Enumeration of (p, q) – parking functions

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Abstract

Parking functions are central in many aspects of combinatorics. We define in this communication a generalization of parking functions which we call $(p_1, \ldots, p_k)$ – *parking functions*. We give a characterization of them in terms of parking functions and we show that they can be interpreted as recurrent configurations in the sandpile model for some graphs. We also establish a correspondence with a Lukasiewicz language, which enables to enumerate $(p_1, \ldots, p_k)$ – parking functions as well as increasing ones.

Résumé

Les suites de parking se sont révélées être au centre de différents problèmes combinatoires. Nous introduisons ici des k-uplets de suites qui les généralisent, et dont nous montrons qu'ils peuvent être interprétés comme les configurations récurrentes de l'automate du tas de sable sur certains graphes. Nous établissons également une correspondance avec un langage de Lukasiewicz, ce qui nous permet d'obtenir des résultats d'énumération.

1 Introduction

Since parking functions were introduced more than thirty years ago in the context of hashing algorithm analysis ([12,13]), they gained a preponderant place in combinatorics of labelled objects. As shown by the elegant proof due to Pollack (see [8]), they are enumerated by *Cayley* numbers n^{n-2}, that play

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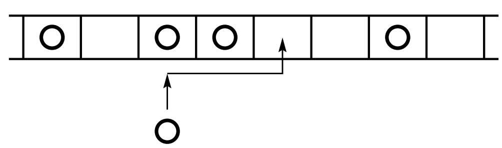
towards labelled objects the same role as *Catalan* numbers $\frac{1}{n+1}\binom{2n}{n}$ towards unlabelled ones: parking functions can actually be considered as a labelled version of Dyck paths. Many bijections are now known between parking functions and combinatorial objects such as Cayley trees, factorizations of a circular permutation as a minimal product of transpositions in $\mathfrak{S}_n$, maximal chains in the lattice of noncrossing partitions ([6,9]), or cells in the Shi hyperplane arrangement ([17,18]). More recently, parking functions were found to be also useful in algebraic combinatorics ([10]).

Among the many definitions of parking functions, we use the following one: a *parking function of length* n is a sequence $u = u_1\ u_2\ \ldots\ u_n$ of n non-negative integers such that there exists a permutation $\sigma = \sigma_1\ \sigma_2\ \ldots\ \sigma_n$ in $\mathfrak{S}_n$ *(strictly) larger* than u (which we denote $\sigma > u$), *i.e.* satisfying, for any index i, $\sigma_i > u_i$. This permutation σ will be said to be a *certificate* for u. For instance, 3 0 1 3 1 is a parking function since the permutation 4 1 2 5 3 is a certificate for it; on the other hand, 0 3 2 3 2 is not a parking function.

This terminology is motivated by the following greedy *parking algorithm*, which helps to explain in everyday words the notion of open adressing. Consider a one-way road with n parking slots numbered 0 through $n-1$; n cars arrive one at a time at the head of the road, that plan to park along this road. Each driver has a preferred parking place in mind, to which he proceeds. He parks there if it is free, but otherwise he has to drive ahead and park in the next empty slot.

EXAMPLE: Let $n = 8$. Consider a situation in which the 4 first cars are parked in places 0, 2, 3 and 6, and suppose that car number 5 tries to park in place 2; unfortunately, places 2 and 3 are occupied, so it has to drive ahead up to place 4.



The algorithm succeeds if each driver finds a parking place, and fails otherwise. Parking functions are exactly the preference functions for which the parking algorithm succeeds.

It was observed in [3] that there is a very simple bijection between parking functions of length n and some assignments of values to the vertices of the complete graph K_{n+1} called *recurrent configurations* in the sandpile model ([4]) introduced in statistical physics and considered by some combinatorialists as the *chip firing game* ([1]). Since recurrent configurations may be defined for any graph in which a vertex is distinguished as the sink, it seems reasonable to examine recurrent configurations of other families of graphs. The first one which comes in mind is that of the complete k – partite graphs, but choosing a sink breaks symmetry; hence we consider the family of complete $(k+1)$ – partite graphs of type $K_{p_1,p_2,\ldots,p_k,1}$, the lonely vertex being the sink.

It turns out that corresponding configurations have many similarities with parking functions, and can actually be considered as a generalization of them. These $(p_1,\ldots,p_k)$ – *parking functions* are k-tuples $(u_1,\ldots,u_k)$ of sequences of non-negative integers satisfying some combinatorial conditions which are detailed below.

Since general case is not substantially different, we concentrate on the particular case $k=2$ for the sake of readability. We prove that the number of (p,q) – parking functions is

$$(p+q+1)(p+1)^{q-1}(q+1)^{p-1},$$

and that the number of increasing ones is the Narayana number

$$\frac{1}{n+1}\binom{n+1}{p}\binom{n+1}{q}$$

where $n=p+q$.

The paper is organized as follows: we first define (p,q) – parking functions in an elementary way without any reference to the sandpile model and give some characterizations of them. We recall some simple facts about the physical model and indicate the scheme of a possible proof for the enumerative result. Then, we use conjugacy on certain words to obtain directly this result. Finally, we extend it to the case of increasing (p,q) – parking functions. Afterwards, we state the corresponding results in the general case of k – partite parking functions. Some perspectives for future investigations are suggested at the end of the paper.

