Central Characters and Conjugacy Classes of the Symmetric Group

or *On some Conjectures of J. Katriel*

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Résumé. Nous démontrons plusieurs conjectures dues à Jacob Katriel sur les classes de conjugaison de $\mathfrak{S}_n$. La première exprime, pour un partage fixé ρ de la forme $r1^{n-r}$, les valeurs propres (ou caractères centraux) ω_ρ^λ en terme des contenus de λ. Tandis que Katriel a conjecturé une forme générique et un algorithme pour calculer les coefficients indéterminés, nous fournissons une formule explicite. La seconde conjecture (présentée au SFCA'98 à Toronto) donne une forme générale pour l'expression d'une classe de conjugaison en terme d'opérateurs élémentaires. Nous la prouvons en utilisant une description en termes d'opérateurs différentiels sur les polynômes symétriques. Finalement nous étendons partiellement nos résultats sur ω_ρ^λ à des partages ρ quelconques.

Abstract. This article addresses several conjectures due to Jacob Katriel concerning conjugacy classes of $\mathfrak{S}_n$ viewed as operators acting by multiplication. The first one expresses, for a fixed partition ρ of the form $r1^{n-r}$, the eigenvalues (or central characters) ω_ρ^λ in terms of contents of λ. While Katriel conjectured a generic form and an algorithm to compute missing coefficients, we provide an explicit expression. The second conjecture (presented at FPSAC'98 in Toronto) gives a general form for the expression of a conjugacy class in terms of *elementary* operators. We prove it using a convenient description by differential operators acting on symmetric polynomials. To conclude, we partially extend our results on ω_ρ^λ to arbitrary partitions ρ.

1 Introduction

Although our aim is to prove Katriel's conjectures, we do not use his notations through this text: instead we keep closer to Macdonald's textbook [13] and provide translations when necessary.

Notations. Let $\lambda = (\lambda_1, \lambda_2, \dots \lambda_k)$ be a partition of weight n and length $\ell(\lambda) = k$, *i.e.* a finite non increasing sequence of positive integers $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_k > 0$

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summing up to n. We write $\lambda \vdash n$ or $|\lambda| = n$ and $\lambda = 1^{\ell_1} 2^{\ell_2} \cdots n^{\ell_n}$ when ℓ_i parts of λ are equal to i ($i = 1 \ldots n$) (we shall consistently use greek letters for partitions and their parts and corresponding latin letters for the multiplicity notation). We denote by $\mathcal{C}_\lambda$ the conjugacy class indexed by the partition λ and by $\lambda(\sigma)$ the cycle-type of a permutation σ. Let $z_\lambda = 1^{\ell_1}\ell_1! 2^{\ell_2}\ell_2! \cdots n^{\ell_n}\ell_n!$, so that $|\mathcal{C}_\lambda| = n!/z_\lambda$.

Let $\mathbb{Q}[\mathfrak{S}_n]$ be the group algebra of the symmetric group over the rational numbers field $\mathbb{Q}$, and let $\mathcal{Z}_n$ be the center of this group algebra. The formal sum of the permutations in a conjugacy class $\mathcal{C}_\lambda$ belongs to $\mathcal{Z}_n$, and the set $\{K_\lambda\}_{\lambda \vdash n}$ of these formal sums forms a linear basis for the center $\mathcal{Z}_n$. Similarly, the irreducible characters of the symmetric group $\mathfrak{S}_n$ are indexed by partitions of weight n and can be considered as elements of $\mathcal{Z}_n$, in which their family $\{\chi^\lambda\}_{\lambda \vdash n}$ also forms a linear basis. For λ and μ two partitions of n we denote by χ^λ_μ the evaluation of the character χ^λ on any permutation of the class $\mathcal{C}_\mu$. In particular, $\chi^\lambda_{1^n} = n!/h_\lambda$ where h_λ is the hook-length product of λ. The character table $[\chi^\lambda_\mu]$ gives natural formulae for changes of basis in $\mathcal{Z}_n$: $\chi^\lambda = \sum_{\mu \vdash n} \chi^\lambda_\mu K_\mu$ and $K_\mu = \sum_{\lambda \vdash n} (\chi^\lambda_\mu / z_\lambda)\chi^\lambda$.

Partitions are usually represented by their Ferrers's diagrams. The content of a cell $x = (i,j) \in \lambda$ is $c(x) = i - j$. In his conjectures, Katriel introduces the *content power-sums* $\mathfrak{p}_k(\lambda)$ defined for $k > 0$ by $\mathfrak{p}_k(\lambda) = \sum_{x \in \lambda} c(x)^k$ and for $\nu \vdash n$ by $\mathfrak{p}_\nu(\lambda) = \prod_i \mathfrak{p}_{\nu_i}(\lambda)$. These $\mathfrak{p}_k(\lambda)$ are the classical power-sum symmetric functions p_k, evaluated on the alphabet $\{c(x) \mid x \in \lambda\}$.

A partition is *reduced* if it contains no part equal to 1. For $\lambda = 1^{\ell_1} 2^{\ell_2} \cdots k^{\ell_k}$, we denote by $\bar{\lambda}$ the reduced partition $2^{\ell_2} \ldots k^{\ell_k}$. The reduced cycle-type of a permutation or of a conjugacy class is defined accordingly.

