Some Uses of Infinitary Intersection Types as Sequences

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**Invariants of Execution**

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Another use of denotations: equating or separating programs i.e. two states that have different denotations cannot be instances of the same program.
Types as Invariants of Execution

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- **Types**: check *statically* (without reducing) that a term is normalizable (soundness of a type system).

- Typing: assigning formulas (called *types*) to variables.
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- When a type system enjoys subject reduction and expansion, types are execution invariants (and they usually provide us with models of $\lambda$-calculus).
NON-TERMINATING PROGRAMS

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- Many possible definitions or variants of sound non termination
  Klop and alii[95], Endrullis, Polonsky and alii[15]
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The collapse of System S on System R is surjective.
Every multiset based derivation is the collapse of a sequence based derivation. No loss of expressivity while resorting to S.
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  - $[a, b, b] + [a, c] := [a, a, b, b, c]$
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  - $(x_k)_{k \in K}$ where $K \subset \mathbb{N} \setminus \{0, 1\}$ and $\forall k \in K$, $x_k \in K$
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    \[(2 \cdot a, 3 \cdot b, 8 \cdot a)\]
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  - $(2 \cdot a, 3 \cdot b, 8 \cdot a) \uplus (4 \cdot a, 9 \cdot c) = (2 \cdot a, 3 \cdot b, 4 \cdot a, 8 \cdot a, 9 \cdot c)$
  - $(2 \cdot a, 3 \cdot b, 8 \cdot a) \uplus (3 \cdot b, 9 \cdot c)$ not defined (incompatibility).
Plan

Klop’s Question

Gardner/de Carvalho’s ITS $\mathcal{R}_0$

The Infinitary Calculus $\Lambda^{001}$

Truncation and Approximability

Sequences as Intersection Types

Answer to Klop’s Problem

Complete Unsoundness of S

Surjectivity of Collapse

Representation Theorem
HEREDITARY HEAD-NORMALIZATION

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$$\lambda x_1 \ldots x_p . x u_1 \ldots u_q \quad (p, q \geq 0)$$
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  - Inductively, a term is WN if it is HN and all the head arguments are themselves WN.
HEREDITARY HEAD-NORMALIZATION

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$$\lambda x_1 \ldots x_p. x u_1 \ldots u_q \quad (p, q \geq 0)$$

<table>
<thead>
<tr>
<th>head variable</th>
<th>head arguments</th>
</tr>
</thead>
</table>

► A term is **head-normalizing (HN)** if it can be reduced to a HNF (in a finite number of steps)

► **Coinductively**, a term is **hereditary head-normalizing (HHN)** if it can be reduced to a HNF and all the head arguments are themselves HHN.
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- Can a *coinductive* ITS characterize the set of HHN terms?
Answering Klop’s Question...

- Present the key notions of truncations and approximability (meant to avoid irrelevant derivations).
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▶ Present the key notions of **truncations** and **approximability** (meant to avoid **irrelevant** derivations).

▶ Understand why **commutative intersection** is **unfit** to express those key notions.
Answering Klop’s Question...

- Present the key notions of **truncations** and **approximability** (meant to avoid *irrelevant* derivations).

- Understand why **commutative intersection** is **unfit** to express those key notions.

- Present the coinductive type assignment system $S$: intersection types are **sequences** of types, instead of **sets** of types (idempotent intersection fw.) or **multisets** of types (regular non-idempotent fw.).
Plan

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Gardner/de Carvalho’s ITS $\mathcal{R}_0$

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Complete Unsoundness of Surjectivity of Collapse

