A BIJECTION FOR LOOPLESS TRIANGULATIONS OF A POLYGON WITH INTERIOR POINTS

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Abstract. Loopless triangulations of a polygon with k vertices in $k+2n$ triangles (with interior points and possibly multiple edges) were enumerated by Mullin in 1965, using generating functions and calculations with the quadratic method.

In this article we propose a simple bijective construction of Mullin's formula. The argument rests on *conjugation of trees*, a variation of the cycle lemma designed for planar maps. In the much easier case of loopless triangulations of the sphere ($k = 3$), we recover and prove correct an unpublished construction of the second author.

Résumé. Les triangulations sans boucles d'un polygone à k côtés en $k+2n$ triangles (avec des points intérieurs et éventuellement des arêtes multiples) ont été énumérées par Mullin en 1965, à l'aide de séries génératrices et de la méthode quadratique.

Dans cet article, nous proposons une construction bijective simple de la formule de Mullin. L'argument repose sur la *conjugaison d'arbres*, une variation sur le lemme cyclique adaptée à l'énumération des cartes planaires. Dans le cas beaucoup plus facile des triangulations ($k = 3$), nous retrouvons et démontrons une construction esquissée par le second auteur.

1. Introduction

In 1965, R.C. Mullin published the following formula for the number of planar loopless triangulations of a rooted k-gon into $k+2n$ triangles (see below for precise definitions):

$$T^*_{k,n} \;=\; |\mathcal{T}^*_{k,n}| \;=\; \frac{2^{n+2}(2k+3n-1)!(2k-3)!}{(n+1)!(2k+2n)!(k-2)!^2} \tag{1}$$

for all $k \geqslant 2$ and $n \geqslant 0$ (see [Mul65] or [GJ83, p145]), which extends the well-known formula for triangulations of a k-gon without interior points:

$$T^*_{k,-1} \;=\; |\mathcal{T}^*_{k,-1}| \;=\; \frac{(2k-4)!}{(k-1)!(k-2)!} \tag{2}$$

for all $k \geqslant 3$. By duality this formula also accounts for the number of rooted non-separable planar maps with a root vertex of degree k and $k+2n$ vertices all of degree 3.

In his work, R.C. Mullin was closely following the seminal steps of W.T. Tutte in his *census* papers [Tut62a, Tut62b, Tut63]. In particular Formula (1) extends Tutte's formula

$$T_n \;=\; T^*_{3,n-2} \;=\; \frac{2^{n+1}(3n)!}{n!(2n+2)!} \tag{3}$$

for rooted loopless triangulations of the sphere with $2n$ triangles (or non-separable cubic maps with $2n$ vertices). The proof itself relies, following Tutte, on a recursive decomposition of triangulations that yields a recurrence for their number. Encoding the latter into generating functions then allows for a solution through the quadratic method and a few pages of calculus.

Ever since their discovery, efforts have been made to find derivations reflecting the elegant and simple product form of this and other formulas of Tutte for planar maps. In particular a construction based on the *conjugation of trees* principle was proposed in the second author's PhD thesis [Sch98] for Formula (3) and a few other formulas of Tutte (all, bipartite, non-separable maps). A new generalization of both Tutte's formula and a formula of Hurwitz was also proved along these lines to enumerate planar constellations [BMS00].

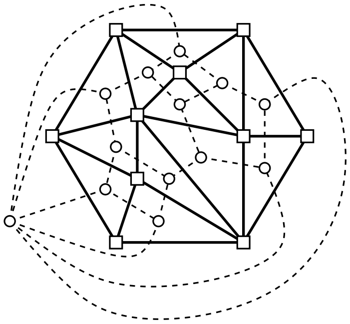

FIGURE 1. A triangulation of an hexagon and its dual

However two parameter formulas for triangulations like (1) seem to resist conjugation of trees. In this article we introduce a slight variation of the family under consideration, which cardinality can be easily deduced from $T^*_{k,n}$, and that appears more suitable for bijective constructions.

In view of this family $\mathcal{T}_{k,n}$, Mullin's formula reads

$$T_{k,n} = |\mathcal{T}_{k,n}| = \frac{2^{n+2}}{2k+2n}\binom{2k-2}{k}\binom{2k+3n}{n+1}. \tag{4}$$

The purpose of the present article is to provide a bijective construction of the latter formula. A main ingredient of our construction is again the *conjugation of trees* principle, and this confirms the adequacy of this approach to the bijective enumeration of planar maps.

However the bijection involves two new ingredients with respect to the treatment of Tutte's formulas. On the one hand, a *special* vertex is introduced in the construction, that allows to account for parameter k of Mullin's formulas. On the other hand, as opposed to the case of constellations [BMS00], the inverse construction does not rely on breadth-first search. Instead, in order to deal with non-separability, one has to resort on more difficult recursive arguments.

The rest of the article is organized as follows: after we establish Formula (4) for the cardinality of $\mathcal{T}_{k,n}$, we exhibit a simple family $\mathcal{E}_{k,n}$ of trees (balanced blossom trees), that are clearly enumerated by the same formula, and we define in a few lines an application φ from $\mathcal{E}_{k,n}$, that we claim onto $\mathcal{T}_{k,n}$. Then comes the harder part, as often with bijections, namely the proof for the unbeliever that the image of the application φ is indeed $\mathcal{T}_{k,n}$ and that it is one-to-one.

2. THE ENUMERATIVE FORMULA FOR ROOTED LOOPLESS TRIANGULATIONS

2.1. **Definitions around planar maps.** Let us make more precise the definitions of the objects under consideration. A (planar) map is a two-cell embedding of a connected planar graph into the sphere considered up to orientation preserving homeomorphisms of the sphere. Multiple edges are allowed. The degree of a vertex or a face is the number of (sides of) edges incident to that vertex or face. The vertex-degree of a face is the number of vertices incident to that face.

