for the first time in this chapter, motivated in Section 10.3.2.

10.3.1 Schreier graphs and polynomial-activity automata

There are many ways to define the Schreier graphs of a group acting on some set. Schreier graphs are essentially a generalization of Cayley graphs: let G be a group generated by S and acting on a set X, the vertices of its Schreier graph (depending on S) are the elements of X, and there is an edge $x \to y$ if y is the image of x under the action of some element of S. By considering the action of the group on itself by right-multiplication, this graph coincides with its Cayley graph. In this chapter, the considered group is always an automaton group which acts on the regular rooted tree, leading to the following definitions:

Definition 10.46. Let $\mathscr{A} = (Q, \Sigma, \delta, \rho)$ be a Mealy automaton. Let u be a finite or infinite word on Σ. The *Schreier graph* of $\langle\, \mathscr{A} \,\rangle$ *pinpointed* by u is the orbit of u under the action of $\langle\, \mathscr{A} \,\rangle$.

The *finite Schreier graph of level n* of $\langle\, \mathscr{A} \,\rangle$, $n \in \mathbb{N}$, is the union of the Schreier graphs pinpointed by the length n words and the *infinite Schreier graph* of $\langle\, \mathscr{A} \,\rangle$ is the union of the Schreier graphs pinpointed by the infinite words.

Pinpointed infinite Schreier graphs were for instance used by R. Grigorchuk and V.V. Nekrashevych to prove the existence of amenable actions for non-amenable groups [33].

The *growth* of the infinite Schreier graph pinpointed by some infinite word w is the sequence $(\gamma_n(w))_{n\in\mathbb{N}}$, where $\gamma_n(w)$ is the number of vertices in the closed ball $B(w,n)$ of radius n centered at w:

$$B(w,n) = \{v \in \Sigma^* \mid \exists q \in Q^{\leq n}, \rho_q(w) = v\} .$$

The growth of a group (which can be seen as the growth of its Cayley graph) has been studied for a long time now (see an introduction to the growth problem in Subsection 10.3.4). In this section we explore the growth of Schreier graphs of automaton groups, in the flavor of the work by I. Bondarenko [6], focusing on polynomial-activity automata.

A Mealy automaton has *polynomial activity* if its non-trivial cycles are pairwise disjoint (a trivial cycle is a state with a loop inducing the identity). Its *degree* is m if the largest number of these cycles connected by a path is $m+1$. A polynomial-activity automaton of degree -1 is *bounded*: any non-trivial cycles are disjoint and not connected by a directed path. The *degree* of a state of a Mealy automaton is the degree of the part of this automaton accessible from it. Note that the set of polynomial-activity automata of degree m forms a group for the usual product of Mealy automata.

The class of polynomial-activity automata is important within Mealy automata: it contains many interesting examples (let us mention the well-known automata of Grigorchuk, and of Gupta-Sidki) and its subfamily of bounded automata has interesting deciding properties, for example order and conjugacy [5], which are undecidable in the general framework of Mealy automata [53, 22].

We give here a property of Schreier graphs of polynomial-activity automata.

Theorem 10.47. *The infinite pinpointed Schreier graphs of a polynomial-activity automaton have subexponential growth.*

Sketch of the proof. Let $\mathscr{A} = (Q, \Sigma, \delta, \rho)$ be a polynomial-activity Mealy automaton of degree m. By passing at the right power of Σ, we can assume that $\mathscr{A}$ satisfies the following properties:

1. for every state q and every letter a, the state $\delta_a(q)$ either belongs to a cycle or has degree less than the degree of q;
2. for every state q in a non-trivial cycle, there exists a letter a such that $\delta_a(q) = q$.

If q is some state in some power $\mathscr{A}^n$ and $v \in \Sigma^*$ is a word, it is clear from the fact that $\mathscr{A}$ has polynomial activity of degree m that $\delta_v(q)$ is either a state in the same cycle as q, or it is not and then its degree is less than the degree of q. In fact, the following property, called **(P)**, can be proven to be true using quite technical considerations (see [6]): there exists a constant C such that for any state q of $\mathscr{A}^n$ and any word $v \in \Sigma^*$ of length greater than $|v| \geq C(\log n)^{(m+1)}$, either $\delta_v(q) = p_1 \cdots p_n$ and each state p_i of $\mathscr{A}$ has degree at most $m-1$, or $\delta_v(q)$ belongs to a loop, that is: there exists a letter $a \in \Sigma$ such that $\delta_{va}(q) = \delta_v(q)$. We let k denote the least integer greater to $C(\log n)^{(m+1)}$.