Representation Theorem
**Typing Rules of \( R_0 \) (Gardner/de Carvalho)**

**Types** \((\tau, \sigma_i)\): \( \tau, \sigma_i := 0 \in \mathcal{O} \mid [\sigma_i]_{i \in I} \to \tau \).  

**Context** \((\Gamma, \Delta)\): assign *intersection* types to variables.

\[
\begin{align*}
\frac{}{\Gamma, x : [\tau] \vdash x : \tau} & \quad \text{ax} \\
\frac{\Gamma \vdash t : [\sigma_i]_{i \in I} \to \tau}{\Gamma \vdash \lambda x.t : [\sigma_i]_{i \in I} \to \tau} & \quad \text{abs} \\
\frac{\Gamma \vdash t : [\sigma_i]_{i \in I} \to \tau \quad (\Delta_i \vdash u : \sigma_i)_{i \in I}}{\Gamma + i \in I \Delta_i \vdash t_1 u : \tau} & \quad \text{app}
\end{align*}
\]

**Examples:**

\[
\begin{align*}
\frac{}{\vdash \lambda x.x : [\tau] \to \tau} & \quad \text{abs} \\
\frac{x : [\tau] \vdash x : \tau}{\vdash \lambda x.x : [\tau] \to \tau} & \quad \text{ax} \\
\frac{x : [\tau] \vdash x : \tau}{\vdash \lambda y.x : [] \to \tau} & \quad \text{ax} \\
\end{align*}
\]
ALTERNATIVE PRESENTATION

Standard presentation

\[
\begin{align*}
\text{ax} & : x : [[\alpha, \beta, \alpha] \to \alpha] \vdash x : [\alpha, \beta, \alpha] \to \alpha \\
\text{ax} & : x : [\alpha] \vdash x : \alpha \\
\text{ax} & : x : [\beta] \vdash x : \beta \\
\text{ax} & : x : [\alpha] \vdash x : \alpha \\
\text{abs} & : x : [\alpha, \beta, \alpha, [\alpha, \beta, \alpha] \to \alpha] \vdash xx : \alpha \\
\vdash & \lambda x. xx : [\alpha, \beta, \alpha, [\alpha, \beta, \alpha] \to \alpha] \to \alpha
\end{align*}
\]
ALTERNATIVE PRESENTATION

Alternative presentation

- Indicate the arity of application rules.

\[ \lambda x \cdot xx \]

\[ x @ (x x x x) \]

\[ \lambda x, \beta, \alpha \rightarrow \alpha \]
**Alternative Presentation**

Alternative presentation

\[ [\alpha, \beta, \alpha] \rightarrow \alpha \]

\[
\begin{array}{c}
\lambda x. xx \\
\alpha \\
\beta \\
\alpha \\
\end{array}
\]

- Indicate the arity of application rules.
- Indicate the types given in axiom leaves.
**ALTERNATIVE PRESENTATION**

Alternative presentation

- Indicate the arity of application rules.
- Indicate the types given in axiom leaves.
- Compute the type of the term.

\[ [\alpha, \beta, \alpha] \rightarrow \alpha \]

\[ \lambda x. xx \rightarrow [\alpha, \beta, \alpha, [\alpha, \beta, \alpha] \rightarrow \alpha] \rightarrow \alpha \]
Alternative Presentation

Alternative presentation

\[ [\alpha, \beta, \alpha] \to \alpha \]

\[ \lambda x \]

\[ \lambda x.xx \]

\[ [\alpha, \beta, \alpha, [\alpha, \beta, \alpha] \to \alpha] \to \alpha \]

Where does this \( \alpha \) come from?

- Indicate the arity of application rules.
- Indicate the types given in axiom leaves.
- Compute the type of the term.
**Alternative Presentation**

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- Indicate the arity of application rules.
- Indicate the types given in axiom leaves.
- Compute the type of the term.

From this axiom rule?

\[ [\alpha, \beta, \alpha] \rightarrow \alpha \]

\( \alpha \quad \beta \quad \alpha \)

\( x \quad x \quad x \quad x \)

\( \lambda x \)

\( \lambda x.xx \)

\( [\alpha, \beta, \alpha, [\alpha, \beta, \alpha] \rightarrow \alpha] \rightarrow \alpha \)

Where does this \( \alpha \) come from?
**ALTERNATIVE PRESENTATION**

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From this axiom rule? Or this one?

\[ [\alpha, \beta, \alpha] \rightarrow \alpha \]

\[ \alpha \quad \beta \quad \alpha \]

\[ x \quad x \quad x \quad x \]

\[ @ \]

\[ \lambda x \]

\[ \lambda x.xx \]

\[ [\alpha, \beta, \alpha, [\alpha, \beta, \alpha] \rightarrow \alpha] \rightarrow \alpha \]

Where does this \( \alpha \) come from?