A planar map is non-separable if it contains no cut-vertex, that is to say no vertex that can be cut into two vertices (each taking part of the edges) in a way that the resulting graph would not be connected anymore.

A map is rooted if one edge is chosen and oriented, which ensures that the considered object has a trivial automorphism group. The startpoint of the root (edge) and the face on its right hand are called respectively root vertex and root face. By convention, this face will be chosen to be the infinite face when representing maps in the plane.

The dual M^* of a map M is obtained from M by putting a vertex in each face of M and an edge of M^* across each edge of M. If M is rooted, the root edge of M^* is the dual of the root edge of M, oriented in such a way that the root vertex of M^* is the dual of the root face of M. This construction is clearly involutive on unrooted maps (see Figure 1).

2.2. **Rooted loopless triangulations.** A triangulation is a planar map such that each face has degree 3. We will only consider loopless triangulations, hence faces are "real" triangles, in the sense that their vertex-degree is 3.

A loopless triangulation of a rooted k-gon is a planar map such that the root face has vertex-degree k while all other faces have degree 3. A rooted triangulation of a k-gon is the same thing except that the root face need not be the distinguished face of degree k. The terminology refers to the possibility, in order to draw the map in the plane, to take the k-gon as infinite face.

A loopless triangulation of a k-gon has $k+2n$ triangles for some integer $n \geqslant -1$, and hence $2k+3n$ edges and $k+n+1$ vertices (k exterior and $n+1$ interior ones). Let $\mathcal{T}_{k,n}$ be the set of rooted loopless triangulations of a k-gon into $k+2n$ triangles. Then

$$k\,T_{k,n} \;=\; 2(2k+3n)\,T^*_{k,n},$$

as immediately follows upon considering doubly rooted triangulations with one root on the polygon and the other anywhere: these can be regarded either as rooted loopless triangulations of a k-gon in which some edge of the k-gon is distinguished (and implicitly oriented in such a way that the k-gon is the corresponding root face), or as loopless triangulations of a rooted k-gon in which some edge is distinguished and oriented.

Hence Mullin's formula becomes

$$T_{k,n} \;=\; 2^{n+3}\frac{(2k+3n)!(2k-3)!}{k\,(n+1)!(2k+2n)!(k-2)!^2},$$

and can be rewritten as previously claimed:

$$T_{k,n} \;=\; \frac{2^{n+2}}{2k+2n}\binom{2k-2}{k}\binom{2k+3n}{n+1}.$$

This formula stands for any $k \geqslant 2$ and any $n \geqslant -1$: it specializes correctly for $k \geqslant 3$, $n=-1$, according to Formula 2; as for the degenerate case $k=2$, $n=-1$, which can only be interpreted as the case of a loop at the special vertex, it boils down to 1.

Observe that $T_n = n\;T_{3,n-2}$: it expresses the fact that a map in $\mathcal{T}_{3,n-2}$ can be regarded as a rooted loopless triangulation with n triangles among which one is distinguished.

2.3. **Dual family.** A cubic map is a map with all vertices of degree 3, and a near-cubic map is a map with all vertices of degree 3, except maybe one. Let $\mathcal{C}_n$ and $\mathcal{C}_{k,n}$ be respectively the set of non-separable cubic maps with $2n$ vertices and the set of non-separable near-cubic maps with a special vertex of degree k and $k+2n$ vertices of degree 3. They are respectively the dual sets of $\mathcal{T}_n$ and $\mathcal{T}_{k,n}$.

3. The constructive census of triangulations

In this section we construct a set of simple objects counted by $T_{k,n}$ and a transformation of these objects that we claim is a bijection onto $\mathcal{T}_{k,n}$.

Terminology for trees. All the trees we are interested in are planted plane trees. In the context of planar maps, it is convenient to define a plane tree as a planar map with only one face, although this is equivalent to classical recursive definitions. Planted means that one vertex of degree 1 is distinguished and called the root.

We shall consider an enriched terminology for trees, with two kinds of vertices of degree 1, *buds* and *leaves*, three kinds of edges, *links*, *inner edges* and *stems*, and three kinds of vertices of larger degrees, *generic*, *pathological* and *special.* Buds and leaves are always incident to stems (as opposed to links or edges) and in pictures, buds are represented by arrows. This terminology reflects the very different roles played by otherwise similar items and hopefully makes things clearer once accepted...

(a) Generic vertices (b) Special vertex

FIGURE 2. From trees to blossom trees

3.1. **Planted plane trees.** The first remark is that the following binomial coefficient, taken from Formula (4),

$$A_{k,n} = \binom{2k+3n}{n+1} = \frac{1}{2k+3n+1}\binom{2k+3n+1}{1, n+1, 2k+2n-1}$$

is the number of planted plane trees with (see also Figure 4.a)

- one special vertex of degree $2k-2$,
- $n+1$ generic vertices, of degree 4,
- $2k+2n$ leaves (including the root) and their $2k+2n$ stems,
- and $n+1$ inner edges connecting the generic and special vertices.

This is nothing but the classical formula for planted plane trees with given numbers of vertices of each degree ([GJ83, p113]). Let us call $\mathcal{A}_{k,n}$ the family of these trees.

Formula (4) now reads

$$T_{k,n} = \frac{2}{2k+2n} 2^{n+1} \binom{2k-2}{k} A_{k,n}, \tag{7}$$

and one can recognize in this formula, the appearance of the numbers of leaves, generic vertices and edges incident to the special vertex.

3.2. **Blossom trees.** Let proceed with the interpretation of the formula by considering the factor

$$B_{k,n} = 2^{n+1}\binom{2k-2}{k} A_{k,n}.$$

Since a tree A of $\mathcal{A}_{k,n}$ has $n+1$ generic vertices of degree 4, the factor 2^{n+1} can be interpreted as the number of ways to select two opposite corners on each generic vertex, while the binomial factor appears as the number of ways to select $k-2$ of the $2k-2$ edges incident to the special vertex.