The proof of the theorem is now by induction on the degree m. For $m = -1$, as the non-trivial cycles of the automaton are disjoint and not connected by a directed path, the number of vertices of each of its pinpointed infinite Schreier graphs is bounded.

Now, suppose that every pinpointed infinite Schreier graph of any polynomial-activity automaton of degree less than m has subexponential growth, less than or equal to $|\Sigma|^{C_1(\log n)^m}$ for some constant C_1. Let $\mathscr{A}$ be a polynomial-activity automaton of degree m and $w = a_1 a_2 \cdots \in \Sigma^\omega$ be an infinite word. Let us look at the growth of the Schreier graph pinpointed by w, and in particular at the balls $B(w,n)$.

In what follows, we divide the word w in two factors: a length k prefix (where k is introduced above) whose contribution to the orbit of w is finite, and an infinite suffix $v = a_{k+1} a_{k+2} \cdots$ whose contribution to the orbit of w is shown to be subexponential through Property **(P)**.

Any word of $B(w,n)$ consists then of a length k prefix which is the image of $a_1 \cdots a_k$ by some element of $\langle\, \mathscr{A} \,\rangle$ and an infinite suffix which is the image of v by some element of

$$\mathscr{N}_{(n,k)} = \{ \delta_{a_1 \cdots a_k}(q) \mid q \in Q^{\leq n} \} .$$

We have a first bound for the size of the considered ball:

$$|B(w,n)| \leq |\Sigma|^k \cdot |\mathscr{N}_{(n,k)}(v)| ,$$

where $\mathscr{N}_{(n,k)}(v)$ denotes the orbit of v under the action of $\mathscr{N}_{(n,k)}$.

Let $\mathscr{B}_n$ denote now the set of the states of $\mathscr{A}^{\leq n}$ of degree less than $m-1$: $\mathscr{B}_n$ satisfies the induction property and so the size of $\mathscr{B}_n(v)$, the orbit of v under the action of $\mathscr{B}_n$, is bounded by $|\Sigma|^{C_1(\log n)^m}$. Furthermore, by Property **(P)**, for each $q \in \mathscr{N}_{(n,k)} \setminus \mathscr{B}_n$, there exists a letter a such that $\delta_a(q) = q$. Hence

$$\mathscr{N}_{(n,k)} \subseteq \bigcup_{a\in\Sigma} \mathscr{N}^a_{(n,k)} \cup \mathscr{B}_n \,,$$

where $\mathscr{N}^a_{(n,k)}$ is the subset of $\mathscr{N}_{(n,k)}$ formed by the elements which are stabilized by the action induced by a.

We consider now the orbit of v under the action of $\mathscr{N}^a_{(n,k)}$ for some letter a.

For $v = a^{\omega}_{k+1}$, there exist some letter $b \in \Sigma$ and some state $p \in Q^{|q|}$ such that:

$$\rho_q(v) = \begin{cases} b^{\omega} & \text{for } q \in \mathscr{N}^{a_{k+1}}_{(n,k)}\,, \\ b\rho_p(a^{\omega}_{k+1}) & \text{otherwise}\,. \end{cases}$$

Hence

$$|\mathscr{N}^{a_{k+1}}_{(n,k)}(v)| \leq |\Sigma| \quad \text{and} \quad |\mathscr{N}^{b}_{(n,k)}(v)| \leq |\Sigma|\cdot \mathscr{B}_n(v)\,,$$

for any letter $b \neq a_{k+1}$. For $v = a^{\ell}_{k+1}cv_1$ with $c \neq a_{k+1}$, we obtain similarly:

$$|\mathscr{N}^{a_{k+1}}_{(n,k)}(v)| \leq |\Sigma|^2 \quad \text{and} \quad |\mathscr{N}^{b}_{(n,k)}(v)| \leq |\Sigma|\cdot|\mathscr{B}_n(a^{\ell-1}_{k+1}cv_1)|\,,$$

for any letter $b \neq a_{k+1}$. The conclusion follows. □

10.3.2 Schreier tries and reversible automata

Until very recently, the Schreier graphs of an automaton (semi)group were seen individually, with no links between them. The notion of an *orbit tree* [41, 42] gives a new, more dynamical, vision of the whole set of finite Schreier graphs for a (semi)group generated by a reversible Mealy automaton. This notion stands on the connected components of the powers of a reversible Mealy automaton. In addition to the above remarks on these components (Remarks 10.1, 10.2, 10.3, and 10.4), let us add the following one:

Remark 10.48. It is known from [41] that a reversible automaton generates a finite semigroup if and only if the sizes of the connected components of its powers are uniformly bounded. It is straightforward to adapt the proof to show that a reversible automaton generates a finite semigroup if and only if the sizes of the minimizations of the connected components of its powers are uniformly bounded.