2 Definition

We first give some notations and conventions which we adopt throughout the paper. Let p and q be two positive integers, and $n=p+q$; for any couple (a,b) of integers such that $a\leqslant b$, $[\![a,b]\!]$ denotes the set of *integers* between a and b:

$$[\![a,b]\!]=\{x|a\leqslant x\leqslant b\}.$$

A (p,q) – *sequence* is a pair (u,v) of sequences of non-negative integers with respective lengths p and q such that

$$\forall i \in [\![1,p]\!], \quad u_i \in [\![0,q]\!] \quad \text{and} \quad \forall j \in [\![1,q]\!], \quad v_j \in [\![0,p]\!].$$

Their set is denoted $\mathcal{S}_{p,q}$.

We define a partial order $\preccurlyeq$ on pairs of sequences of respective lengths p and q: for any two such pairs (u,v) and (u',v'), $(u,v) \preccurlyeq (u',v')$ if for all indices i and j, $u_i \leqslant u'_i$ and $v_j \leqslant v'_j$.

As the set $\mathcal{P}_n$ of parking functions, we define the set $\mathcal{P}_{p,q}$ of (p,q) – *parking functions* as an ideal for some order (here $\preccurlyeq$), determined by its maximal elements. These can be described thanks to permutations in $\mathfrak{S}_{p+q}$. Let us associate to any permutation $\sigma = \sigma_1\ \sigma_2\ \ldots\ \sigma_n$ in $\mathfrak{S}_n$ a (p,q) – sequence (x_σ, y_σ) (somewhat similar to its inversion table): for any index $i \leqslant p$, let x_i denote the i-th letter of x_σ; then x_i is the number of letters less than σ_i among the q last ones of σ, and for any index $j \leqslant q$, the j-th letter y_j of y_σ is the number of letters less than σ_{p+j} among the p first ones of σ. More formally,

$$\forall i \leqslant p,\ x_i = \left|\{1 \leqslant j \leqslant q \mid \sigma_{p+j} < \sigma_i\}\right|$$

and

$$\forall j \leqslant q,\ y_j = \left|\{1 \leqslant i \leqslant p \mid \sigma_i < \sigma_{p+j}\}\right|.$$

Example: Let $p = 5$, $q = 4$, and $\sigma = \underbrace{3\ 6\ 5\ 2\ 8}_{p}\ \underbrace{4\ 1\ 9\ 7}_{q}$.

For any $i \leqslant 5$, x_i is the number of elements less than σ_i in $\{4,1,9,7\}$, and symmetrically for the y_j's. Hence $(x_\sigma, y_\sigma) = (1\ 2\ 2\ 1\ 3,\ 2\ 0\ 5\ 4)$. Remark that σ is not uniquely determined by (x_σ, y_σ), since, for instance, if $\tau = 2\ 5\ 6\ 3\ 8\ 4\ 1\ 9\ 7$, then $(x_\tau, y_\tau) = (x_\sigma, y_\sigma)$.

Definition 1 *A (p,q) – sequence (u,v) is a (p,q) – parking function if there exists a permutation σ in $\mathfrak{S}_{p+q}$ such that $(u,v) \preccurlyeq (x_\sigma, y_\sigma)$. We say that the permutation σ is a* certificate *for (u,v).*

There is a more intuitive way to introduce (p,q) – parking functions, translating the definition into a parking quiz. Suppose that p blue cars and q red ones have to park in a one – way street with n parking slots. Each driver i of a blue car asks to have at least u_i red ones parked before him and each driver j of a red car asks to have at least v_j blue ones parked before him. The (p,q) – sequence (u,v) is a (p,q) – parking function if there exists a parking that satisfies all the wishes of the drivers.

Example: $(0\ 0\ 1\ 2\ 2,\ 0\ 2\ 3\ 5)$ is a $(5,4)$ – parking function, since the following parking suits.

O	X	O	O	X	O	O	X	X

The following remark is elaborated on in Section 6:

Remark 2 *$\mathcal{P}_{p,q}$ is invariant under the action of $\mathfrak{S}_p \times \mathfrak{S}_q$ on (p,q) – sequences. This means that corresponding unlabelled objects exist, whose set is isomorphic to that of* increasing *(p,q) – parking functions.*

3 Characterization

3.1 In terms of parking functions

We show in what way (p,q) – parking functions themselves can be considered as a generalization of usual parking functions. We first give a straightforward criterion for checking whether (u,v) is a (p,q) – parking function.

Let $u = u_1\ u_2\ \ldots\ u_p$ be an integer sequence. The *rank* function of u is the mapping

$$\rho_u : \begin{cases} [\![1,p]\!] \longrightarrow \mathbb{N} \\ \quad i \quad \longmapsto \Big|\{1 \leqslant j \leqslant p \mid u_j < u_i\}\Big| + \Big|\{1 \leqslant j < i \mid u_j = u_i\}\Big| \end{cases}$$

i.e. for any index i, $\rho_u(i)$ is the number of indices j such that either $u_i > u_j$ or $j < i$ and $u_j = u_i$.

Hence ρ_u is such that the numbers $u_i + \rho_u(i)$ are all distinct and satisfy

$$\forall i,j \leqslant p, \quad u_i < u_j \implies u_i + \rho_u(i) < u_j + \rho_u(j).$$

Let $\vec{u}$ be the sequence of length p whose i-th element is $u_i + \rho_u(i)$. Then, on the one hand, $\rho_{\vec{u}}(i) = \Big|\{1 \leqslant j \leqslant p \mid \vec{u}_j < \vec{u}_i\}\Big|$ for any index i, and on the other hand, $\rho_u = \rho_{\vec{u}}$.