Given such a selection, let us apply the transformation of Figure 2.a to generic vertices and, that of Figure 2.b to the special vertex. Each generic vertex is expanded into two vertices of degree 4 joined by a generic link, each carrying a bud. The selected edges on the special vertex are transformed to make room for a special link and two buds attached to a pathological vertex of degree 4. Observe that in these constructions buds always immediately precede links in counterclockwise direction around created vertices.

The set $\mathcal{B}_{k,n}$ of trees that are constructed in this manner from trees of $\mathcal{A}_{k,n}$ is of course of cardinality $B_{k,n}$. We call them *blossom trees.* By construction blossom trees are exactly the planted plane trees with (see also Figure 3 left, or Figure 4.b)

- one special vertex incident to $k-2$ special links and k edges;
- $k-2$ pathological vertices of degree 4, incident to the $k-2$ previous links, each carrying two buds right before the link in counterclockwise order;
- $2n+2$ generic vertices of degree 4, organized in $n+1$ pairs connected by generic links, each vertex carrying one bud right before the link in counterclockwise order;
- $2k+2n$ leaves, $2k+2n-2$ buds, and their $4k+4n-2$ stems,
- $n+1$ inner edges connecting some generic, pathological or special vertices.

Formula (4) now reads

$$T_{k,n} = \frac{2}{2k+2n} B_{k,n}, \tag{8}$$

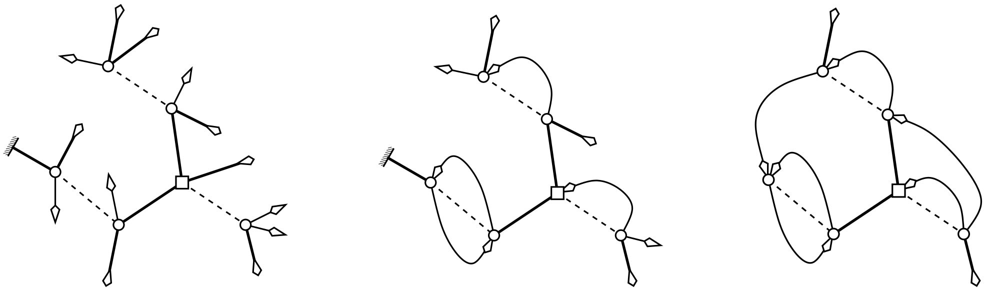

FIGURE 3. The partial closure of an unbalanced blossom tree

making it inviting to distinguish two leaves among the $2k+2n$.

3.3. **Balanced blossom trees.** The *partial closure* of a blossom tree B consists in the following greedy procedure (see Figure 3 or Figure 4.d). Start with $\dddot{B}^{(0)} = B$, $i = 1$.

(1) Find a bud b_i and a leaf ℓ_i such that, walking from b_i to ℓ_i around the infinite face of $\dddot{B}^{(i-1)}$ in counterclockwise direction, no other bud or leaf is met.
(2) Fuse b_i, ℓ_i and their stems into an edge m_i so as to create a bounded face around the previous walk. In particular this new bounded face contains no bud or leaf.
(3) Call $\dddot{B}^{(i)}$ the resulting map and, if it still contains buds, increment i and return to Step (1).

Observe that the latter loop continues until there is no more free bud. The operation in Step (2) is called the *matching* of b and ℓ, and the resulting edge is called a *matching edge.*

The result of this partial closure is a planar map $\dddot{B} = \dddot{B}^{(2k+2n-2)}$ with $k+2n$ vertices of degree 4, one special vertex of degree $2k-2$, and two remaining leaves that we call *free* in the infinite face. This map $\dddot{B}$ is independent of the exact order in which buds and leaves have been matched, (exactly like in a balanced parenthesis word, there is a partial order of inclusion of pairs, and the freedom of the algorithm lies in ordering incomparable pairs).

A blossom tree is called *balanced* if its root is one of the two leaves that remain free throughout partial closure. Let $\mathcal{E}_{k,n}$ be the balanced subset of $\mathcal{B}_{k,n}$. Two blossom trees are called *conjugated* if they can be obtained one from another simply by changing the root leaf. The resulting conjugacy classes of $\mathcal{B}_{k,n}$ are naturally associated with unplanted trees. Matchings between buds and leaves only depend on the conjugacy class of the blossom tree, hence we can also consider the partial closure of an unplanted tree.

Now consider a blossom tree B with root leaf r and ℓ one of the two free leaves of B. Taking now ℓ as root of B, we obtain a balanced blossom tree with a second distinguished leaf r. This yields[1]:

$$2\,B_{k,n} \;=\; (2k+2n)\,E_{k,n}$$

where $E_{k,n}$ denote the number of balanced blossom trees.

As a consequence, Formula (4) finally reads

$$T_{k,n} \;=\; E_{k,n},$$

and we are lead to seek a bijection between triangulations and balanced blossom trees.