- Indicate the arity of application rules.
- Indicate the types given in axiom leaves.
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Subject Reduction Property for $M_0$

If $\Pi \triangleright \Gamma \vdash t : \tau$ and $t \to t'$, then $\exists \Pi' \triangleright \Gamma \vdash t' : \tau$
**Subject Reduction Property for $\mathcal{M}_0$**

If $\Pi \triangleright \Gamma \vdash t : \tau$ and $t \rightarrow t'$, then $\exists \Pi' \triangleright \Gamma \vdash t' : \tau$

$$(\lambda x.r)s \rightarrow r[s/x]$$

\[\begin{align*}
\Pi_r \\
\vdots \\
\Gamma, x : [\sigma_i]_{i \in I} \vdash r : \tau \\
\Gamma \vdash \lambda x.r : [\sigma_i]_{i \in I} \rightarrow \tau \\
\Gamma + \sum_{i \in I} \Delta_i \vdash (\lambda x.r)s : \tau
\end{align*}\]
**Subject Reduction Property for \( \mathcal{M}_0 \)**

If \( \Pi \vdash \Gamma \vdash t : \tau \) and \( t \rightarrow t' \), then \( \exists \Pi' \vdash \Gamma \vdash t' : \tau \)

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(\lambda x. r)s \rightarrow r[s/x]
\]

\[\]

Axiom leaves typing \( x \) inside \( \Pi_r \)

\[
\Pi_r \\
\left( x : [\sigma_i] \vdash x : \sigma_i \right)_{i \in I} \\
\Gamma, x : [\sigma_i]_{i \in I} \vdash r : \tau \\
\Gamma \vdash \lambda x. r : [\sigma_i]_{i \in I} \rightarrow \tau \\
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\]

\[
\begin{array}{c}
\Pi_r \\
\vdash (x : [\sigma_i] \vdash x : [\sigma_i]_{i \in I})_{i \in I}
\end{array}
\]

\[
\begin{array}{c}
\Pi_i \\
\vdash \lambda x.r : [\sigma_i]_{i \in I} \rightarrow \tau
\end{array}
\]

\[
\begin{array}{c}
\Delta_i \vdash s : [\sigma_i]_{i \in I}
\end{array}
\]

\[
\Gamma + \sum_{i \in I} \Delta_i \vdash (\lambda x.r)s : \tau
\]
Subject Reduction Property for $\mathcal{M}_0$

If $\Pi \triangleright \Gamma \vdash t : \tau$ and $t \rightarrow t'$, then $\exists \Pi' \triangleright \Gamma \vdash t' : \tau$

$$(\lambda x. r)s \rightarrow r[s/x]$$
**Subject Reduction Property for $M_0$**

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**Subject Reduction Property for** \( M_0 \)

If \( \Pi \triangleright \Gamma \vdash t : \tau \) and \( t \rightarrow t' \), then \( \exists \Pi' \triangleright \Gamma \vdash t' : \tau \)

\[
(\lambda x.r)s \rightarrow r[s/x]
\]

\[
\Pi_r \quad \begin{pmatrix}
\Pi_i \\
\vdots
\end{pmatrix} \\
\Delta_i \vdash s : \sigma_i \\
\vdots
\]

\[
\Gamma + \sum_{i \in I} \Delta_i \vdash r[s/x] : \tau
\]

**Vocabulary:**
We say each *association* (between \( x \)-axiom leaves and arg-derivations) yields a *derivation reduct* \( \Pi' \) typing \( r[s/x] \).
**Subject Reduction Property for $M_0$**