Multiplication by a conjugacy class. Let λ be a partition of n. We are interested in the element ω^λ of $\mathcal{Z}_n$ which is defined ([13, p126]) by

$$\forall \rho \vdash n, \qquad \omega^\lambda(\rho) = \omega^\lambda_\rho = \frac{h_\lambda}{z_\rho}\chi^\lambda_\rho.$$

These elements are called *central characters* or *eigenvalues* by J. Katriel[1]. As the family $\{\chi^\lambda/h_\lambda\}_{\lambda \vdash n}$ forms a basis of orthogonal idempotents in $\mathcal{Z}_n$, we have

$$\forall \rho, \lambda \vdash n, \qquad K_\rho \cdot \chi^\lambda = \sum_{\mu \vdash n} \frac{\chi^\mu_\rho}{z_\mu}(\chi^\mu \cdot \chi^\lambda) = \omega^\lambda_\rho \chi^\lambda,$$

explaining why the evaluation ω^λ_ρ may be called the eigenvalue of the conjugacy class $\mathcal{C}_\rho$ associated to the eigenvector χ^λ. Here we consider K_ρ or $\mathcal{C}_\rho$ as an operator acting on $\mathcal{Z}_n$ by multiplication. The multiplicative structure of $\mathcal{Z}_n$ has been largely studied in terms of *connexion coefficients* [2, and ref. therein], also called *structure constants* [3, and ref. therein]. These coefficients are defined for

[1] For a partition $\rho \vdash n$ and an irreducible representation Γ, indexed by the partition $\gamma \vdash n$, our ω^γ_ρ is denoted λ^Γ_ρ in [8].

all triples of partitions (λ, μ, ν) of n by

$$K_\lambda \cdot K_\mu = \sum_{\nu \vdash n} \mathfrak{a}_{\lambda,\mu}^{\nu} K_\nu.$$

The first set of conjectures that we consider is [8, Conj.1–2] (see also [5]). In these articles, Katriel suggests that for a fixed integer r, the eigenvalues $\omega^\lambda_{r1^{n-r}}$ are given by evaluations of a polynomial in $\mathbb{Q}[n][p_1, \ldots, p_{r-1}]$ on the contents power-sums $\mathfrak{p}_k(\lambda)$. The first values ($r = 2, 3, 4$) were computed by Frobenius himself (1901). Ingram [4] later computed other values of ω^λ_μ. In [6], an algorithm is given, which is used in [7] to produce numerical expressions supporting these conjectures up to $r = 18$. Theorem 1 gives an explicit expression for the polynomials considered by Katriel and proves some of their conjectured properties.

In a second set of conjectures, considered in Section 3, Katriel looks for expressions of the conjugacy classes as sums of *elementary* operators. He requires that these expressions depend only on the reduced cycle-type. These conjectures were presented at FPSAC'98 [11] and are derived from previous weaker conjectures [9, 10, and ref. therein]. In order to state more easily Katriel's formulae, we use a representation of the action of conjugacy classes on $\mathcal{Z}_n$ by an action of differential operators on the space of symmetric functions. Once stated in this form (theorem 2), these conjectures are relatively easy to prove. Our approach is reminiscent of that of Goulden and Jackson (see [2] and reference therein).

Finally in Section 4, we consider a third set of conjectures [8, Conj.3–6], which extend the first ones, from partitions of the form $r1^{n-r}$ to arbitrary partitions ρ. Unlike the case $r1^{n-r}$, we have not found an explicit expression of ω_ρ for arbitrary ρ in terms of the p_i. However we derive from Theorem 2 a proof of a weak version of [8, Conj.4] on the general form of ω_ρ, which implies [8, Conj.5].

Numerous examples of decompositions of ω_ρ or K_ρ for small ρ are found *e.g.* in [8]. Thus we did not include a large number of examples. We instead provide `Maple` procedures based on our theorems at `http://www.loria.fr/~schaeffe`.

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2 Central characters for partitions $r1^{n-r}$

Throughout this section, let r denote a fixed integer. Let the content weight of a partition ν be $w(\nu) = |\nu| + 2\ell(\nu)$.

In order to state our polynomial expression for $\omega_{r1^{n-r}}$, we define two formal power series in the indeterminate Y with polynomial coefficients in n: first,

$$P^r_j(Y,n) = (1+nY)^{-j} + (1+(n+r-1)Y)^{-j} - (1+(n+r)Y)^{-j} - (1+(n-1)Y)^{-j} \quad (1)$$

and $P^r_\nu(Y,n) = \prod_i P^r_{\nu_i}(Y,n)$, so that $P^r_\nu(Y,n)$ is the power sum symmetric function p_ν evaluated on the alphabet $-Y_{n+r} - Y_{n-1} + Y_{n+r-1} + Y_n$, where $Y_n = \frac{1}{1+nY}$.

Second,

$$Q^r(Y,n) = \frac{(-1)^r}{r^2}\prod_{i=0}^{r-1}(1+(n+i)Y)\left(\frac{(1+(n+r)Y)(1+(n-1)Y)}{(1+(n+r-1)Y)(1+nY)}\right)^n . \quad (2)$$

Finally, let $Q_i^r(n)$ be the coefficient of Y^i in $Q^r(Y,n)$ and $P^r_{\nu,i}(n)$ the coefficient of $Y^{2\ell(\nu)+i}$ in $P^r_\nu(Y,n)$.The polynomials $Q_i^r(n)$ and $P^r_{\nu,i}(n)$ have respective degree $i-1$ and i. With these notations, let $\Omega_r \in \mathbb{Q}[n][p_1,\dots,p_{r-1}]$ be the polynomial:

$$\Omega_r(n,p_1,\dots,p_{r-1}) = \sum_{\substack{k,\nu \\ k+w(\nu)=r+1}} \frac{(-1)^{|\nu|}p_\nu}{z_\nu}\sum_{i+j=k} Q_i^r(n)P^r_{\nu,j}(n).$$

We shall prove the following theorem.