In the second part of this section, we will deal with labeled trees. There will be several possible label sets for these trees, but we need to set up some common terminology. All our trees are rooted, *i.e.* , with a selected vertex called the *root*. We will visualize the trees traditionally as growing down from the root. A *path* is a (possibly infinite) sequence of adjacent edges without backtracking from top to bottom. A path is said to be *initial* if it starts at the root of the tree. A *branch* is an infinite initial path. The lead-off vertex of a non-empty path e is denoted by $\top(e)$ and its terminal vertex by $\bot(e)$ whenever the path is finite.

The *level of a vertex* is its distance to the root and the *level of an edge* or *a path* is the level of its initial vertex.

The vertices of an orbit tree of a reversible Mealy automaton are the connected component of its powers, *i.e.* , the finite Schreier graphs of its dual.

In fact, when the automaton is reversible, its powers have a particular form: each connected component of its $(n+1)$-th power, for some integer n, can be seen as several copies of some connected component of its n-th power. More precisely, we have the following property:

Property 10.49. [39] Let $\mathscr{A}$ be a reversible Mealy automaton with state set Q, and n be a positive integer. The following links appear between the connected components of $\mathscr{A}^n$ and $\mathscr{A}^{n+1}$:

1. The length n prefixes of the states of a connected component of $\mathscr{A}^{n+1}$ belong to the same connected component in $\mathscr{A}^n$.
2. If u and v are two states of the same connected component in $\mathscr{A}^n$ $(u, v \in Q^n)$, then any connected component of $\mathscr{A}^{n+1}$ contains as many states prefixed by u as states prefixed by v.

The *orbit tree* $t(\mathscr{A})$ of the dual of a Mealy automaton $\mathscr{A} = (Q, \Sigma, \delta, \rho)$ has vertices the connected components of the powers of $\mathscr{A}$, and the incidence relation built by adding an element of Q: for any non-negative integer n, the connected component of a word $u \in Q^n$ is the parent of the connected component(s) of ux, for any $x \in Q$. This notion has been described in [50] for more general actions on trees and leads in this context to a graph. In the case of rooted trees (which is the only one we consider in this chapter), this graph is a tree as proved in [20]. To avoid the heaviness of saying "the orbit tree of the dual of the Mealy automaton", we introduce here a new terminology: if $\mathscr{A}$ is a Mealy automaton, we call the orbit tree of its dual its *Schreier trie*. Indeed, this tree is related to Schreier graphs as each of its levels is a finite Schreier graph of its dual, and it is a trie by looking at the states of the connected components labeling its vertices.

In [20], the vertices of a Schreier trie are labeled by the size of the connected component. Following [41], we use a different though equivalent labeling: if $\mathscr{C}$ is the parent of $\mathscr{D}$ in the orbit tree, we label the edge $\mathscr{C} \to \mathscr{D}$ by the ratio $\frac{|\mathscr{D}|}{|\mathscr{C}|}$ which is known to be an integer by Property 10.49.

Each vertex of $t(\mathscr{A})$ is labeled by a connected automaton with state set in Q^n, where n is the level of this vertex in the tree. By a minor abuse, we can consider that each vertex is labeled by a finite language in Q^n, or even by a word in Q^n.

Let u be a (possibly infinite) word over Q. The *path of* u in the Schreier trie $t(\mathscr{A})$ is the unique initial path going from the root through the connected components of the prefixes of u; u can be called *a representative* of this path (we can say equivalently that this path is *represented* by u or that the word u *represents* the path).

A branch of a Schreier trie is *active* if it has infinitely many coefficients greater than 1.

The Schreier trie $t(\mathscr{A})$ is built emphasizing the prefix-relation. Nevertheless, once this tree obtained, it is interesting to highlight some paths that are suffix compatible. A *1-self-liftable* (finite or infinite) path in this tree is a path such that each representant of a level has a suffix in the previous level of the same path. Note that this definition is equivalent by replacing "each" by "some" because of the reversibility of the automaton $\mathscr{A}$. For example, the path represented by q^ω, where q is a state of $\mathscr{A}$, is 1-self-liftable.

It is quite direct to obtain the following result.