EXAMPLE: If $u = 4\ 0\ 3\ 4\ 2$, then $\rho_u = 3\ 0\ 2\ 4\ 1$ and $\vec{u} = 7\ 0\ 5\ 8\ 3$.

Proposition 3 *A (p,q) – sequence (u,v) is a (p,q) – parking function if and only if the concatenation of the sequences $\vec{u}$ and $\vec{v}$ is a parking function.*

PROOF. Let $w = \vec{u} \cdot \vec{v}$, *i.e.* $w_1 \ldots w_p = \vec{u}$ and $w_{p+1} \ldots w_{p+q} = \vec{v}$. Let $\sigma = \sigma_1\ \sigma_2\ \ldots\ \sigma_n$ be a permutation in $\mathfrak{S}_n$ satisfying the following *monotony* condition:

$$\forall i,j \leqslant n, \quad w_i < w_j \implies \sigma_i < \sigma_j.$$

We prove that σ is a certificate for w if and only if it is a certificate for (u,v).

Consider indeed its associated (p,q) – sequence (x_σ, y_σ), and let $x_\sigma = x_1 \dots x_p$. Then, for any $i \in [\![1,p]\!]$, $x_i = \Big|\{1 \leqslant j \leqslant q \mid \sigma_{p+j} < \sigma_i\}\Big|$. On the other hand, $\rho_u(i) = \rho_{\vec{u}}(i) = \Big|\{1 \leqslant k \leqslant p \mid w_k < w_i\}\Big| = \Big|\{1 \leqslant k \leqslant p \mid \sigma_k < \sigma_i\}\Big|$. Since σ is a permutation, it implies that $x_i + \rho_u(i) = \sigma_i - 1$, hence:

$$\forall i \in [\![1,p]\!],\ x_i - u_i = \sigma_i - 1 - w_i.$$

Symmetrically, let $y_\sigma = y_1 \dots y_q$. For all $j \in [\![1,q]\!]$, $y_j - v_j = \sigma_{p+j} - 1 - w_{p+j}$.

Hence $w < \sigma$ if and only if $(u,v) \leqslant (x_\sigma, y_\sigma)$.

To end the proof, just observe that any parking function or (p,q) – parking function has a monotonous certificate, hence this condition on σ is not restrictive. □

A corollary of this Proposition is that the set of parking functions may be considered as the diagonal of the set of bipartite parking functions:

Proposition 4 *A sequence $u = u_1\, u_2 \dots u_n$ is a parking function if and only if (u,u) is an (n,n) – parking function.*

PROOF. Clearly if u is a parking function, it is certified by the bijection $i \mapsto \rho_u(i) + 1$. Hence a sequence u is a parking function if and only if, for any $i \in [\![1,n]\!]$, $u_i \leqslant \rho_u(i)$, *i.e.* if and only if, for any $i \in [\![1,n]\!]$, $\vec{u}_i \leqslant 2\rho_u(i)$.

Let w be the square of $\vec{u}$ for concatenation. Remark that $\vec{u}$ is a sequence of n distinct elements, with rank function ρ_u. So w has rank function ρ_w given by:

$$\forall i \in [\![1,n]\!],\ \rho_w(i) = 2\rho_u(i) \ \text{ and } \ \rho_w(n+i) = 2\rho_u(i) + 1.$$

According to the above arguments, w is a parking function if and only if $w_{n+i} = w_i \leqslant 2\rho_u(i)$ for any $i \in [\![1,n]\!]$, which gives the result. □

3.2 In terms of Lukasiewicz languages

We define a mapping φ from the set of (p,q) – sequences $\mathcal{S}_{p,q}$ into the free monoid over the alphabet $A = \{a,b\} \times \{a,b\}$, and give a necessary and sufficient condition on $\varphi(u,v)$ for (u,v) to be a (p,q) – parking function.

More precisely, let (u,v) be a (p,q)-sequence; then $\varphi(u,v) = (\varphi_q(u), \varphi_p(v))$, where $\varphi_q(u)$ is a word on the alphabet $\{a,b\}$ with p occurrences of a and $q+1$ of b defined in the following way: consider the increasing rearrangement $\tilde{u}$ of u, then $\varphi_q(u)$ is such that the i-th occurrence of the letter a in it is preceded by $\tilde{u}_i$ occurrences of b; $\varphi_p(v)$ is defined symmetrically.

EXAMPLE: $\varphi(1\ 5\ 0\ 2,\ 4\ 0\ 3\ 4\ 2) = (ababab bbab,\ abbababaab)$.

Observe that the positions of the occurrences of a in $\varphi_q(u)$ and $\varphi_p(v)$ are the elements of $\vec{u}$ and $\vec{v}$ respectively.