If $\Pi \vdash \Gamma \vdash t : \tau$ and $t \rightarrow t'$, then $\exists \Pi' \vdash \Gamma \vdash t' : \tau$

$$(\lambda x.r)s \rightarrow r[s/x]$$

\[
\begin{pmatrix}
\Pi_r \\
\Pi_i \\
\vdots \\
\Delta_i \vdash s : \sigma_i \\
\vdots
\end{pmatrix}
\]

$\Gamma + \sum_{i \in I} \Delta_i \vdash r[s/x] : \tau$

**Observation:**
If a type $\sigma$ occurs several times in $[\sigma_i]_{i \in I}$, there can be several associations, each one yielding a possibly different derivation reducts $\Pi'$. 
NORMALIZABILITY RESULTS

Proposition
A term is HN iff it is typable in $\mathcal{R}_0$. 

Proposition
A term is WN iff it is typable in $\mathcal{R}_0$ by using an unforgetful judgment.

Definition
A judgement $\Gamma \vdash t : \tau$ is unforgetful if there is no negative occurrence of $\square$ in $\Gamma$ and no positive occurrence of $\square$ in $\tau$. 

\[ \square \text{ occurs negatively in } \tau \Rightarrow \square \text{ occurs positively in } \sigma \]

\[ \text{If } \square \text{ occurs negatively in } \sigma_2 \text{ then } \square \text{ occurs positively in } \sigma_1, \sigma_2, \sigma_3 \rightarrow \tau \text{ and so on.} \]
NORMALIZABILITY RESULTS

Proposition
A term is HN iff it is typable in $\mathcal{R}_0$.

Proposition
A term is WN iff it is typable in $\mathcal{R}_0$ by using an **unforgettable** judgment.
NORMALIZABILITY RESULTS

Proposition
A term is HN iff it is typable in $R_0$.

Proposition
A term is WN iff it is typable in $R_0$ by using an **unforgettable** judgment.

Definition
A judgement $\Gamma \vdash t : \tau$ is **unforgettable** if there is no negative occurrence of $[\ ]$ in $\Gamma$ and no positive occurrence of $[\ ]$ in $\tau$. 
NORMALIZABILITY RESULTS

Proposition
A term is HN iff it is typable in \( R_0 \).

Proposition
A term is WN iff it is typable in \( R_0 \) by using an unforgetful judgment.

Definition
A judgement \( \Gamma \vdash t : \tau \) is unforgetful if there is no negative occurrence of [ ] in \( \Gamma \) and no positive occurrence of [ ] in \( \tau \).

- [ ] occurs negatively in [ ] → \( \tau \)
- If [ ] occurs negatively in \( \sigma_2 \) then [ ] occurs positively in \([\sigma_1, \sigma_2, \sigma_3] \rightarrow \tau\) and so on.
Plan

Klop’s Question

Gardner/de Carvalho’s ITS $R_0$

The Infinitary Calculus $\Lambda^{001}$

Truncation and Approximability

Sequences as Intersection Types

Answer to Klop’s Problem

Complete Unsoundness of Surjectivity of Collapse

Representation Theorem
\(\infty\)-TERMS

- Variable \(x\)
- Abstraction \(\lambda x. u\)
- Application \(u v\)
\( \infty \)-TERMS

\[
\begin{align*}
\text{Variable} & \quad x \\
\text{Abstraction} & \quad \lambda x.u \\
\text{Application} & \quad u \ v
\end{align*}
\]

▶ **Position**: finite sequence in \( \{0, 1, 2\}^* \), e.g. \( 0 \cdot 0 \cdot 2 \cdot 1 \cdot 2 \).
\(\infty\text{-TERMS}\)

- **Variable** \(x\)
- **Abstraction** \(\lambda x.u\)
- **Application** \(u \, v\)

- **Position**: finite sequence in \(\{0, 1, 2\}^*\), e.g. \(0 \cdot 0 \cdot 2 \cdot 1 \cdot 2\).
- **Applicative Depth (a.d.)**: number of \(\downarrow\)-edges e.g.