Theorem 1 (part of [8, Conj. 1]). *For $n \geqslant r$, and for all partitions λ of n,*

$$\omega^\lambda_{r1^{n-r}} = \Omega_r(n,\mathfrak{p}_1(\lambda),\dots,\mathfrak{p}_{r-1}(\lambda)).$$

Moreover, for $n<r$, and all partitions λ of n, $\Omega_r(n,\mathfrak{p}_1(\lambda),\dots,\mathfrak{p}_{r-1}(\lambda)) = 0$.

Corollary 1 ([8, Conj. 2]). *The coefficient of p_{r-1} in Ω_r is 1.*

Although we have an explicit form for the inner sum in the definition of Ω_r, we have been unable to prove the following cancellations:

Conjecture 1 (Remaining from [8, Conj. 1]). In the above definition of Ω_r, the inner sum, which is a polynomial in $\mathbb{Q}[n]$ of degree at most k, is null if k is odd, and of degree $k/2$ if k is even.

Proof. Let us borrow the following result from [4] (reproduced in [13, p118]). Let $\rho = r1^{n-r}$, λ be a partition of n, μ be its shifted partition $\mu_i = \lambda_i + n - i$ for $1 \leqslant i \leqslant n$ and $\varphi(X) = \prod_{i=1}^n (X-\mu_i)$. Then

$$\omega^\lambda_{r1^{n-r}} = \frac{h_\lambda}{z_{r1^{n-r}}}\chi^\lambda_\rho = \frac{-1}{r^2}\sum_{i=1}^n \mu_i(\mu_i-1)\dots(\mu_i-r+1)\frac{\varphi(\mu_i-r)}{\varphi'(\mu_i)}, \quad (3)$$

which is also the coefficient of X^{-1} in the expansion of

$$\frac{-1}{r^2}X(X-1)\dots(X-r+1)\frac{\varphi(X-r)}{\varphi(X)}$$

in descending powers of X. Changing the sign of X, the latter expression can be explicitly written as

$$\frac{(-1)^r}{r^2}\prod_{i=0}^{r-1}(X+n+i)\prod_{i=1}^n \frac{X+r+\mu_i}{X+r+n-i}\prod_{i=1}^n\frac{X+n-i}{X+\mu_i}. \quad (4)$$

Recall now that the content polynomial $c_\lambda(X)$ of the partition λ is the polynomial in the indeterminate X defined ([13, p15]) by

$$c_\lambda(X) = \prod_{x\in\lambda}(X + c(x)).$$

From [13, p. 15], for $\xi_i = \lambda_i + m - i$, $1 \leqslant i \leqslant m$, we have

$$\frac{c_\lambda(X+m)}{c_\lambda(X+m-1)} = \prod_{i=1}^{m}\frac{X+\xi_i}{X+m-i}.$$

On one hand, if we take $m = n$, we get $\xi_i = \mu_i$ for $1 \leqslant i \leqslant n$. On the other hand, if $m = n+r$, then $\xi_i = r + \mu_i$ for $1 \leqslant i \leqslant n$ and $\xi_{n+i} = r - i$ for $0 \leqslant i < r$. Therefore expression (4) can be rewritten as

$$\frac{(-1)^r}{r^2}\prod_{i=0}^{r-1}(X+n+i)\frac{c_\lambda(X+n+r)}{c_\lambda(X+n+r-1)}\,\frac{c_\lambda(X+n-1)}{c_\lambda(X+n)} \tag{5}$$

Upon setting $X = 1/Y$, we obtain $\omega^\lambda_{r1^{n-r}}$ as a coefficient in a Taylor expansion:

$$\omega^\lambda_{r1^{n-r}} \;=\; [Y^{r+1}]Q^r(Y,n)\cdot L^r_\lambda(Y,n) \tag{6}$$

where $Q^r(Y,n)$ is given by formula (2) and

$$\begin{aligned} L^r_\lambda(Y,n) &= \prod_{x\in\lambda}\frac{\left(1+\frac{c(x)}{1+(n+r)Y}Y\right)\left(1+\frac{c(x)}{1+(n-1)Y}Y\right)}{\left(1+\frac{c(x)}{1+(n+r-1)Y}Y\right)\left(1+\frac{c(x)}{1+nY}Y\right)} \\ &= \Lambda_Y(X_{n+r}+X_{n-1}-X_{n+r-1}-X_n)\,, \end{aligned} \tag{7}$$

where the right hand side of the last identity is the exterior power Λ_Y of a disjoint union of alphabets in Λ-ring notation with $X_n = \left\{\frac{c(x)}{1+nY} \mid x \in \lambda\right\}$ (see [12, Chap. 1]). This exterior power can then be expanded into power-sums by the formula

$$L^r_\lambda(Y,n) = \sum_\nu z_\nu^{-1}p_\nu\left(-X_{n+r} - X_{n-1} + X_{n+r-1} + X_n\right)(-Y)^{|\nu|}. \tag{8}$$

The alphabet X_n factors into $\{c(x) \mid x \in \lambda\}\cdot(1+nY)^{-1}$, so that

$$p_\nu\left(-X_{n+r}-X_{n-1}+X_{n+r-1}+X_n\right) \;=\; \mathfrak{p}_\nu(\lambda)\cdot P^r_\nu(Y,n)$$

where the $P^r_\nu(Y,n)$ are the power sums defined by formula (1). Therefore from (6) and (8), $\omega^\lambda_{r1^{n-r}}$ is given by the coefficient of Y^{r+1} in

$$\sum_\nu \frac{(-1)^{|\nu|}\mathfrak{p}_\nu(\lambda)}{z_\nu}Y^{|\nu|}P^r_\nu(Y,n)Q^r(Y,n). \tag{9}$$