Property 10.50. The sequence of the labels on a 1-self-liftable path decreases.

In a more general way, we say that an edge e of the tree is *liftable* to an edge f if each state of $\perp(e)$ admits a state of $\perp(f)$ as a suffix.

We extend the notion of children to edges: the *children* of some edge e are the edges f such that $\perp(e)$ and $\top(f)$ coincide. We can consider the children of an edge that are liftable to it and call them *legitimate* children.

Order and finiteness problems

The order problem and the finiteness problem are strongly related to structural properties of the Schreier trie of a reversible automaton:

Proposition 10.51. *The semigroup generated by a reversible Mealy automaton is finite if and only if the connected components of its powers have bounded size, that is, if and only if the number of labels greater than 1 in a branch of its Schreier trie is uniformly bounded.*

Sketch of the proof. Let $\mathscr{A}$ be a reversible Mealy automaton.

A connected component of a power of $\mathscr{A}$ is an orbit of the action of the semigroup generated by its dual. Hence if the sizes of these components are not bounded, the dual automaton generates an infinite semigroup, and so does $\mathscr{A}$ by Theorem 10.7. If these sizes are bounded, there exist two different powers of $\mathscr{A}$ that are equivalent, and $\mathscr{A}$ generates a finite semigroup. $\square$

Note that by Proposition 10.6, this result holds for the group generated by an invertible-reversible Mealy automton.

Proposition 10.52. *A state q of an invertible-reversible Mealy automaton induces an action of finite order if and only if the connected components of the powers of q have bounded sizes,* i.e. *if and only if the branch of the Schreier trie represented by q^ω is not active.*

Sketch of the proof. If q induces an action of finite order n, then q^n acts like the identity, as well as all the states in its connected component by Remark 10.3. Hence this connected component generates a finite group and the result follows by Proposition 10.51. If the connected components of the powers of q are bounded, eventually two of them are isomorphic, with the powers of q labeling the same state, hence q induces an action of finite order. $\square$

If j is a subtree of a Schreier trie, a *j-word* is a representative of one of its vertices. A *cyclic j-word* is a word whose all powers are representative of vertices of j.

10.3.3 The Burnside problem

The Burnside problem is a famous, long-standing question in group theory. In 1902, W. Burnside asked if a finitely generated group whose all elements have finite order—henceforth called a *Burnside group*—is necessarily finite [9].

The question stayed open until E.S. Golod and I. Shafarevitch exhibit in 1964 an infinite group satisfying Burnside's conditions [27, 28], hence solving the general Burnside problem. In the early 60's, V.M. Glushkov suggested using automata to attack the Burnside problem [24]. Later, S.V. Alešin [2] in 1972 and then R. Grigorchuk [29] in 1980 gave simple examples of automata generating infinite Burnside groups.

It is remarkable that all known examples of infinite Burnside automaton groups are generated by non-reversible Mealy automata. It has been proven in fact that two specific subfamilies of invertible-reversible Mealy automata cannot generate infinite Burnside groups: non-bireversible automata [26] and connected automata of prime size [25]. Both proofs rely on the construction of a particular branch in the Schreier trie of the automaton.

Let $\mathscr{A} = (Q, \Sigma, \delta, \rho)$ be an invertible-reversible Mealy automaton and $t(\mathscr{A})$ be its Schreier trie.

When the automaton is not bireversible.

In this case, the automaton $\mathscr{A}$ has at least one non-bireversible connected component, say $\mathscr{B}$, and all the 1-self-liftable branchs of the Schreier trie of $\mathscr{B}$ are active (see Lemma 10.53 below), hence all the elements of the semigroup generated by $\mathscr{B}$ have infinite order by Proposition 10.52.

Lemma 10.53. *The 1-self-liftable branchs of the Schreier trie of a connected invertible-reversible non-bireversible Mealy automaton are all active.*

Sketch of the proof. Because of the non-bireversibility of the automaton and of all of the connected components of its power, there exists no 1-self-liftable branch with a label 1. □

When the automaton is connected of prime size.

Of course, if the Schreier trie of this automaton has at least one active 1-self-liftable branch, the technique developped in the previous case can apply. But nothing ensures that the existence of such a branch. So we have to develop an alternative strategy in case $\mathscr{A}$ has no active 1-self-liftable branch. This strategy can be summarized into two steps:

step 1 exhibit a subtree of the Schreier trie whose labels from some level on are 1: hence there is only a finite number of applications induced by the language of the labels of this subtree;

step 2 prove that the action induced by a word of states has a uniform bounded power equivalent to the action induced by some word in the above language.