\[
\text{ad}(1 \cdot 2 \cdot 2 \cdot 0 \cdot 2 \cdot 1 \cdot 2) = 4
\]
001-TERMS

\( \Lambda^{001} \): the set of \( \infty \)-terms \( t \) s.t.:

\[ \text{br is an infinite branch of } t \Rightarrow \text{ad(br)} = \infty. \]
**001-TERMS**

\[ \Lambda^{001} \]: the set of \( \infty \)-terms \( t \) s.t.:

\[ \text{br} \text{ is an infinite branch of } t \Rightarrow \text{ad}(\text{br}) = \infty. \]

\[ f^\omega := f(f(f(\ldots))))) \]

i.e. \( f^\omega = f(f^\omega) \) (fixpoint)

Infinite rightward branch
001-TERMS

Λ^{001}: the set of $\infty$-terms $t$ s.t.:

$br$ is an infinite branch of $t \Rightarrow \text{ad}(br) = \infty.$

- Start from $b \in \text{supp}(t)$
001-TERMS

$\Lambda^{001}$: the set of $\infty$-terms $t$ s.t.:

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- Move $\uparrow$ or $\searrow$
  a.d. does not increase
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001-TERMS

\( \Lambda^{001} \): the set of \( \infty \)-terms \( t \) s.t.:

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- Start from \( b \in \text{supp}(t) \)
- Move \( \uparrow \) or \( \downarrow \)
  a.d. does not increase
- A leaf \( b_0 \) must be reached
Definition
A reduction sequence $t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \ldots \rightarrow t_n \rightarrow \ldots$ is strongly converging if it is of finite length or if $\lim \text{ad}(b_n) = \infty$. 

**Strong Convergence**
**Strong Convergence**

$$\Delta_f := \lambda x. f(xx)$$

$$\Delta_f \Delta_f: "Curry"$$

$$\Delta_f \Delta_f \rightarrow f(\Delta_f \Delta_f) \rightarrow f^2(\Delta_f \Delta_f) \rightarrow f^3(\Delta_f \Delta_f) \rightarrow f^4(\Delta_f \Delta_f) \rightarrow \ldots \rightarrow \infty \ f^\omega$$
**Strong Convergence**

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**Strong Convergence**

\[ \Delta_f := \lambda x.f(xx) \quad \Delta_f \Delta_f: "Curry" \]

\[ \Delta_f \Delta_f \to f(\Delta_f \Delta_f) \to f^2(\Delta_f \Delta_f) \to f^3(\Delta_f \Delta_f) \to f^4(\Delta_f \Δ_f) \to \ldots \to \infty f^\omega \]
**Strong Convergence**

- Unstable Part
- Increase of a.d.

$t_0$
Strong Convergence

Unstable Part

Increase of a.d.

$t_1$
**Strong Convergence**

- **Stabilized Part**
- **Unstable Part**
- **Increase of a.d.**

In the diagram, the stabilized part is indicated by the lower portion of the triangle, and the unstable part is represented by the upper portion. The line marked $t_2$ separates these two regions, indicating a critical point in the convergence process.
Strong Convergence

Stabilized Part

Unstable Part

Increase of a.d.

$t_3$
**Strong Convergence**

- **Stabilized Part**
- **Unstable Part**
- **Increase of a.d.**

$t$ ...
Strong Convergence

Stabilized Part

Unstable Part

$t_{50}$

Increase of a.d.
Strong Convergence

- Stabilized Part
- Unstable Part
- Increase of a.d.

$t...$
Strong Convergence

Stabilized Part

Unstable Part

Increase of a.d.

$t$ ...

Stabilized Part

Unstable Part
Strong Convergence

- Stabilized Part
- Unstable Part

Increase of a.d.

$t_{1000}$
Strong Convergence

Stabilized Part

Unstable Part

Increase of a.d.

$t...$
**Strong Convergence**

- Stabilized Part
- Unstable Part
- Increase of a.d.