Both words $\varphi_q(u)$ and $\varphi_q(v)$ have length $p+q+1$. Hence we may consider $\varphi(u,v)$ as a word on the alphabet $A = \{(a,a),(b,a),(a,b),(b,b)\}$. For any word w in A^* and any letter $(x,y) \in A$, $|w|$ and $|w|_{(x,y)}$ denote respectively the length of w and the number of occurrences of (x,y) in w. We define:

$$|w|_a = 2|w|_{(a,a)} + |w|_{(a,b)} + |w|_{(b,a)} = |w| + |w|_{(a,a)} - |w|_{(b,b)}$$
$$|w|_b = 2|w|_{(b,b)} + |w|_{(a,b)} + |w|_{(b,a)} = |w| + |w|_{(b,b)} - |w|_{(a,a)}.$$

Then $|\varphi(u,v)|_a = p+q$ and $|\varphi(u,v)|_b = p+q+2$.

We define a morphism Δ from A^* to $\mathbb{Z}$ by setting:

$$\Delta(a,a) = 1$$
$$\Delta(a,b) = \Delta(b,a) = 0$$
$$\Delta(b,b) = -1$$

With this notation,

$$\Delta(\varphi(u,v)) = |\varphi(u,v)|_{(a,a)} - |\varphi(u,v)|_{(b,b)} = -1.$$

Proposition 5 *The pair (u,v) is a (p,q)-parking function if and only if $\varphi(u,v)$ satisfies the following condition:*

$$\Delta(w) \geqslant 0 \ \text{ for any factorization } \varphi(u,v) = ww' \ \text{ such that } w' \neq \varepsilon,$$

i.e. *if and only if $\varphi(u,v)$ belongs to the (Lukasiewicz) language $\mathcal{L}$ defined by the equation*

$$\mathcal{L} = (a,a)\cdot\mathcal{L}^2 + (a,b)\cdot\mathcal{L} + (b,a)\cdot\mathcal{L} + (b,b),$$

where + denotes union and $\cdot$ concatenation.

(For generalities about Lukasiewicz languages, see [14]).

Observe that, for any word w in A^*, $\Delta(w) \geqslant 0$ if and only if $|w|_a \geqslant |w|_b$, or equivalently $|w|_a \geqslant |w|$.

PROOF. By Proposition 3, (u, v) is a (p, q) – parking function if and only if $\vec{u} \cdot \vec{v}$ is a parking function. Let $i \leqslant p + q$, and w be the prefix of length i of $\varphi(u, v)$. There are as many occurrences of a in w as elements less than or equal to i in $\vec{u} \cdot \vec{v}$:

$$\begin{aligned} |w|_a &= |\{j \mid \vec{u}_j \leqslant i\}| + |\{j \mid \vec{v}_j \leqslant i\}| \\ &= |\{j \mid (\vec{u} \cdot \vec{v})_j \leqslant i\}| \end{aligned}$$

But $\vec{u} \cdot \vec{v}$ is a parking function if and only if $|\{j \mid (\vec{u} \cdot \vec{v})_j \leqslant i\}| \geqslant i$, that is, if and only if $|w|_a \geqslant i = |w|$.

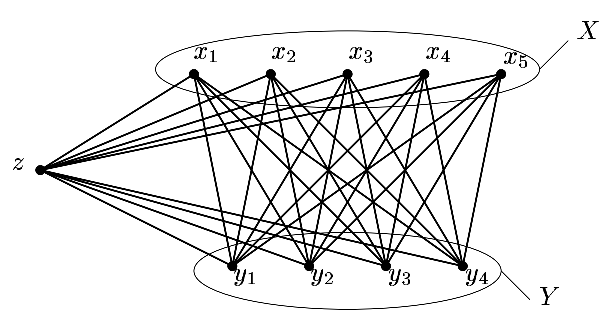
But this means exactly $\Delta(w) \geqslant 0$. □

4 Sandpiles on $K_{p,q,1}$

The sandpile model is defined as an evolution process on the configurations of a graph, *i.e.* on the mappings from the set of vertices of a graph into $\mathbb{N}$. We consider here the graph $K_{p,q,1}$ and we show that (p, q) – parking functions correspond to the *recurrent* configurations of this model. This gives a proof for their enumeration.

The complete tripartite graph $K_{p,q,1}$ has three subsets of vertices of respective sizes p, q and 1, $X = \{x_1, x_2, \ldots, x_p\}$, $Y = \{y_1, y_2, \ldots, y_q\}$, and $\{z\}$, and its set of edges is $(X \times Y) \cup (\{z\} \times (X \cup Y))$.

EXAMPLE: $K_{5,4,1}$:



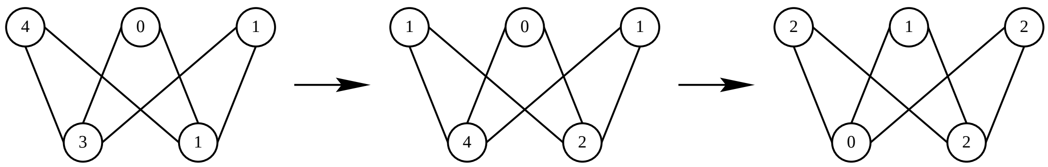
A *configuration* of this graph is an assignment of non-negative integers to each vertex in $X \cup Y$. Hence a configuration is a pair (u, v) of sequences of respective lengths p and q. The integer u_i (resp. v_j) may be considered as a number of grains of sand lying in vertex x_i (resp. y_j).

A *toppling* of vertex $x_i \in X$ occurs if $u_i > q$; in that case, the new configuration (u', v') is such that:

- $u'_i = u_i - q - 1$,
- $\forall k \leqslant p,\ k \neq i \implies u'_k = u_k$,
- $\forall j \leqslant q,\ v'_j = v_j + 1$.