3.4. **Case of $\mathcal{T}_n$.** A similar (but much simpler) construction provides an interpretation of Tutte's enumerative formula for the set $\mathcal{T}_n$ of loopless triangulations with n triangles, that can be rewritten in the following way:

$$T_n \;=\; \frac{2}{2n+2}\,2^n\,\frac{1}{2n+1}\binom{3n}{n}. \tag{9}$$

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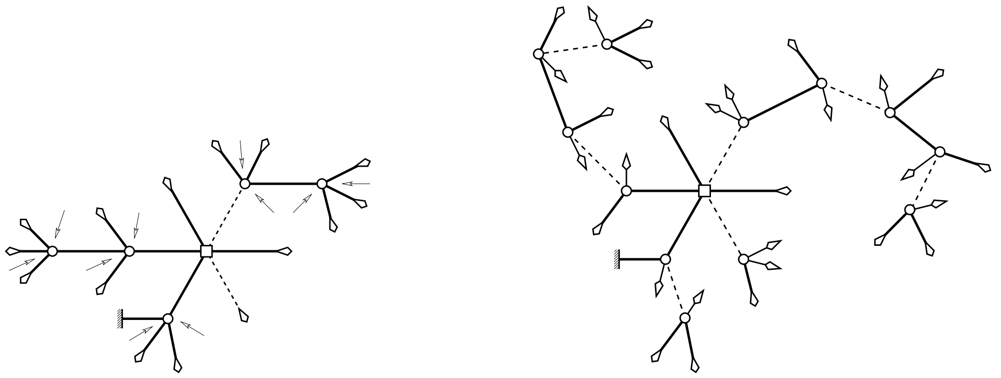
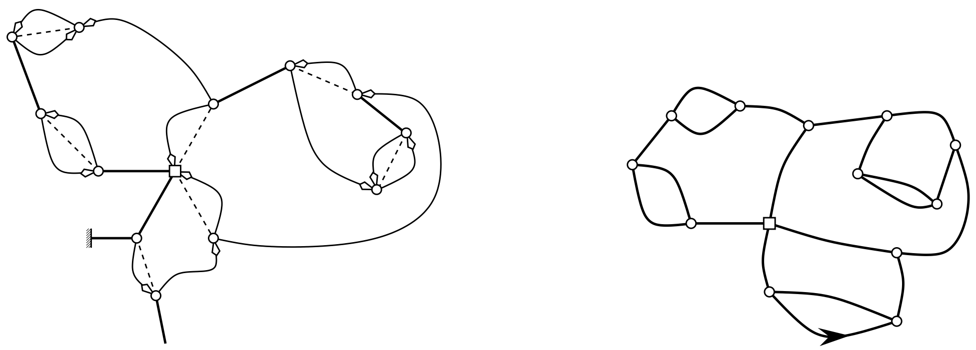
[1] Observe that this relation is the translation for conjugacy classes of trees of the *cycle lemma* for conjugacy classes of Łukasiewicz words. This lemma, initially due to Dworetzki and Motzkin, underlies Raney's combinatorial proof of the Lagrange inversion formula [Lot97, Chap. 11]. This analogy motivates our choice of terminology.



(a) A tree in $\mathcal{A}_{4,4}$, in which some corners and edges are distinguished,

(b) the corresponding (balanced) blossom tree,



(c) its partial closure, indicating the two free leaves

(d) and the corresponding rooted near-cubic map.

FIGURE 4. An example of complete closure

The coefficient $\frac{1}{2n+1}\binom{3n}{n}$ is the number of planted plane ternary trees with n internal nodes, that is trees with n generic vertices of degree 4, $n-1$ inner edges and $2n+2$ stems and leaves (including the root). The blossom trees obtained from these trees by the transformation of Figure 2.a have $2n$ generic vertices, their n links and $2n$ buds, $2n+2$ leaves, $4n+2$ stems and $n-1$ inner edges. Let $\mathcal{B}_n$ be the set of these blossom trees without special vertex. After the partial closure of any of these trees, two leaves remain unmatched, so the ratio of balanced blossom trees in $\mathcal{B}_n$ is $\frac{2}{2n+2}$. Hence the corresponding subset $\mathcal{E}_n$ has cardinality

$$E_n = \frac{2}{2n+2} \cdot 2^n \cdot \frac{1}{2n+1}\binom{3n}{n} = T_n.$$

Notations. In the following, $\mathcal{E}$ denotes the set of all balanced blossom trees, and $\mathcal{U}$ the set of all *unplanted* blossom trees (with or without special vertex). Any tree in $\mathcal{U}$ corresponds to one or two trees in $\mathcal{E}$, depending on its automorphism group.

3.5. **The complete closure.** In fact the bijection was almost already completely described. Let us define the *complete closure* φ as an application defined on the set $\mathcal{E}$: given B a tree of $\mathcal{E}$,

(1) construct the partial closure $\underline{\ddddot{B}}$ of B,
(2) remove all the links and call $\overline{B}$ the result,
(3) fuse the two remaining stems of $\overline{B}$ into a root edge oriented away from the root of B,

the resulting rooted planar map is $\varphi(B)$. (See Figure 4 for a complete example.)

Our main result, to be proved in the rest of the paper, is the following theorem.

Theorem 1. *The complete closure* φ *is a bijection from the set* $\mathcal{E}_{k,n}$ *(resp.* $\mathcal{E}_n$*) of balanced blossom trees onto the set* $\mathcal{C}_{k,n}$ *(resp.* $\mathcal{C}_n$*) of (near-)cubic maps and by duality onto the set* $\mathcal{T}_{k,n}$ *(resp.* $\mathcal{T}_n$*) of triangulations.*

The proof is twofold. First we prove that the complete closure of a tree is indeed a non-separable (near-)cubic map. Then we prove that the application is one-to-one.

4. THE CLOSURE OF A BALANCED BLOSSOM TREE

Let B be a balanced blossom tree of $\mathcal{E}_{k,n}$. In this section we prove that the complete closure $\varphi(B)$ is indeed a non-separable near-cubic map with the expected number of vertices of each kind.

The vertices of degree 4 of B, either generic or pathological, are incident to exactly one link. After Step (2) of the complete closure, they result in vertices of degree 3. As for the special vertex, it is incident to $k-2$ links and k edges so that it yields a vertex of degree k in $\varphi(B)$. The rooted planar map $\varphi(B)$ hence contains a vertex of degree k and $k+2n$ vertices of degree 3. As a consequence, $\varphi(B)$ belongs to $\mathcal{C}_{k,n}$ if and only if it is non-separable, a fact we shall now prove. Similarly, if B belongs to $\mathcal{E}_n$, $\varphi(B)$ belongs to $\mathcal{C}_n$ if and only if it is non-separable.