Then by E.I. Zelmanov's result [56, 57], the conclusion comes.

Assume that $t(\mathscr{A})$ has no active 1-self-liftable branch.

Jungle trees

We first build the tree of step 1. This tree, called *jungle tree*, starts with a linear part whose labels decrease and eventually ends as a regular tree with all labels 1.

Definition 10.54. Let e be a finite initial 1-self-liftable path such that:

- the lowest (*i.e.* the last) edge of e has at least two legitimate children;
- every of its legitimate children has label 1.

The *jungle tree* $j(e)$ of e is the subtree of $t(\mathscr{A})$ build as follows:

- it contains the path e — its *trunk*;
- it contains the regular tree rooted by $\perp(e)$, and formed by all the edges which are descendant of $\perp(e)$ and liftable to the lowest edge of e.

The *arity* of this jungle tree is the number of legitimate children of $\perp(e)$. Since every legitimate child has label 1, it is also the label of the last edge of e.

Words in $\perp(e)$ are called *stems*. They have all the same length which is the length of the trunk of $j(e)$.

A tree is a *jungle tree* if it is the jungle tree of some finite initial 1-self-liftable path.

Note that: (i) $t(\mathscr{A})$ has at least one jungle tree, since $\mathscr{A}$ has no active self-liftable branch by hypothesis; (ii) $t(\mathscr{A})$ has finitely many jungle trees.

Any jungle tree answers step 1. Let us look closer at the language of j-words, for some jungle tree j whose trunk has length n. In particular, the existence of cyclic j-words is ensured by the simple fact that any j-word of length $n\times(1+|Q|^n)$ admits a cyclic j-word as a factor. Besides, every cyclic j-word induces an action of finite order, bounded by a uniform constant depending on j, by Proposition 10.52.

From now on, j denotes a jungle tree of $\mathscr{A}$, whose trunk has length n.

The set of stems of j has several interesting properties.

Proposition 10.55. *The relation over stems $u \sim v$, defined by: there exists $s \in Q^*$ such that usv is a j-word and ρ_{us} acts like the identity on Σ^*, is an equivalence relation.*

Sketch of the proof. Because of the construction of the tree j, the j-words have a lot of good combinatorial properties (proved in [25]):

- any factor of a j-word is itself a j-word;
- if uv is a j-word, with $|v| \geq n$, what can follow uv in j is independant from u. In particular, if vw is also a j-word, then so is uvw;
- if $t, u, v \in Q^*$ are such that tuv is a j-word, then there exists $w \in Q^*$ such that $tuvwu$ is also a j-word (which means that if you are walking on a j-word and you have already seen some factor, you can find eventually this same factor);

□

Proposition 10.56. *The set of states which appear as first letter of a stem in a $\sim$-class has a cardinal which is greater than or equal to 2 and divides the number of states of $\mathscr{A}$.*

Corollary 10.57. *If $\mathscr{A}$ has a prime size, all the states appear as first letter of a stem in a fixed $\sim$-class.*

The main tool of this section is the following one:

Proposition 10.58. *Let $\mathscr{A} = (Q, \Sigma, \delta, \rho)$ be a connected bireversible Mealy automaton of prime size, with no active self-liftable branch. Let j be a jungle tree of its Schreier trie, and u some (possibly empty) j-word. Then for any state $q \in Q$, there exists $w \in Q^*$ such that uwq is a j-word and ρ_w acts like the identity of Σ^*.*

Proof. Let s be a stem such that us is a j-word: there exists a stem x with first letter q in the $\sim$-class of s, from Corollary 10.57, *i.e.* there exists $v \in Q^*$ such that svx is a j-word and ρ_{sv} acts like the identity of Σ^*. Conclusion comes from the fact that what can follow a stem depend only on its length n suffix. □

We now have all elements to prove the main result of this section.

Theorem 10.59. *A connected invertible-reversible Mealy automaton of prime size cannot generate an infinite Burnside group.*

Proof. Let $\mathscr{A}$ be a connected invertible-reversible Mealy automaton of prime size. If $\mathscr{A}$ is not bireversible, step 1 apply.

If $\mathscr{A}$ is bireversible and its Schreier trie has an active self-liftable branch, as seen above, the method of step 1 is still valid and $\langle \mathscr{A} \rangle$ has an element of infinite order.