The missing grain of sand is supposed to have fallen in the sink z. A toppling of vertex $y_j \in Y$ is defined similarly.

EXAMPLE: Two successive topplings on $K_{3,2,1}$ (for convenience's sake, the sink z has been omitted):



Definition 6 *A configuration (u, v) is* stable *if no vertex can topple,* i.e. *if (u, v) is a (p, q) – sequence. A stable configuration (u, v) is* recurrent *if it can be obtained by a sequence of topplings from a configuration (u', v') such that, for any $i \leqslant p$, $u'_i > q$, and for any $j \leqslant q$, $v'_j > p$.*

Proposition 7 *A configuration (u, v) is recurrent if and only if the pair (u', v') defined by:*

$$\forall i \leqslant p,\ u'_i = q - u_i \quad \text{and} \quad \forall j \leqslant q,\ v'_j = p - v_j$$

is a (p, q) – parking function.

PROOF. We use a characterization due to Dhar [4] of recurrent configurations. In the case of $K_{p,q,1}$, the criterion claims that (u, v) is recurrent if and only if the addition of 1 to each u_i and each v_j leads to a sequence of topplings in which each vertex topples exactly once. It is easy to verify that the order in which the vertices topple gives a permutation which is a certificate for (u', v') and vice versa. □

Majumdar and Dhar [16] have also shown that recurrent configurations on a graph are in one-to-one correspondence with its spanning trees. This supplies a proof of the enumeration formula announced in Section 1, either by using known results on the number of spanning trees of multipartite graphs or by reproving them by arguments along the lines of Joyal's method [11,15] for Cayley's formula. We do not detail this proof since we give another one in the next Section, that has the advantage of providing also a proof for the enumeration of increasing (p, q) – parking functions.

5 Enumeration by conjugacy

In order to enumerate (p, q) – parking functions, we establish a relationship between conjugacy on words and conjugacy on integer sequences, then we use a generalization due to L. Chottin of the so-called cyclic lemma ([2]).

Recall that two words w and w' are *conjugates* if $w = w_1 w_2$ and $w' = w_2 w_1$. Let ε denote the empty word, then any word w has $|w|$ factorizations $w = w_1 w_2$ such that $w_1 \neq \varepsilon$, which we call *proper* factorizations in the sequel. Hence it has at most $|w|$ conjugates (one of which is equal to w, for $w_2 = \varepsilon$). The number of different conjugates of w divides $|w|$ and each conjugate is due to the same number of distinct factorizations (see [14], p. 8).

We define a related notion of conjugacy for integer sequences:

Definition 8 *Let $u = u_1\ u_2\ \dots\ u_p$ be an integer sequence such that for any index i, $0 \leqslant u_i \leqslant q$, and let k belong to $[\![0, q]\!]$. The k-th q –* conjugate *of u is the sequence $s_q^k(u)$ defined by:*

$$\forall i \in [\![1, p]\!],\ s_q^k(u)_i = u_i + k \mod q + 1.$$

We denote by $\tilde{s}_q^k(u)$ the increasing rearrangement of $s_q^k(u)$. Observe that the $q + 1$ q – conjugates of a sequence are all different, but this is not the case for the sequences $\tilde{s}_q^k(u)$. However, each *increasing* q – conjugate corresponds to the same number of q – conjugates of u.

Proposition 9 *The mapping φ_q induces a bijection between q – conjugates of u and proper factorizations $w_1 w_2$ of $\varphi_q(u)$ such that w_1 ends with a letter b (and realizes a bijection between increasing q – conjugates of u and conjugates of $\varphi_q(u)$ ending with a letter b).*

PROOF. Let $w = \varphi_q(u)$, then for any k in $[\![0, q]\!]$, there is a unique factorization $w = w_1 w_2$ such that w_1 ends with a b and $|w_2|_b = k$. Clearly

$$\varphi_q\left(s_q^k(u)\right) = w_2 w_1.$$

□

Definition 10 *Let (u, v) be a (p, q) – sequence; we define its* conjugates *as the pairs $(s_q^i(u), s_p^j(v))$, for all i and j in $[\![0, q]\!]$ and $[\![0, p]\!]$ respectively.*

The following Proposition is a special case of Theorem 4.2 of [2]. We give its proof for the sake of completeness.

Proposition 11 *For any pair (w', w'') of words in $\{a,b\}^*$ such that $|w'|_a = q$, $|w''|_a = p$ and $|w'| = |w''| = p+q+1$, there exist exactly $p+q+1$ ways of properly factorizing w' and w'' into $w'_1w'_2$ and $w''_1w''_2$ so that $(w'_2w'_1, w''_2w''_1)$ belongs to $\mathcal{L}$.*

PROOF. There are $(p+q+1)^2$ pairs of proper factorizations of w' and w'', which we gather in $p+q+1$ classes with respect to the value of $|w'_1| - |w''_1|$ mod $p+q+1$. Each class is constituted of one pair of the type $w = (w'_1\varepsilon, w''_1w''_2)$ and its factorizations as a word of A^*. The cyclic lemma due to Dvoretzky and Motzkin ([5], sometimes attributed to Raney) claims that there is only one proper factorization w_1w_2 of w such that w_2w_1 belongs to $\mathcal{L}$. Since each class corresponds to a different word w and since there are $p+q+1$ classes, this ends the proof. □

Note that only $(p+1)(q+1)$ among the $(p+q+1)^2$ ways of properly factorizing w' and w'' are such that both w'_1 and w''_1 end with a letter b. Moreover, any of the $p+q+1$ decompositions whose existence is claimed by Proposition 11 satisfies this condition.