Consider now the coefficient of Y^i in $P_j^r(Y,n)$: from (1),

$$P_j^r(Y,n) = \sum_{i\geqslant 0} \binom{j+i-1}{i} \left(n^i + (n+r-1)^i - (n+r)^i - (n-1)^i\right)(-Y)^i. \quad (10)$$

Terms of degree 0 or 1 in Y cancel, and the coefficient of Y^2 is $-2r\binom{j+1}{2}$. This implies that the lowest degree of $P_\nu^r(Y,n)$ is $2\ell(\nu)$ and justifies the choice of $P_{\nu,i}^r(n) = [Y^{2\ell(\nu)+i}]P_\nu^r(Y,n)$. With $w(\nu) = |\nu| + 2\ell(\nu)$, the coefficient of Y^{r+1} in formula (9) gives $\omega^\lambda_{r1^{n-r}} = \Omega_r(n, \mathfrak{p}_1(\lambda), \ldots, \mathfrak{p}_{r-1}(\lambda))$.

From (10), the polynomial $P_{j,i}^r$ clearly has degree i in n, so that $P_{\nu,i}^r(n)$ has degree i. Expansions using the binomial theorem show that $Q_i^r(n)$ is a polynomial of degree $i-1$ without constant term.

For the second part of the theorem, observe that, starting from formula (3), all manipulations are valid including when $n < r$. But in Formula (3), the nullity for $n < r$ is immediate: in the summand, either $m_i < r$, and the falling power does the job, or there exists j such that $m_i = m_j + r$.

And finally, for the corollary, observe that the contribution of $Q^r(Y,n)$ is $(-1)^r/r^2$ by (2), while that of $P_{r-1}^r(Y,n)$ has been established to be $-2r\binom{r}{2}$. With $z_{(r-1)} = r-1$, the coefficient 1 is found. □

3 Elementary operators

3.1 Symmetric functions

Let $\mathbf{x}=\{x_1, x_2, \ldots\}$ be a set of indeterminates and let $\Lambda = \Lambda_{\mathbb{Q}}[\mathbf{x}]$ be the ring of symmetric functions in $\{x_1, x_2, \ldots\}$ over the field $\mathbb{Q}$ of rational numbers. The usual scalar product $<,>$ on Λ is defined on the linear basis $\{p_\lambda\}_\lambda$ by:

$$\forall\, \lambda, \mu \qquad < p_\lambda, p_\mu >= z_\lambda \delta_{\lambda,\mu}.$$

We need the following differential operators known in the litterature as Hammond's operators (see [14] or [13]):

Definition 1. *For each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, let ${p_\lambda}^\perp$ be the adjoint operator to the multiplication by $p_\lambda(\mathbf{x})$ with respect to the scalar product $<,>$:*

$$\forall\, f, g \in \Lambda \qquad < p_\lambda f, g >=< f, {p_\lambda}^\perp g > .$$

The operator ${p_\lambda}^\perp$ is conveniently described as a differential operator on the basis $\{p_\lambda(\mathbf{x})\}_\lambda$:

$${p_\lambda}^\perp = \lambda_1\lambda_2\cdots\lambda_k \frac{\partial^k}{\partial p_{\lambda_1}\partial p_{\lambda_2}\ldots\partial p_{\lambda_k}}.$$

The use of such operators in relation with connexion coefficients is not new and can be found for instance in [2]. We are interested in representing the multiplication by a conjugacy class as an action of an operator on the space of symmetric functions: more precisely we look for operators G_α, satisfying

$$\forall \beta, \gamma \vdash |\alpha|, \qquad [q_\gamma]\ G_\alpha \cdot q_\beta \ = \ [K_\gamma]\ K_\alpha \cdot K_\beta$$

where $\{q_\lambda = p_\lambda / z_\lambda\}_\lambda$ is an orthonormal basis of Λ. Here is a trivial way to do this:

Definition 2. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition of n. For any permutation $\rho \in \mathcal{C}_\lambda$, define the operator $G_\lambda : \Lambda^n \to \Lambda^n$ by*

$$G_\lambda \;=\; \frac{1}{z_\lambda} \sum_{\sigma \in \mathfrak{S}_n} p_{\lambda(\rho\sigma)} {p_{\lambda(\sigma)}}^{\perp} \;=\; \frac{1}{z_\lambda} \sum_{\nu,\mu \vdash n} \frac{\mathfrak{a}^{\nu}_{\lambda,\mu}}{z_\mu} p_\nu {p_\mu}^{\perp}. \tag{11}$$

From this definition and the orthogonality relation $p_\mu^\perp p_\lambda = z_\lambda \delta_{\lambda,\mu}$ for partitions of the same weight, it is immediate that for any partitions λ, μ of n

$$G_\lambda \cdot q_\mu = \sum_{\gamma \vdash n} \mathfrak{a}^{\nu}_{\lambda,\mu} q_\nu.$$

The operators G_λ are not interesting because their definition uses the structure constants $\mathfrak{a}^{\nu}_{\lambda,\mu}$ which they are meant to produce; however they provide an easy introduction to what we mean by representing the multiplication by the action of an operator on symmetric functions.

Our aim is to define a more interesting family H of such operators, satisfying for all partitions α,

$$\forall \beta, \gamma \vdash |\alpha|, \qquad [q_\gamma]\ H_{\bar{\alpha}} \cdot q_\beta \;=\; [K_\gamma]\ K_\alpha \cdot K_\beta.$$

Observe here that we require H, in some loose sense, to be defined in terms of the reduced partition $\bar{\alpha}$ and not of α.