$\cdots$
STRONG CONVERGENCE

Stabilized Part

Unstable Part

Increase of a.d.

\( t \ldots \)
Strong Convergence

Stabilized Part

Increase of a.d.
**Strong Convergence**

Conclusion
Strong Convergence

Conclusion

A strongly converging reduction sequence (s.c.r.s) allows us to define its limit.
**Infinitary Normalization**

- The notions of redex and head-normalizability do not change.
**Infiniterary Normalization**

- The notions of redex and head-normalizability do not change.

- The NF of $\Lambda^{001}$ are generated by the *coinductive* grammar:

  \[ t = \lambda x_1 \ldots \lambda x_p.x t_1 \ldots t_q \quad (p, q \geq 0) \]
INFINITARY NORMALIZATION

- The notions of redex and head-normalizability do not change.

- The NF of $\Lambda^{001}$ are generated by the coinductive grammar:

  $$ t = \lambda x_1 \ldots \lambda x_p.x \, t_1 \ldots \, t_q \quad (p, q \geq 0) $$

Definition (Infinitary WN)

A 001-term is WN if it can be reduced to a NF through at least one s.c.r.s.
**Inferential Normalization**

▶ The notions of redex and head-normalizability do not change.

▶ The NF of $\Lambda^{001}$ are generated by the *coinductive* grammar:

$$t = \lambda x_1 \ldots \lambda x_p.x t_1 \ldots t_q \quad (p, q \geq 0)$$

**Definition (Inferential WN)**

A 001-term is WN if it can be reduced to a NF through at least one s.c.r.s.

▶ Thus, a (finite) term is HHN iff it is 001-WN.
Plan

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Representation Theorem
Truncation (Figures)

$\Pi' \triangleright \Gamma \vdash f^\omega : 0$

Every Variable is Typed

$\Gamma = f : [[[0] \to o]_\omega$ (infinite multiplicity)
**Truncation (Figures)**

Π' can be **truncated** into Π₄':

\[
\begin{align*}
[0] & \rightarrow 0 \\
[0] & \rightarrow 0 \\
[0] & \rightarrow 0 \\
[0] & \rightarrow 0 \\
[0] & \rightarrow 0 \\
[0] & \rightarrow 0
\end{align*}
\]
Π' can be **truncated** into Π'_4:

\[ [0] \rightarrow 0 \]
Truncation (Figures)

\( \Pi' \) can be truncated into \( \Pi'_4 \):
TRUNCATION (Figures)

\( \Pi' \) can be **truncated** into \( \Pi_3' \):

\[
\begin{align*}
[ ] & \rightarrow 0 \\
[0] & \rightarrow 0 \\
[0] & \rightarrow 0 \\
& \rightarrow 0 \\
& \rightarrow 0 \\
& \rightarrow 0 \\
& \rightarrow 0 \\
& \rightarrow 0
\end{align*}
\]
**Truncation (Figures)**

\( \Pi' \) can be **truncated** into \( \Pi'_3 \):

![Diagram showing truncation process with nodes labeled and arrows indicating direction of truncation.](image-url)
**Truncation (Figures)**

$f^\omega$ may be replaced by $f^3(\Delta_f \Delta_f)$ in $\Pi'_3$, yielding $\Pi^3_3$:

![Diagram showing the truncation process](image-url)
\( \Pi_3^3 \) may be expanded 3 times, yielding \( \Pi_3 \supset \Delta_f \Delta_f : \)

\[ \begin{align*}
\lbrack & \rbrack \rightarrow 0 \\
\lbrack 0 & \rbrack \rightarrow 0 \\
\lbrack 0 & \rbrack \rightarrow 0
\end{align*} \]
TRUNCATION (Figures)

Back to $\Pi_4'$, level 4 truncation of $\Pi'$:
TRUNCATION (FIGURES)

$f^\omega$ may be replaced by $f^4(\Delta_f \Delta_f)$ in $\Pi_3'$, yielding $\Pi_4^4$:
TRUNCATION (FIGURES)