Theorem 12 *Exactly $(p+q+1)$ among the $(p+1)(q+1)$ conjugates of any (p,q) – sequence (u,v) are (p,q) – parking functions.*

PROOF. Each conjugate of (u,v) corresponds to a different way of properly factorizing $\varphi_q(u)$ and $\varphi_p(v)$ into $w'_1w'_2$ and $w''_1w''_2$ so that w'_1 and w''_1 end with a b. By Proposition 11, exactly $p+q+1$ of them are such that $(w'_2w'_1, w''_2w''_1)$ belongs to $\mathcal{L}$, and by Proposition 5, this is the condition for (u,v) to be a (p,q) – parking function. □

Corollary 13 *The number of (p,q) – parking functions is*

$$(p+q+1)\ (p+1)^{q-1}\ (q+1)^{p-1}.$$

PROOF. The ratio of (p,q) – parking functions among the $(p+1)^q(q+1)^p$ (p,q) – sequences is $\dfrac{p+q+1}{(p+1)(q+1)}$ by Theorem 12. □

6 Increasing (p,q) – parking functions

A (p,q) – sequence (u,v) is *increasing* if, for all $i < p$ and $j < q$, $u_i \leqslant u_{i+1}$ and $v_j \leqslant v_{j+1}$. The set $\mathcal{I}_{p,q}$ of increasing (p,q) – parking functions clearly constitutes a system of representatives of the orbits of the action of $\mathfrak{S}_p \times \mathfrak{S}_q$ on $\mathcal{P}_{p,q}$.

In this section we give two different proofs of the following result:

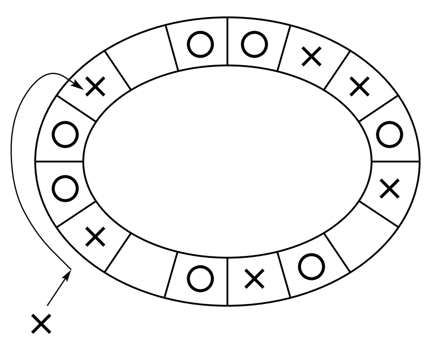
Proposition 14 *The number of increasing (p,q) – parking functions is*

$$\frac{p+q+1}{(p+1)(q+1)}\binom{p+q}{p}\binom{p+q}{q}.$$

Note that these numbers are usually known as Narayana's numbers and have many interpretations. For instance, they enumerate plane trees with $n+2$ vertices and $p+1$ leaves, or noncrossing partitions of $[\![1, n+1]\!]$ into $p+1$ blocks.

The first proof is an adaptation of Pollack's one for counting parking functions:

PROOF 1. Consider a circular parking lot with $p+q+1$ slots numbered clockwise 0 to $p+q$. The corresponding parking algorithm is similar to the usual one, except that preference $p+q$ is allowed and treated like any other: if slot $p+q$ is occupied, the car moves clockwise to the first empty slot.



The mapping $(u,v) \mapsto \vec{u} \cdot \vec{v}$ realizes a one-to-one correspondence between $\mathcal{I}_{p,q}$ and sequences w of length $p+q$ such that:

- $\forall i \leqslant p+q,\ 0 \leqslant w_i \leqslant p+q,$
- $\forall i < p+q,\ i \neq p \implies w_i < w_{i+1},$
- the parking process leaves slot number $p+q$ unoccupied.

Clearly for any preference function satisfying the two first conditions, one slot is left empty, and by symmetry there are exactly as many sequences with a given empty slot as with any other.

Hence the number of increasing (p, q) – parking functions is

$$\frac{1}{p+q+1}\binom{p+q+1}{p}\binom{p+q+1}{q},$$

which is of course equal to the expected result. □

The second proof shows that this result is also a straightforward consequence of Theorem 12:

PROOF 2. Consider classes of $\mathcal{S}_{p,q}$ closed by conjugacy and by the action of $\mathfrak{S}_p \times \mathfrak{S}_q$. Let C be one of these classes; C is a disjoint union of orbits under the action of $\mathfrak{S}_p \times \mathfrak{S}_q$. All these orbits have the same cardinality, since the action of an element of $\mathfrak{S}_p \times \mathfrak{S}_q$ is the same on all conjugates of a sequence, and each one contains exactly one increasing (p, q) – sequence. Moreover, either all elements of a given orbit are (p, q) – parking functions or none of them is. Hence, in C, the ratio of increasing (p, q) – parking functions among the increasing (p, q) – sequences is equal to the ratio of (p, q) – parking functions among (p, q) – sequences. Finally, since C is a disjoint union of conjugacy classes in which the ratio of (p, q) – parking functions is the same (by Theorem 12), the ratio of increasing (p, q) – parking functions among the increasing (p, q) – sequences in C, is equal to

$$\frac{p+q+1}{(p+1)(q+1)}.$$

Since this is true for any class C, their ratio among the total number of increasing (p, q) – sequences is the same. But the number of increasing (p, q) – sequences is

$$\binom{p+q}{p}\binom{p+q}{q},$$

thus ending the proof. □

Remark that another generalization of parking functions called k – *valet functions* is defined in [9], whose particular case $k = 2$ is isomorphic to increasing (p, q) – parking functions.