Therefore we can assume that $\mathscr{A}$ is bireversible and its Schreier trie has no active self-liftable branch. Let us show that $\langle \mathscr{A} \rangle$ is finite. Let j be some jungle tree of $t(\mathscr{A})$. As in [41] we prove that for any word $u \in Q^*$, ρ_u has some uniformly bounded power which acts like some cyclic j-word.

Let $u \in Q^*$. We prove by induction that any prefix of u induces the same action as some j-word. It is obviously true for the empty prefix. Fix some $k < |u|$ and suppose that the prefix v of length k of u induces the same action as some j-word s. Let $x \in Q$ be the $(k+1)$-th letter of u. By Corollary 10.58, there exists a j-word w inducing the identity, such that swx is a j-word. But vx and swx induce the same action ; the result follows. Hence we obtain a j-word $u^{(1)}$ inducing the same action as u.

By the very same process, we can construct, for any $i \in \mathbb{N}$, a j-word $u^{(i)}$ inducing the same action as u, such that $u^{(1)}u^{(2)} \dots u^{(i)}$ is a j-word. Since the set Q^n is finite there exist $i < j$, $j - i \leq |Q|^n$, such that $u^{(i)}$ and $u^{(j)}$ have the same prefix of length n. Take $v = u^{(i)}u^{(i+1)} \dots u^{(j-1)}$: v is a cyclic j-word and induces the same action as u^{j-i}. As seen before, the fact that v is a cyclic j-word implies that the order of its induced action ρ_v is bounded by a constant depending only on j, hence so does ρ_u (with a different constant, but still depending only on j). Consequently, every element of $\langle \mathscr{A} \rangle_+$ has a finite order, uniformly bounded by a constant, whence, as $\langle \mathscr{A} \rangle_+$ is residually finite, by Zelmanov's theorem [56, 57], $\langle \mathscr{A} \rangle_+$ is finite, which concludes the proof. □

10.3.4 Growth and level-transitivity

In this subsection, we give a negative answer to the Milnor problem on the existence of groups of intermediate growth for a very particular class of automaton groups: the ones generated by an invertible-reversible Mealy automaton whose Schreier trie has a unique branch.

This family of groups contain in particular automaton groups which are branch groups, one of the three classes into which the class of just infinite groups is naturally decomposed [31, 3].

Growth

Let H be a semigroup generated by a finite set S. The *length* of an element g of the semigroup, denoted by $|g|$, is the length of its shortest decomposition:

$$|g| = \min\{n \mid \exists s_1, \dots, s_n \in S, g = s_1 \cdots s_n\} .$$

The *growth function* γ_H^S of the semigroup H with respect to the generating set S enumerates the elements of H with respect to their length:

$$\gamma_H^S(n) = |\{g \in H ; |g| \leq n\}| .$$

The *growth functions* of a group are defined similarly by taking symmetrical generating sets.

The growth functions corresponding to two generating sets are equivalent [44], so we may define the *growth* of a group or a semigroup as the equivalence class of its growth functions. Hence, for example, a finite (semi)group has a bounded growth, while an infinite abelian (semi)group has a polynomial growth, and a non-abelian free (semi)group has an exponential growth.

It is quite easy to obtain groups of polynomial or exponential growth. Answering a question of J. Milnor [45], R. Grigorchuk gave the very first example of an automaton group of intermediate growth [30]: faster than any polynomial, slower than any exponential (see Grigorchuk automaton in Figure 10.1).

Level-transitivity

The action of a (semi)group generated by an invertible Mealy automaton $\mathscr{A} = (Q, \Sigma, \delta, \rho)$ is *level-transitive* if its restriction to Σ^n has a unique orbit, for any n (this notion is equivalently called *spherically transitive* [32]). From a dual point of view it means that the powers of the dual $\mathfrak{d}\mathscr{A}$ are connected, *i.e.* its Schreier trie has a unique branch.

The level-transitivity of an automaton semigroup has some influence on the growth of the semigroup generated by the dual automaton.

Theorem 10.60. *[40] The semigroup generated by an invertible-reversible Mealy automaton whose Schreier trie has a unique branch has exponential growth.*

Note that the exponential growth of the semigroup generated by an invertible Mealy automaton implies the exponential growth of the group generated by this same automaton.