3.2 Restricted permutations

In order to define our operators $H_{\bar{\alpha}}$, we need some elementary results on restricted permutations. For a subset S of $[n] = \{1, \dots, n\}$, and a permutation $\sigma \in \mathfrak{S}_n$, let $\sigma_{|S}$ be the permutation of the elements of S such that, for all $i \in S$, $\sigma_{|S}(i) = \sigma^k(i)$ where k is the least positive integer such that $\sigma^k(i)$ is in S.

The idea rests on the following observation: let ρ and σ be permutations in $\mathfrak{S}_n$, and consider the product $\tau = \rho\sigma$. Then, if $[n] \setminus S$ contains only fix points of ρ, we have $\tau_{|S} = \rho_{|S}\sigma_{|S}$, and conversely τ can be obtained from $\tau_{|S}$ by inserting after each $i \in S$ the block that separates i and $\sigma_{|S}(i)$ in the decomposition of σ into disjoint cycles.

Example. If $\rho_{|3} = (1\ 2\ 3)$ and $\sigma = (1\ aaa\ 2\ bbb)\ (3\ ccc)$, then $\sigma_{|3} = (1\ 2)\ (3)$, $\tau_{|3} = (1\ 3)\ (2)$ and $\tau = (1\ aaa\ 3\ ccc)\ (2\ bbb)$.

Given a permutation σ_0 of $\mathfrak{S}_p$, and $n \geqslant p$, the permutation σ_0 can be extended naturally to a permutation of $\mathfrak{S}_n$ by adding fix points. Therefore any $\sigma_0 \in \mathfrak{S}_p$ acts by left multiplication on $\mathfrak{S}_n$. We need one last definition: given a partition $\alpha = (\alpha_1, \dots, \alpha_\ell)$, the canonical permutation of type α is the permutation of cycle type α whose k-th cycle is $(\alpha_1 + \ldots + \alpha_{k-1} + 1, \dots, \alpha_1 + \ldots + \alpha_k)$.

3.3 The operator $H_{\bar{\alpha}}$ and Katriel's notations

We give two equivalent definitions of H.

Definition 3. *Let $\bar{\alpha}$ be a reduced partition of weight p. Let ρ_0 be the canonical permutation of cycle type $\bar{\alpha}$. Then the operator $H_{\bar{\alpha}} : \Lambda \to \Lambda$ is defined by:*

$$H_{\bar{\alpha}} = \frac{1}{z_{\bar{\alpha}}} \sum_{\sigma_0 \in \mathfrak{S}_p} \sum_{i_1, \dots, i_p \geqslant 1} p_{\gamma'} \cdot p^{\perp}_{\beta'} \tag{12}$$

where β' is the cycle type of any permutation τ obtained from σ_0 by inserting $i_j - 1$ elements after each $j \in \{1, \dots, p\}$, and γ' is the cycle type of $\rho_0 \tau$.

The fact that the cycle type γ' depends only on the integers $i_1, \dots, i_p$, and not on the elements we choose to insert in σ_0, is a consequence of the previous discussion on restricted permutations.

This operator is closely related to Katriel's bracket operators (which are not completely rigorously defined). A simple variation on his notation is:

$$\langle\!\langle i_1 + i_2; i_3 \mid i_1; i_2 + i_3 \rangle\!\rangle \quad \text{stands for} \quad \sum_{i_1, i_2, i_3 \geqslant 1} p_{[i_1+i_2, i_3]} p^{\perp}_{[i_1, i_2+i_3]},$$

where the brackets $[\,,]$ denote multisets of integers (*i.e.* partitions). A further simplification of this notation (even closer to Katriel's) is to replace each variable by its index and write sums as cycles:

$$\langle\!\langle (1,2)(3) \mid (1)(2,3) \rangle\!\rangle \quad \text{stands for} \quad \langle\!\langle i_1 + i_2; i_3 \mid i_1; i_2 + i_3 \rangle\!\rangle$$

Let us rewrite Definition 3 with this notation:

Definition 4. *Let $\bar{\alpha}$ be a reduced partition of weight p. Let ρ_0 be the canonical permutation of cycle type $\bar{\alpha}$. Then*

$$H_{\bar{\alpha}} = \frac{1}{z_{\bar{\alpha}}} \sum_{\sigma_0 \in \mathfrak{S}_p} \langle\!\langle \rho_0 \sigma_0 \mid \sigma_0 \rangle\!\rangle. \tag{13}$$

Finally, Katriel conjectured a symmetry in the coefficients, which allows the introduction of a last notation:

$$\langle P \mid Q \rangle \quad \text{stands for} \quad \langle\!\langle P \mid Q \rangle\!\rangle + \langle\!\langle Q \mid P \rangle\!\rangle$$

Examples. We keep the intermediate notation which we find more descriptive.