Observe that, since the matching of buds and leaves only depends on the conjugacy class of a blossom tree, the non-separability of the complete closure is indeed a property of the underlaying unplanted tree, and not of the balanced rooting. In the sequel, for convenience's sake, we consider an unplanted blossom tree U in $\mathcal{U}$.

A preliminary observation is that any separating vertex of degree 3 is incident to a separating edge. It is thus sufficient to prove on the one hand that $\varphi(U)$ has no separating edge (Section 4.1 to 4.4) and on the other hand that the possible special vertex is not separating (Section 4.5).

4.1. A preliminary lemma on the structure of blossom trees.

Consider U a blossom tree of $\mathcal{U}$ and $e=(v_1,v_2)$ an inner edge or a link of A. The *decomposition* of U at e consists in cutting e in its middle, so as to create two new leaves ℓ_1 and ℓ_2, attached by two stems e_1 and e_2 respectively to v_1 and v_2. As a result, the tree U yields two subtrees $U_1(e)$ and $U_2(e)$, respectively containing v_1 and v_2. A leaf ℓ of U is said *incoming* with respect to e if, in the partial closure of U, it is free or matched to a bud b that does not belong to the same subtree as ℓ (with respect to e). By extension, the matching edge (b,ℓ) is also called incoming with respect to e.

The following lemma is immediate upon counting leaves and buds in each subtree and considering the cyclic orders around v_1 and v_2.

Lemma 1. *Let* U *be a blossom tree and* e *an inner edge or a link of* U*.*

- *If* e *is an inner edge, then* $U_1(e)$ *and* $U_2(e)$ *are well formed blossom trees, with two more leaves than buds (including* ℓ_1 *and* ℓ_2*), and thus at least one incoming leaf each.*
- *If* e *is a link between two generic vertices, then* $U_1(e)$ *and* $U_2(e)$ *contain two more leaves than buds (including* ℓ_1 *and* ℓ_2*). Moreover in the partial closure of* U*, the bud adjacent to* v_1 *in* $U_1(e)$ *is matched with an incoming leaf of* $U_2(e)$*.*
- *If* e *is a link incident to the special vertex (assumed in* $U_1(e)$*), then* $U_1(e)$ *has four more leaves than buds and* $U_2(e)$ *has as many buds as leaves (including* ℓ_1 *and* ℓ_2*). As a consequence, in the closure of* U*, the two buds adjacent to* v_2 *in* $U_2(e)$ *are matched to two incoming leaves of* $U_1(e)$*, and* $U_2(e)$ *has at least two incoming leaves.*

4.2. The incremental complete closure.

Let us now consider an application of the (greedy) partial closure procedure of Section 3.3 to U, resulting into the map $\ddot{U}$ through the sequences b_i, ℓ_i, m_i and $\ddot{U}^{(i)}$, for $i \geqslant 1$. Given a matching edge m, obtained from (b,ℓ), we define $e(m)$ to be the unique link incident to the vertex adjacent to b. By construction, for each link e of A there are exactly two indices $j<i$ such that $e(m_i)=e(m_j)=e$. Let us call these indices, the *dates* of e. Finally define a planar map $\underline{U}^{(i)}$ by deleting from $\ddot{U}^{(i)}$ every *generic* link which largest date is less or equal to i. In other terms, $\underline{U}^{(i)}$is constructed from $\underline{U}^{(i-1)}$ by adding m_i and removing $e(m_i)$ if it is generic and the other matching edge m_j such that $e(m_j)=e(m_i)$ satisfies $j<i$. Let $\underline{U}$ be the resulting map.

The following technical lemma precisely describes the evolution of connectedness in $\underline{U}^{(i)}$.

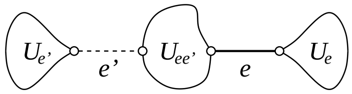


FIGURE 5. Simultaneous decomposition of U at e and e'

Lemma 2. *For all $i \geqslant 0$ the planar map $\underline{U}^{(i)}$ is connected. Moreover for any link or inner edge e of U, the graphs induced in $\underline{U}^{(i)}$ respectively by the vertices of $U_1(e)$ and by the vertices of $U_2(e)$ are connected.*

Proof. The lemma is obviously true for the tree $\underline{U}^{(0)} = U$. Assume now the lemma true for indices until $i-1$ and consider the construction of $\underline{U}^{(i)}$ from $\underline{U}^{(i-1)}$. Let $e' = e(m_i)$ and j be the other date of e'.

There is a deletion only if e' is a generic link and $j < i$. In this case, observe first that, according to Lemma 1, the matching edge m_i connects a vertex of $U_1(e')$ to a vertex of $U_2(e')$ so that by induction hypothesis $\underline{U}^{(i)}$ remains connected upon deleting e'.

Then consider another link or inner edge e and the decompositions of U at e and e': performing both decompositions yields three subtrees, U_e, $U_{ee'}$ and $U_{e'}$ where the indices refer to incidences with e and e' (see also Figure 5). In $\underline{U}^{(i-1)}$, the graphs induced respectively by U_e and $U_{e'}$ are connected by induction hypothesis. The deletion of e' does not touch the graph induced by U_e so that we only have to deal with the graph induced by $U_{e'} \cup U_{ee'}$. Since e' is generic, one of m_i or m_j is incident to the endpoint of e' in $U_{ee'}$ and has its other endpoint (the leaf) in $U_{e'}$. Since $U_{e'}$ is connected, the deletion of e' does not disconnect $U_{e'} \cup U_{ee'}$.