The Nerode classes of two consecutive powers of its state set are linked in the following way:

Lemma 10.61. *Let $\mathscr{A} = (Q, \Sigma, \delta, \rho)$ be an invertible-reversible Mealy automaton whose Schreier trie has a unique branch. Let $(C_i)_{1 \le i \le k}$ be the Nerode classes of Q^n for some n, and D be a Nerode class of Q^{n+1}. We have*

$$D = \bigcup_{q \in Q_D} C_{i_{q,D}} q \quad \text{and} \quad D = \bigcup_{q \in Q'_D} q C_{i'_{q,D}} ,$$

where $Q_D \subseteq Q$ and $Q'_D \subseteq Q$ have the same cardinality, and the $(i_{q,D})_{q \in Q_D}$ on the one hand and the $(i'_{q,D})_{q \in Q_D}$ on the other are pairwise distinct.

The automata $\mathfrak{m}(\mathscr{A}^n)$ and $\mathfrak{m}(\mathscr{A}^{n+1})$ have the same size if and only if $Q_D = Q'_D = Q$.

Theorem 10.60 can now be proven by observing that a relatively immediate consequence of Lemma 10.61 is that the sequence $(|\mathfrak{m}(\mathscr{A}^n)|)_{n \ge 0}$ increases strictly and exponentially.

The next theorem improves Theorem 10.60 for Mealy automata of prime size.

Theorem 10.62. *[39] The semigroup generated by an invertible-reversible Mealy automaton of prime size whose Schreier trie has a unique branch is free on the automaton state set.*

The idea is to bound the sizes of the Nerode classes in the powers of the Mealy automaton.

For the next three lemmas, let $\mathscr{A} = (A, \Sigma, \delta, \rho)$ be a reversible p-state Mealy automaton, p prime, whose Schreier trie is formed by a unique branch. By Proposition 10.51, $\mathscr{A}$ generates an infinite semigroup.

Lemma 10.63. *There cannot exist two equivalent words of different length in Q^*.*

Proof. For each m, $\mathscr{A}^m$ is connected, and so any two words of length m are mapped one onto the other by an element of $\langle\, \mathfrak{d}\mathscr{A}\, \rangle_+$.

Let u and v be two equivalent words of different lengths, say $|u| < |v|$. Every word of length $|v|$ is then equivalent to a word of length $|u|$: if w is of length $|v|$, then $w = \delta_t(v)$ for some $t \in \Sigma^*$, and, by Remark 10.2, w is equivalent to $\delta_t(u)$ of length $|u|$. By Remark 10.48, the semigroup $\langle\, \mathscr{A}\, \rangle_+$ is finite, which is impossible. □

Lemma 10.64. *All the Nerode classes of a given power Q^m have the same size, which happens to be a power of p.*

The proof of this lemma is direct from Remark 10.1.

Lemma 10.65. *There cannot exist two equivalent words of the same length in Q^*.*

Proof. Let u and v be two different equivalent words of the same length $n+1$. Let us prove by induction on $m > n$ that $\mathfrak{m}(\mathscr{A}^m)$ has at most p^n states.

The automaton $\mathscr{A}^{n+1}$ has p^{n+1} states. The words u and v are in the same Nerode class: by Lemma 10.64, all Nerode classes of Q^{n+1} have at least p elements and $\mathfrak{m}(\mathscr{A}^{n+1})$ has at most p^n states.

Suppose that $\mathfrak{m}(\mathscr{A}^m)$ has at most p^n states. Then, since all Nerode classes have the same size by Lemma 10.64, the induction hypothesis implies that they have at least p^{m-n} elements. Let us look at $[x_1^m]$: it contains

$$x_1^m, u_1, u_2, \ldots, u_{p^{m-n}-1}\,,$$

which are pairwise distinct. Among these words, there is at least one whose suffix in x_1 is the shortest, say u_1 without loss of generality: $p^{m-n} > 1$ and x_1^m has the longest possible suffix in x_1. Hence $[x_1^{m+1}]$ contains the following pairwise distinct $p^{m-n}+1$ words

$$x_1^{m+1}, u_1x_1, u_2x_1, \ldots, u_{p^{m-n}-1}x_1, x_1u_1\,.$$

By Lemma 10.64, $|[x_1^{m+1}]|$ is a power of p, so $|[x_1^{m+1}]| \geq p^{m+1-n}$. As all Nerode classes of Q^{m+1} have the same cardinality, we can conclude that $\mathfrak{m}(\mathscr{A}^{m+1})$ has at most $p^{m+1}/p^{m+1-n} = p^n$ elements, ending the induction.

Consequently, since there is only a finite number of different Mealy automata with up to p^n states, there exist $k < \ell$ such that $\mathfrak{m}(\mathscr{A}^k)$ and $\mathfrak{m}(\mathscr{A}^\ell)$ are equal up to state numbering. Hence the semigroup $\langle\, \mathscr{A}\, \rangle_+$ is finite, which is impossible. □

As a corollary of Lemmas 10.63 and 10.65 we can state the following proposition.