$$\begin{aligned}
H_2 &= \frac{1}{2} \langle\!\langle i_1; i_2 \mid i_1 + i_2 \rangle\!\rangle + \frac{1}{2} \langle\!\langle i_1 + i_2 \mid i_1; i_2 \rangle\!\rangle \;=\; \frac{1}{2} \langle i_1 + i_2 \mid i_1; i_2 \rangle \\
&= \frac{1}{2} \Big(\sum_{i_1, i_2 \geqslant 1} p_{i_1, i_2} p^{\perp}_{i_1 + i_2} + \sum_{i_1, i_2 \geqslant 1} p_{i_1 + i_2} p^{\perp}_{i_1, i_2} \Big) \\
&= \frac{1}{2} p_2 p^{\perp}_{11} + \frac{1}{2} p_{11} p^{\perp}_{2} + p_3 p^{\perp}_{21} + p_{21} p^{\perp}_{3} + p_4 p^{\perp}_{31} \frac{1}{2} p_4 p^{\perp}_{22} + \frac{1}{2} p_{22} p^{\perp}_{4} + \cdots \\
H_3 &= \frac{1}{3} \langle\!\langle i_1 + i_2 + i_3 | i_1, i_2, i_3 \rangle\!\rangle + \frac{1}{3} \langle\!\langle i_1, i_2, i_3 | i_1 + i_2 + i_3 \rangle\!\rangle + \frac{1}{3} \langle\!\langle i_1 + i_2 + i_3 | i_1 + i_2 + i_3 \rangle\!\rangle \\
&\quad + \frac{1}{3} \langle\!\langle i_1 + i_2, i_3 | i_1, i_2 + i_3 \rangle\!\rangle + \frac{1}{3} \langle\!\langle i_1 + i_3, i_2 | i_2, i_1 + i_3 \rangle\!\rangle + \frac{1}{3} \langle\!\langle i_2 + i_3, i_1 | i_3, i_1 + i_2 \rangle\!\rangle
\end{aligned}$$

$$H_3 = \langle\!\langle i_1{+}i_2,i_3 | i_1,i_2{+}i_3 \rangle\!\rangle + \frac{1}{3}\langle i_1{+}i_2{+}i_3 | i_1,i_2,i_3 \rangle + \frac{1}{3}\langle\!\langle i_1{+}i_2{+}i_3 | i_1{+}i_2{+}i_3 \rangle\!\rangle$$
$$= \frac{1}{3} \sum_{\substack{(i_1,i_2,i_3)\geqslant 1\\ n=i_1+i_2+i_3}} \left(p_n p^{\perp}_{(i_1,i_2,i_3)} + p_{(i_1,i_2,i_3)} p^{\perp}_n + p_n p^{\perp}_n + 3 p_{(i_1+i_3,i_2)} p^{\perp}_{(i_1+i_2,i_3)} \right)$$

$$H_{2^2} = \frac{1}{8}\langle i_1;i_2;i_3;i_4 | i_1{+}i_2;i_3{+}i_4 \rangle + \frac{1}{4}\langle\!\langle i_1{+}i_2;i_3;i_4 | i_1;i_2;i_3{+}i_4 \rangle\!\rangle$$
$$+ \frac{1}{4}\langle\!\langle i_1{+}i_2;i_3{+}i_4 | i_1{+}i_3;i_2{+}i_4 \rangle\!\rangle + \langle\!\langle i_1{+}i_2{+}i_3;i_4 | i_1;i_2{+}i_3{+}i_4 \rangle\!\rangle$$
$$+ \frac{1}{2}\langle i_1{+}i_2{+}i_3{+}i_4 | i_1{+}i_2;i_3;i_4 \rangle + \frac{1}{4}\langle\!\langle i_1{+}i_2{+}i_3{+}i_4 | i_1{+}i_2{+}i_3{+}i_4 \rangle\!\rangle$$

Katriel's *global conjecture* in [11] is that K_α "=" $H_{\bar\alpha}$. More formally, we shall prove the following theorem.

Theorem 2 (Global Conjecture). *Let α, β and γ be partitions of n, then*

$$[K_\gamma]\ K_\alpha \cdot K_\beta \;=\; [q_\gamma]\ H_{\bar\alpha} \cdot q_\beta .$$

Observe that applying a permutation of indices in an elementary bracket operator does not change it. Collecting equivalent terms to form a sum over "distinct contributions" (as we did in the examples), we immediately obtain from (13) a proof of Katriel's *central conjecture* on the resulting coefficients.

Proof. Let p be the weight of $\bar\alpha$, ρ_0 the associated canonical permutation, and ρ its natural extension in $\mathfrak{S}_n$. On one hand,

$$[K_\gamma]\ K_\alpha \cdot K_\beta \;=\; \frac{|\mathcal{C}_\alpha|}{|\mathcal{C}_\gamma|}\ [K_\alpha]\ K_\beta \cdot K_\gamma \;=\; \frac{z_\gamma}{z_\alpha}\ \mathrm{Card}\,\{(\sigma,\tau)\in \mathcal{C}_\beta\times\mathcal{C}_\gamma \mid \rho\sigma=\tau\}$$
$$= \frac{z_\gamma}{z_\alpha} \sum_{\sigma_0\in\mathfrak{S}_p} \mathrm{Card}\left\{(\sigma,\tau)\in \mathcal{C}_\beta\times\mathcal{C}_\gamma \mid \sigma_{|p}=\sigma_0,\ \tau_{|p}=\rho_0\sigma_{|p}\right\}$$