□

4.3. Separating edges and generic links.

Lemma 3. *The only separating edges in $\underline{U}$ are inner edges of U that separate the two free leaves.*

Proof. Consider a matching edge m, and let $e = e(m)$ and m' be the second matching edge with $e(m') = e$. Then Lemma 1 asserts that m and m' are incoming with respect to e. In view of Lemma 2, their respective endpoints on both sides of e can be connected to construct a cycle containing m and m'. Moreover, if e is special, the same argument provides a cycle through e and, for instance, m.

Let now e be an inner edge of A that is still a separating edge in $\underline{U}$. Consider the decomposition of U at e. No matching edge connects a vertex of $U_1(e)$ to a vertex of $U_2(e)$ otherwise e would not be separating (Lemma 2). In view of Lemma 1, this implies that there is one free leaf in both subtrees. Step (3) of the complete closure, fusing these two free leaves to form the root, will thus provide a cycle containing e and the root.

□

4.4. Matching edges are not separating in $\varphi(U)$.

Lemma 4. *The only separating edges in $\overline{U}$ are inner edges of U that separate the two free leaves.*

Proof. If U has no special vertex, $\overline{U} = \underline{U}$, hence this lemma is equivalent to Lemma 3.

Now suppose that U has a special vertex with degree $2k-2$. In order to show that the removal of the special links from $\underline{U}$ does not make any matching edge separating, it is sufficient to prove that any two faces that are merged by removing some special links have no common matching edge.

Let us consider the $2k-2$ subtrees of U at the special vertex v, more precisely defined as the subtrees not containing v in the decomposition of U at any edge or link incident to v. We call such a subtree generic or pathological depending on whether it is attached to the special vertex by an inner edge or by a link. At any step i of the construction of Section 4.2, Lemma 2 ensures that these subtrees induce connected subgraphs of $\underline{U}^{(i)}$, and that, according to Lemma 1, any of them has at least one incoming leaf.

Given a particular ordering of the matching in the application of the partial closure procedure, let us consider the time j at which, for the first time, an incoming edge of a subtree at v is matched.

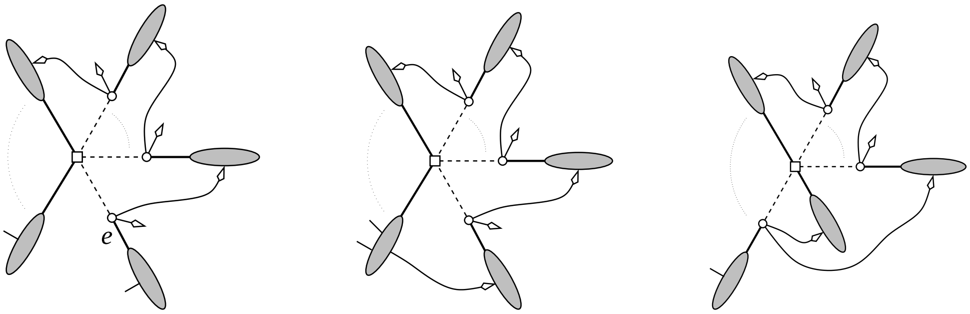


FIGURE 6

This is also the first time that a matching edge is created between two subtrees. Let us now consider an ordering such that j is as large as possible. In this case, at time j, all the matchings that are internal to each subtree have been performed. More precisely, with the notation of Section 4.2, the ordering is such that, for any $i < j$, b_i and ℓ_i belong to the same generic or pathological subtree, and, for any $i \geqslant j$, b_i and ℓ_i belong to different subtrees.

Perform then the construction until Step $j-1$. As already observed, at that moment the subtrees are two by two independent. Moreover, at each pathological vertex, the bud that precedes the link in counterclockwise order is in position to be matched with the first incoming leaf of the next subtree (in counterclockwise order around the special vertex). Every such matching creates a bounded face, which cannot be affected by any further step since it does not contain any generic link.

Once these $k-2$ matchings are performed, only two kinds of buds can be matched in such a way that the created bounded face contains a special link: the first bud (if any) of a generic subtree that precedes a pathological one, or the second bud of a pathological vertex that precedes a sequence of subtrees with no more unmatched leaf (or bud). These matchings also create faces that will not be affected by any further step.

As illustrated by Figure 6 there are three different ways for a group of faces to be merged into one face by the removal of special links. In each case, we need to argue that, as a whole, these faces do not complete a turn around v.

- In the first case (Figure 6 left), a (non-empty) sequence of bounded faces merge with the infinite face of $\underline{U}$. In this case the bounded faces cannot perform a complete turn around v. Hence two non successive faces in the sequence share no edge, and two successive faces share a special link. In any case they do not have a matching or inner edge in common. As for the infinite face, in view of the disposition of buds, it may only be incident twice to the inner edge marked e in the figure. In this case the shaded subtree below e contains exactly one of the two free leaves so that e separates the two leaves.
- The second case involves two generic subtrees and a (non-empty) sequence of pathological ones. Since $k \geqslant 2$, the two generic subtrees are different, and hence again the complete turn is excluded.
- In the third case, there exists a pathological vertex p_1 such that its second bud is matched with a leaf that belongs to a pathological subtree attached on a pathological vertex p_2. It implies that the sequence S of subtrees that follow p_1 and precede p_2 around the special vertex has only one free leaf. In other words, this sequence contains exactly one more generic subtree than pathological ones. Hence the number of involved pathological subtrees is at least the number of involved generic subtrees minus one, so that at least one generic subtree is not involved, hence p_1 is different from any involved pathological vertex that follows S. Hence the complete turn is not performed.

We conclude that matching edges are not separating edges and that all separating inner edges still separate the two leaves. Since all links have been removed, the lemma is proved. □

FIGURE 7

This lemma proves that $\overline{U}$ can be described as an alternating sequence $\overline{U}_1, e_2, \overline{U}_2, \ldots, e_p, \overline{U}_p$ of submaps $\overline{U}_i$ and edges e_i (with p possibly equal to 1), such that $\overline{U}_1$ and $\overline{U}_p$ carry one free leaf each (see Figure 7), and no $\overline{U}_i$ contains a separating edge.