7 Generalization

It is natural to introduce the set $\mathcal{P}_{p_1,p_2,\ldots,p_k}$ of $(p_1, p_2, \ldots, p_k)$ – parking functions in a similar way as (p, q) – parking functions by dividing the elements of a permutation into k intervals instead of two.

We adopt the following notations: let n and k be two positive integers, and $(p_1, \ldots, p_k)$ be a composition of n into k parts, *i.e.* a k-tuple of positive integers such that $p_1 + \cdots + p_k = n$. For any $i \in [\![1, k]\!]$, let $q_i = n - p_i$.

A $(p_1, \ldots, p_k)$ – *sequence* is a k-tuple $(u^{(1)}, \ldots, u^{(k)})$ of integer sequences with respective lengths $p_1, \ldots, p_k$ and such that

$$\forall i \in [\![1, k]\!],\ \forall j \in [\![1, p_i]\!],\ 0 \leqslant u_j^{(i)} \leqslant q_i.$$

For any permutation $\sigma = \sigma_1 \ldots \sigma_n \in \mathfrak{S}_n$, let $x_\sigma = (x^{(1)}, \ldots, x^{(k)})$ be the $(p_1, \ldots, p_k)$ – sequence such that, for any $i \in [\![1, k]\!]$ and any $j \in [\![1, p_i]\!]$,

$$x_j^{(i)} = \left|\{1 \leqslant k \leqslant n \mid k \notin [\![\pi_{i-1} + 1, \pi_i]\!] \text{ and } \sigma_k < \sigma_{\pi_{i-1}+j}\}\right|,$$

where $\pi_0 = 0$ and, for any $i \in [\![1, k]\!]$, $\pi_i = p_1 + \cdots + p_i$.

EXAMPLE: Let $k = 3$, $(p_1, p_2, p_3) = (3, 2, 4)$, and $\sigma = \underbrace{3\ 6\ 5}_{p_1}\ \underbrace{2\ 8}_{p_2}\ \underbrace{4\ 1\ 9\ 7}_{p_3}$.
Then $x_\sigma = (2\ 3\ 3,\ 1\ 6,\ 2\ 0\ 5\ 4)$.

Definition 15 *A $(p_1, \ldots, p_k)$ – sequence u is a $(p_1, \ldots, p_k)$ – parking function if there exists a permutation σ such that $u \preccurlyeq x_\sigma$.*

Observe that the limit case $k = n$ corresponds to usual parking functions, while the case $k = 1$ is degenerated: the only (n) – parking function is a sequence containing n letters 0.

This definition gives rise to developments analogous to above. As proofs in the generic case are essentially the same as in the case $k = 2$, we only state the results.

For instance, $(p_1, \ldots, p_k)$ – parking functions correspond bijectively to recurrent configurations on the complete $(k+1)$ – partite graph $K_{p_1,\ldots,p_k,1}$, hence $\mathcal{P}_{p_1,\ldots,p_k}$ can be put in one-to-one correspondence with the set of its spanning trees.

It can also be obtained that increasing $(p_1, \ldots, p_k)$ – parking functions are isomorphic to k – valet functions on $(p_1, \ldots, p_k)$ defined in [9] .

The characterization in terms of Lukasiewicz languages suits as well:

For any $(p_1,\ldots,p_k)$ – sequence $u=(u^{(1)},\ldots,u^{(k)})$, let

$$\varphi(u)=\left(\varphi_{q_1}(u^{(1)}),\ldots,\varphi_{q_k}(u^{(k)})\right).$$

$\varphi(u)$ may be considered as a word over the alphabet $A_k=\{a,b\}^k$. For any letter w in A_k, let $|w|_a$ denote the number of occurrences of a in it, and $\Delta(w)=|w|_a-1$. This defines a morphism from A_k^* to $\mathbb{Z}$ such that, for any $(p_1,\ldots,p_k)$ – sequence u, $\Delta(\varphi(u))=-1$.

Proposition 5 becomes:

Proposition 16 *Let $u=(u^{(1)},\ldots,u^{(k)})$ be a $(p_1,\ldots,p_k)$ – sequence; u is a $(p_1,\ldots,p_k)$ – parking function if and only if $\varphi(u)$ belongs to the Lukasiewicz language $\mathcal{L}_k$ defined by the equation*

$$\begin{aligned}\mathcal{L}_k &= \sum_{w\in A_k} w\cdot {\mathcal{L}_k}^{|w|_a}\\ &= (a,\ldots,a,a)\cdot {\mathcal{L}_k}^{k}+(a,\ldots,a,b)\cdot {\mathcal{L}_k}^{k-1}+\cdots+(b,\ldots,b,b)\ .\end{aligned}$$

This enables to enumerate $(p_1,\ldots,p_k)$ – parking functions and increasing ones thanks to an argument of conjugacy:

Definition 17 *Let u be a $(p_1,\ldots,p_k)$ – sequence; its* conjugates *are the k-tuples $(s_{q_1}^{i_1}(u^{(1)}),\ldots,s_{q_k}^{i_k}(u^{(k)}))$, for all $j\in[\![1,k]\!]$ and $i_j\in[\![0,q_j]\!]$.*