Our discussion on restricted permutations implies that, for all $\sigma_0\in\mathfrak{S}_p$,

$$\mathrm{Card}\left\{(\sigma,\tau)\in \mathcal{C}_\beta\times\mathcal{C}_\gamma \mid \sigma_{|p}=\sigma_0,\ \tau_{|p}=\rho_0\sigma_{|p}\right\}$$
$$= \sum_{(i_0,\dots,i_p)\in C(\beta,\gamma)} \binom{n-p}{i_0}\cdot(n-p-i_0)!\cdot\frac{i_0!}{z_{\beta-\beta'}}$$

where the sum runs over the compositions $(i_0,\dots,i_p)$ of n such that inserting i_j elements of $\{p+1,\dots,n\}$ after each j of $\{1,\dots,p\}$ in $\sigma_{|p}$ and $\tau_{|p}$ leads to permutations of respective cycle types β' and γ' with the following properties: $\forall i, b'_i\leqslant b_i,\ c'_i\leqslant c_i$ and $\beta-\beta'=\gamma-\gamma'$, where $\beta=1^{b_1}\dots n^{b_n}$ and so on, and $\beta-\beta'$ denotes the partition $1^{b_1-b'_1}\dots n^{b_n-b'_n}$. This simplifies to:

$$[K_\gamma]\ K_\alpha\cdot K_\beta \;=\; \frac{z_\gamma}{z_{\bar\alpha}} \sum_{\bar\sigma\in\mathfrak{S}_p}\ \sum_{(i_0,\dots,i_p)\in C(\beta,\gamma)} \frac{1}{z_{\beta-\beta'}}\ .$$

On the other hand,

$$H_{\bar{\alpha}} \cdot q_\beta \;=\; \frac{1}{z_{\bar{\alpha}}} \sum_{\sigma_0 \in \mathfrak{S}_p} \; \sum_{i_1,\dots,i_p \geqslant 0} p_{\gamma'} \, \frac{\pi_{\beta'}}{z_\beta} \, \frac{\partial p_\beta}{\partial p_{\beta'}}.$$

Since $\dfrac{\partial p_\beta}{\partial p_{\beta'}} = 0$ unless $\beta \geqslant \beta'$, in which case $\dfrac{\partial p_\beta}{\partial p_{\beta'}} = \dfrac{b_1! \dots b_n!}{(b_1 - b_1')! \dots (b_n - b_n')!} \, p_{\beta - \beta'}$, we obtain:

$$H_{\bar{\alpha}} \cdot q_\beta \;=\; \frac{1}{z_{\bar{\alpha}}} \sum_{\bar{\sigma} \in \mathfrak{S}_p} \; \sum_{(i_0,\dots i_p) \in C(\beta)} \frac{p_{\gamma' + \beta - \beta'}}{z_{\beta - \beta'}},$$

where the sums runs over the compositions of n such that $\forall i, b_i' \leqslant b_i$. So finally:

$$[q_\gamma] \;\; H_{\bar{\alpha}} \cdot q_\beta \;=\; \frac{z_\gamma}{z_{\bar{\alpha}}} \sum_{\sigma \in \mathfrak{S}_p} \; \sum_{(i_0,\dots i_p) \in C(\beta,\gamma)} \frac{1}{z_{\beta - \beta'}} \;=\; [K_\gamma] \;\; K_\alpha \cdot K_\beta.$$

□

An immediate consequence of Theorem 2 is that the operators $H_{\overline{\lambda}}$ are self adjoint and therefore their expansions $H_{\overline{\lambda}} = \sum_{\nu,\mu} a^{\nu}_{\lambda,\mu} p_\nu p_\mu^{\perp}$, are symmetric in ν and μ, a fact that can also be proved directly form their definition. This is also part of Katriel's conjecture.

3.4 Families of connexion coefficients

For a reduced partition $\bar{\alpha}$, let $K_{\bar{\alpha}}(n)$ be the sum in $\mathcal{Z}_n$ of all permutations with reduced cycle-type $\bar{\alpha}$ if $n \geqslant |\bar{\alpha}|$, and 0 otherwise.

Let $\bar{\alpha}$, $\bar{\beta}$ be reduced partitions and define the coefficients $\mathfrak{a}^{\bar{\gamma}}_{\bar{\alpha},\bar{\beta}}(n)$ by

$$K_{\bar{\alpha}}(n) \cdot K_{\bar{\beta}}(n) = \sum_{\bar{\gamma}} \mathfrak{a}^{\bar{\gamma}}_{\bar{\alpha},\bar{\beta}}(n) K_{\bar{\gamma}}(n). \tag{14}$$

In [1] it is proved that the $\mathfrak{a}^{\bar{\gamma}}_{\bar{\alpha},\bar{\beta}}(n)$ are polynomials in n. This also follows from Theorem 2: let k (resp. h) be the largest part of $\bar{\beta}$ (resp. $\bar{\gamma}$), and apply the elementary operator $H_{\bar{\alpha}}$ to $q_{\bar{\beta} 1^{n - |\bar{\beta}|}}$; the non-zero contributions are of the form

$$\left[q_{\bar{\gamma} 1^{n - |\bar{\gamma}|}}\right] \;\; p_\mu p_\lambda^{\perp} q_{\bar{\beta} 1^{n - |\bar{\beta}|}}$$

where λ and μ are partitions of length $|\bar{\alpha}|$ having parts of size at most respectively k and h. There are finitely many such partitions and the contribution of this term is a polynomial of degree ℓ_1 in n.

Theorem 2 is a generalisation of this result in the sense that it proves that other families of coefficients are polynomial. For instance, for any reduced partition $\bar{\alpha}$ with even weight, the coefficient

$$\mathfrak{b}_{\bar{\alpha}}(n) = [K_{2^n}] \;\; K_{\bar{\alpha}}(2n) \cdot K_{2^n}$$

is a polynomial in n. The expression of H_{2^2} presented before gives

$$\mathfrak{b}_{2^2}(n) = \frac{1}{4} \;\; [q_{2^n}] \;\; p_{2^2} p_{2^2}^{\perp} q_{2^n} \;\;=\;\; n(n-1).$$

4 Central characters for general partitions

In view of Section 3.4 we have obtained in Theorem 1 the eigenvalues of $K_r(n)$ as polynomials in $\mathbb{Q}[n][p_1, \ldots, p_{r-1}]$. Following a suggestion in [8], we seek similar results for the eigenvalues of $K_{\bar\alpha}(n)$ for any reduced partition $\bar\alpha$.