As a consequence, $\varphi(U)$ has no separating edge.

4.5. The special vertex.

Suppose that U has a special vertex. The following lemma concludes the proof that $\varphi(U)$ is non-separable.

Lemma 5. *The special vertex v is not a separating vertex of $\varphi(U)$.*

Proof. Assume that the special vertex v is separating in $\overline{U}$ (as given by Step (2) of complete closure) and consider a decomposition of $\overline{U}$ into two components $\overline{U}_1$ and $\overline{U}_2$ connected only at v. This decomposition induces a decomposition of $\ddot{U}$: special links connect v to a vertex of $\overline{U}_1$ or $\overline{U}_2$ and do not interfere; once special links are replaced, generic links appear inside bounded faces and hence inside the two components.

In turn the decomposition of $\ddot{U}$ at v induces a decomposition of the tree U into two sequences of subtrees rooted at v such that there is no matching edge from one to the other. Since Lemma 1 provides in particular an incoming leaf on the first tree of both these sequences, these leaves must be the two free leaves of U.

Returning to $\overline{U}$, we conclude that $\overline{U}_1$ and $\overline{U}_2$ each contain one free leaf. Hence v is not a cut vertex anymore after Step (3) of the complete closure. □

5. THE INVERSE CONSTRUCTION

In this section we define by induction on the number of edges a construction which is inverse to the complete closure.

Let us first consider the minimal cases of non-separable (near-)cubic maps with at most two vertices. The case $k = 2$, $n = -1$ is the degenerate case of the loop at a special vertex and corresponds to the tree with one special vertex of degree 2. The case $n = 1$, without special vertex, is the case of a bundle of three edges between two vertices and corresponds to the unique balanced blossom tree with two generic vertices. The case $k = 3$, $n = -1$ is the case of a bundle of three edges between two vertices, one of them being special; the two different rootings of this map correspond to the two balanced rootings of the unique blossom tree with a special vertex of degree 3 and one single pathological vertex.

Now suppose that C is a rooted non-separable (near-)cubic map with at least three vertices among which, possibly, a special vertex of any degree, and the others of degree 3. Let the root edge be oriented from a vertex v_1 to a vertex v_2, and define $\tilde{C}$ by cutting the root edge into two stems with leaves f_1 and f_2. If there exists $B \in \mathcal{E}$ such that $C = \varphi(B)$, then B is necessarily planted on leaf f_1, and reconstructing B consists in recovering links between vertices: these links determine which vertices are generic or pathological, and which stems carry leaves or buds.

In the following, for any map $\tilde{C}$, we determine at least one link between two vertices that exists *necessarily* in any tree U in $\mathcal{U}$ such that $\overline{U} = \tilde{C}$, and we construct two strict submaps of $\tilde{C}$ in which links induce links in $\tilde{C}$. Since such a construction is well defined for any map $\tilde{C}$, it proves that φ is one-to-one.

The construction depends on whether $\tilde{C}$ is separable:

5.1. The map $\tilde{C}$ contains a separating vertex other than v_1 or v_2.

Since C is non-separable, the map $\tilde{C}$ is organized as a chain of non-separable components between v_1 and v_2. In the rest of this section, separating vertices are implicitly supposed to be distinct from v_1 and v_2 and two cases are distinguished.

– First case: the map $\tilde{C}$ has a separating vertex v that is not the special vertex (see Figure 8). In this case v has degree 3 and, as already argued, there is a separating edge e. In view of the discussion of the previous section, if there is a tree U in $\mathcal{U}$ such that $\overline{U} = \tilde{C}$ then e is an inner edge

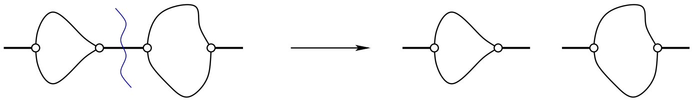


FIGURE 8. Induction with separating edge.

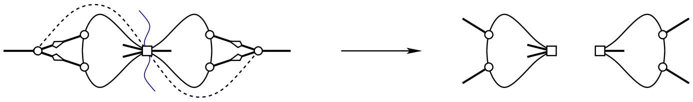


FIGURE 9. Induction with separating special vertex.

of U, and in the decomposition of U at e, the leaves f_1, ℓ_1 and f_2, ℓ_2 are the free leaves of $U_1(e)$ and $U_2(e)$, (so that their partial closure are independent).

Now there is a unique way to recover such a structure. First cut e in $\tilde{C}$ into two stems e_1 and e_2 with leaves ℓ_1 and ℓ_2. The resulting two components of $\tilde{C}$ allow to recover U_1 and U_2 by induction hypothesis and the unique tree U is obtained by fusing back e_1 and e_2 between U_1 and U_2.

– Second case: the special vertex v is the only separating vertex of $\tilde{C}$ (Figure 9). Let C_1 and C_2 be the two non-separable components of $\tilde{C}$ at v. As was already analyzed in Section 4.5, if there is a tree U such that $\overline{U} = \tilde{C}$, then the links or edges incident to v in U are arranged in counterclockwise order into two successive sequences $e_1, \dots, e_p$ with endpoints in C_1 and $e'_1, \dots, e'_q$ with endpoints in C_2, with p and q greater or equal to two in order to avoid separating edges.

Let us prove that the subtree S of U attached to e_1 (resp. to e'_1) is reduced to a special link carrying a pathological vertex. By construction of the two sequences, the incoming leaves of S are free, so that there is at most one such leaf. In view of Lemma 1, there is exactly one. Now if e_1 is an inner edge, Lemma 1 implies that there is no matching edge leaving this subtree, and e_1 is a separating edge of $\tilde{C}$. Therefore e_1 is a special link, which by definition carries a pathological vertex. Finally the subtree cannot be bigger otherwise the pathological vertex would carry an edge and the latter would be separating in $\tilde{C}$.