Proposition 10.66. *Let $\mathscr{A}$ be a reversible Mealy automaton of size p, with p prime. If the Schreier trie of $\mathscr{A}$ has a unique branch, then $\mathscr{A}$ generates a free semigroup of rank p, with the states of $\mathscr{A}$ being free generators of the semigroup.*

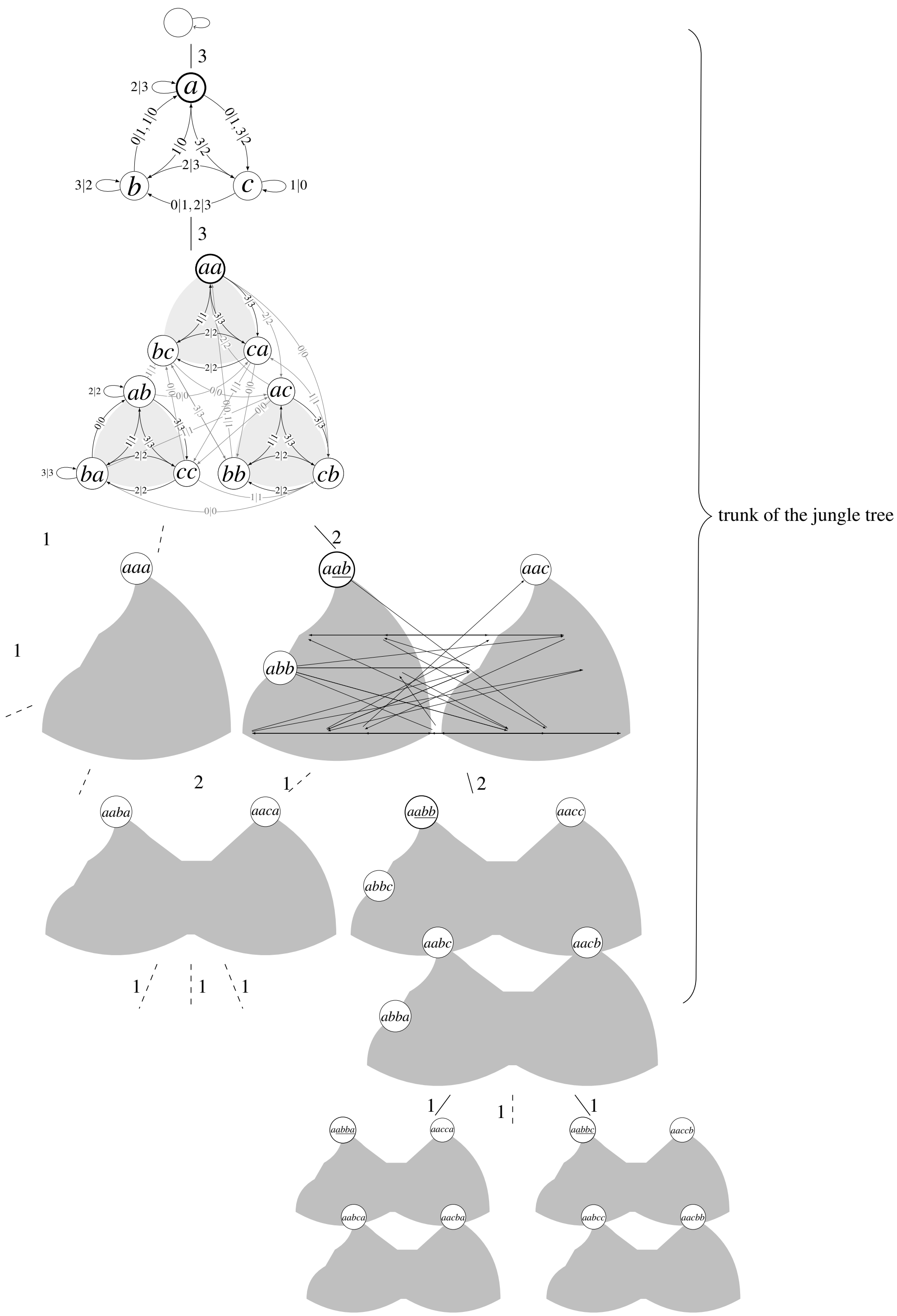


Fig. 10.19 An example of the first levels of a Schreier trie (all edges) and a jungle tree (plain edges). After the trunk the jungle tree consists in a regular binary tree (plain edges).

Index

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