Let $\bar\alpha = (2^{a_2}, \ldots, k^{a_k})$ be a reduced partition. In similarity with the situation in the Farahat-Higman ring structure (see [13] p. 131, ex. 24, 25), we observe from Theorem 2:

$$K_{\alpha_1}(n) \cdots K_{\alpha_\ell}(n) = \left(\prod_{i=2}^{k} a_i!\right) \; K_{\bar\alpha}(n) + \sum_{\bar\beta,\ |\bar\beta|<|\bar\alpha|} c(\bar\beta, n) K_{\bar\beta}(n), \tag{15}$$

from which we deduce

$$K_{\bar\alpha}(n) = \frac{K_{\alpha_1}(n) \cdots K_{\alpha_\ell}(n)}{\prod_i a_i!} + \sum_{\bar\beta,\ |\bar\beta|<|\bar\alpha|} b^{\bar\beta}(n) \prod_i K_{\beta_i}(n) \tag{16}$$

where the partitions $\bar\beta$ also satisfy $|\bar\beta|+\ell(\bar\beta) \leqslant |\bar\alpha|+\ell(\bar\alpha)$: an elementary operator can only increase the length by breaking a cycle, but this operation prevents the insertion of a fix point and reduces accordingly the final weight. Now the $\omega_{\bar\alpha}^{\lambda}$ are eigenvalues of $K_{\bar\alpha}$, *i.e.* for $n > |\bar\alpha|$, and $\lambda \vdash n$, $K_{\bar\alpha}(n) \cdot \chi^\lambda = \omega^\lambda_{\bar\alpha 1^{n-|\bar\alpha|}} \chi^\lambda$. Using Formula (16) we obtain,

$$\omega^\lambda_{\bar\alpha 1^{n-|\bar\alpha|}} = \frac{\omega^\lambda_{\alpha_1 1^{n-\alpha_1}} \cdots \omega^\lambda_{\alpha_\ell 1^{n-\alpha_\ell}}}{\prod_i a_i!} + \sum_{\bar\beta} b^{\bar\beta}(n) \prod_i \omega^\lambda_{\bar\beta_i 1^{n-\bar\beta_i}},$$

so that we are led to define the polynomial $\Omega_{\bar\alpha}$ in $\mathbb{Q}[n][p_1, p_2, \ldots]$ by

$$\Omega_{\bar\alpha} = \frac{\Omega_{\alpha_1} \cdots \Omega_{\alpha_\ell}}{\prod_i a_i!} + \sum_{\bar\beta} b^{\bar\beta}(n) \prod_i \Omega_{\beta_i}, \tag{17}$$

where the sum ranges over reduced partitions $\bar\beta$ with $|\bar\beta| < |\bar\alpha|$. In each Ω_{α_i} in the previous formula, the monomials p_ν satisfy $w(\nu) \leqslant \alpha_i + 1$ so that the monomials p_ν in the product satisfy $w(\nu) \leqslant |\bar\alpha| + \ell(\bar\alpha)$, and it is also the case for the monomials in the $\Omega_{\bar\beta}$. Finally, from Theorem 1 we get the following theorem.

Theorem 3 (Part of [8, Conj. 3]). *Let $\bar\alpha$ be a reduced partition. For $n \geqslant |\bar\alpha|$, and for all partitions λ of n,*

$$\omega^\lambda_{\bar\alpha 1^{n-|\bar\alpha|}} = \Omega_{\bar\alpha}(n, \mathfrak{p}(\lambda)),$$

where $\Omega_{\bar\alpha}$ is a polynomial in $\mathbb{Q}[n][p_1, p_2, \ldots]$ involving only monomials p_ν such that $w(\nu) \leqslant |\bar\alpha| + \ell(\bar\alpha)$.

Corollary 2 ([8, Conj. 4]). *Let $\bar\alpha = (2^{a_2}, \ldots, k^{a_k})$ be a reduced partition, and $\alpha' = (1^{a_2}, \ldots, (k-1)^{a_k})$. Then the coefficient of $p_{\alpha'}$ in $\Omega_{\bar\alpha}$ is*

$$\prod_{i \geqslant 2} \frac{1}{a_i!}.$$

Conjecture 2 (Remaining from [8, Conj. 3]). The coefficient of a monomial p_ν of $\Omega_{\bar{\alpha}}$, which is a polynomial in $\mathbb{Q}[n]$, is in fact null if $k = |\bar{\alpha}| + \ell(\bar{\alpha}) - w(\nu)$ is even, and of degree at most $k/2$ otherwise.

This conjecture is a consequence of Conjecture 1 of the present article, using (17).

Conjecture 3 (From [8, Conj. 5]). The polynomial $\Omega_{\bar{\alpha}}$ vanishes on partitions that are too small, *i.e.* for $n < |\bar{\alpha}|$ and $\lambda \vdash n$, $\Omega_{\bar{\alpha}}(n, \mathfrak{p}(\lambda)) = 0$.

Conclusion. Differential operators on Λ similar to the $H_{\bar{\alpha}}$ can also be defined to describe the decomposition of the inner tensor product, or Kronecker product $\chi^\lambda \otimes \chi^\mu = \sum_\nu t^\nu_{\lambda\mu} \chi^\nu$ of two irreducible representations of $\mathfrak{S}_n$ as a linear combination of irreducible representations. These differential operators are defined using Schur functions and their adjoints.

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