Proposition 18 *For any word $(w^{(1)},\ldots,w^{(k)})$ over A_k such that $|w^{(i)}|_a=q_i$ and $|w^{(i)}|_b=p_i+1$ for any i, exactly $(n+1)^{k-1}$ k-tuples of proper factorizations $w^{(i)}=w_1^{(i)}w_2^{(i)}$ are such that $(w_2^{(1)}w_1^{(1)},\ldots,w_2^{(k)}w_1^{(k)})$ belongs to $\mathcal{L}_k$.*

As a consequence,

Proposition 19 *The ratio of (increasing) $(p_1,\ldots,p_k)$ – parking functions in the set of (increasing) $(p_1,\ldots,p_k)$ – sequences is*

$$\frac{(n+1)^{k-1}}{(n-p_1+1)\cdots(n-p_k+1)}.$$

Hence the number of $(p_1,\ldots,p_k)$ – parking functions and increasing ones are respectively

$$(n+1)^{k-1}\prod_{i=1}^{k}(n-p_i+1)^{p_i-1}$$

and

$$\frac{1}{n+1}\prod_{i=1}^{k}\binom{n+1}{p_i}.$$

8 Perspectives

8.1 Product of transpositions

Since the number of parking functions of length n is equal to the number of decompositions of the $(n+1)$ – cycle $(0\ 1\ 2\ \ldots\ n)$ into a product of n transpositions, it seems natural to seek for an interpretation of the number of (p,q) – parking functions as the number of decompositions of a circular permutation into a product of transpositions satisfying certain conditions.

Edges of K_{n+1} may be considered as transpositions, so each spanning tree of this graph corresponds to a set of n transpositions and hence to $n!$ different factorizations of a circular permutation; and the enumeration follows since each of the $n!$ circular permutations has the same number of decompositions.

Trying to obtain an analogous result for (p,q) – parking functions, we consider all the spanning trees of $K_{p,q,1}$ and compute all the possible products of the transpositions corresponding to their edges. We observe that every circular permutation on $\{0,1,2,\ldots,p+q\}$ is obtained by this way. However the number of times each of these is obtained is not uniform as it was the case for K_{n+1}.

For instance, $K_{3,2,1}$ has 216 spanning trees, each one consisting of 5 edges, which gives a total of 25920 products. The circular permutation (0 1 2 3 4 5) is obtained 131 times, (0 1 2 4 3 5), 211 times, (0 1 4 2 3 5), 261 times, (0 4 1 2 3 5), 186 times, and (0 1 4 2 5 3), 316 times.

8.2 Hyperplane arrangements

The number of parking functions is equal to the number of regions in the Shi arrangement of hyperplanes, and bijections between regions and parking functions are known. It would be interesting to find arrangements of hyperplanes whose numbers of regions is equal to the numbers of (p,q) – parking functions.

References

[1] N. Biggs, *The Tutte polynomial as a growth function*, J. Alg. Comb. **10**, 115–133, 1999.

[2] L. Chottin, *Enumération d'arbres et formules d'inversion de séries formelles*, J. Comb. Th. B. **31**, 23–45, 1981.

[3] R. Cori and D. Rossin, *On the sandpile group of dual graphs*, Eur. J. Comb. **21**, 447–459, 2000.

[4] D. Dhar, *Self-organized critical state of sandpile automaton models*, Phys. Rev. Lett. **64**, 1613–1616, 1990.

[5] A. Dvoretzky and Th. Motzkin, *The asymptotic density of certain sets of real numbers*, Duke Math. J., **14**, 315–321, 1947.

[6] P.H. Edelman and R. Simion, *Chains in the lattice of noncrossing partitions*, Disc. Math. **126**, 107–119, 1994.

[7] D. Foata, *Rearrangements of Words*, in [14].

[8] D. Foata and J. Riordan, *Mappings of acyclic and parking functions*, Aequ. Math. **10**, 10–22, 1974.

[9] J.D. Gilbey and L. H. Kalikow, *Parking functions, valet functions and priority queues*, Discr. Math. **197/198**, 351–373, 1999.

[10] M.D. Haiman, *Conjectures on the quotient ring by diagonal invariants*, J. Alg. Comb. **3**, 17–76, 1994.

[11] A. Joyal, *Une théorie combinatoire des séries formelles*, Adv. in Math. **42**, 1–82, 1981.

[12] A.G. Konheim and B. Weiss, *An occupancy discipline and applications*, Siam J. Appl. Math. **14**, 1266–1274, 1966.

[13] G. Kreweras, *Sur les partitions non croisées d'un cycle*, Disc. Math. **1**, 333–350, 1972.

[14] Lothaire, *Combinatorics on words*, Addison Wesley, Reading (MA), 1983.

[15] G. Labelle, *Une nouvelle démonstration combinatoire des formules de Lagrange*, Adv. in Math. **42**, 217–247, 1981.

[16] S. M. Majumdar and D. Dhar, *Equivalence of the abelian sandpile model and the $q \to 0$ limit of the Potts model*, Physica A. **185**, 129–135, 1992.

[17] R.P. Stanley, *Parking functions and noncrossing partitions*, Elec. J. Comb. **4**, R20, 1997.

[18] R.P. Stanley, *Hyperplane arrangements, parking functions and tree inversions*, Mathematical essays in honor of Gian-Carlo Rota, Progr. Math., 161, Birkhäuser, Boston/Basel/Berlin, 1998, 359–375.