Hence the tree U is decomposed at v into a special link e_1 that carries a pathological vertex, followed by the tree U_1 formed of $e_2, \dots, e_p$ and their subtrees, by a special link e'_1 that carries a pathological vertex, and by the tree U_2 formed of $e'_2, \dots, e'_p$ and their subtrees. Moreover U_1 and U_2 are well formed blossom trees whose free leaves are respectively matched by the buds of the two pathological vertices.

Now there is a unique way to recover such a structure. First v_1 and v_2 are identified as pathological vertices (since they carry the free leaves). Then, deleting v_1 and v_2 from their respective non-separable component yields two maps C_1 and C_2 from which U_1 and U_2 can be recovered by induction hypothesis. The unique tree U is obtained upon recreating the cyclic order around v.

5.2. **The map $\tilde{C}$ has no other separating vertex than v_1 and v_2.** A first easy case is when the special vertex v carries one of the two free leaves of $\tilde{C}$, say f_1 so that $v = v_1$. Then the analysis is exactly the same as the analysis of the case where v is the only separating vertex (second case of the previous section), with the second sequence reduced to a single leaf: v_2 is found to be pathological and upon deleting v_2 and f_1 the induction hypothesis applies to provide a unique reconstruction.

The main case is when the special vertex v is neither v_1 nor v_2. Assume, without loss of generality, that v is not on the counterclockwise path around the infinite face from v_1 to v_2. (Even if v is incident to the infinite face, it cannot appear in both path from v_1 to v_2 and back from v_2 to v_1.) Let us discuss the constraints on a tree U such that $\overline{U} = \tilde{C}$.

Observe first that v_1 cannot be a pathological vertex: even if the special vertex v is incident to the infinite face, this immediately yields a contradiction in the way its buds are to be matched. Hence v_1 is a generic vertex of U. Let F be the bounded face incident to v_1 in $\tilde{C}$. A generic link e

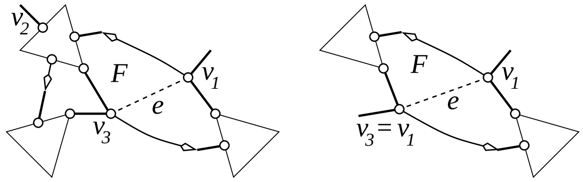


FIGURE 10. Two typical dispositions in the main case of recursion.

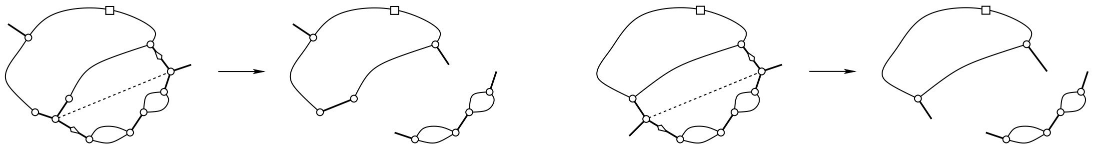


FIGURE 11. Two typical dispositions in the main case of recursion.

joins v_1 to another vertex v_3 in this face F. Let us consider the subtrees attached to v_1 and v_3 in F (Figure 10).

Since vertex v_1 is adjacent to a free leaf f_1 and a bud, it carries a unique (possibly empty) subtree, which precedes f_1 in counterclockwise order. Call this subtree S_1. According to Lemma 1 and in view of the free leaf f_1, the subtree S_1 has only one incoming leaf. The latter is therefore matched by the bud of v_3 and this matching edge is incident to both F and the infinite face. Moreover there is no other edge incident to F on the path from v_2 to v_3 along the infinite face in counterclockwise direction. Indeed this could only be an inner edge (for A to be connected) and Lemma 1 would then impose an incoming leaf in the infinite face between v_2 and v_1.

Consider next the decomposition of U at e and take $U_1(e)$ to contain v_1. In view of its previous definition, the tree S_1 is obtained from $U_1(e)$ upon deleting v_1 and it is a balanced blossom tree. On the other hand, define a tree S_2 from $U_2(e)$ as follows. First delete the bud and the stem inherited from e that are incident to v_3, so that the latter vertex has degree 2. Then smooth this vertex out so as to fuse its two incident edges into one single edge e' (which may be a stem). The result is a tree $U_2(e)$ whose closure leaves e' in the infinite face, and whose free leaves are f_2 and the leaf ℓ_1 matched with the bud of v_1 in U.

Finally there is a unique way to recover the structure (Figure 11). First, taking F to be the bounded face incident to v_1, we dispose of a characterization of vertex v_3 as the first vertex incident to F on the path from v_2 to v_1 around the infinite face in counterclockwise direction. In particular if v_2 is incident to F then $v_3 = v_2$ (as illustrated on the right hand side of Figure 10). Second, the complete closure of the trees S_1 and S_2 are uniquely obtained as follows. Delete f_1 and its stem and cut v_1 so as to create two new leaves ℓ_1 (for the bud of v_1) and ℓ_2 (for the subtree). Detach the edge that follows v_3 along the infinite face from v_2 to v_1: this edge is also incident to F and this operation creates a leaf ℓ_3 in the same component as ℓ_2. Call this component $\tilde{C}_1$. The vertex v_3 remains of degree 2 and can be smoothed so as to fuse its two incident edges into one single edge e that belongs to a second component, $\tilde{C}_2$, that also contains ℓ_1 and v_2. In view of the previous analysis, the two maps $\tilde{C}_1$ and $\tilde{C}_2$ are the images of S_1 and S_2 by complete closure (upon opening the roots). By induction hypothesis, there exists exactly one couple of such trees. From S_1 and S_2 the tree U is readily recovered and the proof is complete.

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