$\Pi_4^4$ may be expanded 4 times, yielding $\Pi_4 \triangleright \Delta_f \Delta_f$:
**Recipe for ∞-Expansion**

**Question:** how do we expand $\Pi' \triangleright f^\omega$, to get $\Pi$, typing $\Delta_f \Delta_f$?
Recipe for $\infty$-Expansion

**Question:** how do we expand $\Pi' \triangleright f^\omega$, to get $\Pi$, typing $\Delta_f \Delta_f$?

We have the idea of **level n truncation** of $\Pi'$ and the idea of **subject substitution** (by a reduct of finite rank, in a finite derivation).
Recipe for $\infty$-Expansion

**Question:** how do we expand $\Pi' \triangleright f^\omega$, to get $\Pi$, typing $\Delta_f \Delta_f$?

We have the idea of **level n truncation** of $\Pi'$ and the idea of **subject substitution** (by a reduct of finite rank, in a finite derivation).

- Truncate $\Pi'$ into $^{f\Pi'}$, finite derivation typing $f^\omega$ (hint: replace an occ. of $[\alpha] \rightarrow \alpha$ by $[] \rightarrow \alpha$).
Recipe for $\infty$-Expansion

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- In $^f\Pi'$, replace $f^\omega$ by $f^n(\Delta_f \Delta_f)$, for $n$ great enough: you get $^f\Pi'_n$. 
**Recipe for ∞-Expansion**

**Question:** how do we expand $\Pi' \triangleright f^\omega$, to get $\Pi$, typing $\Delta_f \Delta_f$?

We have the idea of **level n truncation** of $\Pi'$ and the idea of **subject substitution** (by a reduct of finite rank, in a finite derivation).

- Truncate $\Pi'$ into $^f\Pi'$, finite derivation typing $f^\omega$ (hint: replace an occ. of $[\alpha] \rightarrow \alpha$ by $[] \rightarrow \alpha$).
- In $^f\Pi'$, replace $f^\omega$ by $f^n(\Delta_f \Delta_f)$, for $n$ great enough: you get $^f\Pi'_n$.
- Expand $n$ times $^f\Pi'_n$: you get $\Pi_n$ typing $\Delta_f \Delta_f$. 
Recipe for ∞-Expansion

Question: how do we expand Π’ ⊳ fω, to get Π, typing ΔfΔf?

We have the idea of level n truncation of Π’ and the idea of subject substitution (by a reduct of finite rank, in a finite derivation).

▶ Truncate Π’ into fΠ’, finite derivation typing fω (hint: replace an occ. of [α] → α by [[]] → α).
▶ In fΠ’, replace fω by fn(ΔfΔf), for n great enough: you get fΠ’.
▶ Expand n times fΠ’: you get Πn typing ΔfΔf.
▶ Take the join of all the Πn (while n → ∞): this defines Π, the desired expansion of Π’. 
Unsoundness

- Expanding $\Pi'$, we can get an unforgettable derivation $\Pi$ typing $\Delta_f \Delta_f$. 

Expanding $\Pi'$, we can get an unforgettable derivation $\Pi$ typing $\Delta_f \Delta_f$. 

Derivation $\Pi$ features a type $\rho$ coinductively defined by the fixpoint equation $\rho = [\rho] \omega \rightarrow \rho$.

Type $\gamma$ allows to type $\Delta \Delta_f$. Need for a validity criterion.
Unsoundness

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Unsoundness

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- Type $\gamma$ allows to type $\Delta \Delta$. Need for a validity criterion.
Informally, see a derivation Π as a set of symbols (type variables $o$ or $→$ that we found inside each judgment of $P$).
Approximability (Heuristic)

- Informally, see a derivation $\Pi$ as a set of symbols (type variables $\circ$ or $\to$ that we found inside each judgment of $P$).

- A (finite) approximation $^f\Pi$ of a derivation $\Pi$ is a finite subset of symbols of $\Pi$ which is itself a derivation. We write $^f\Pi \leq \Pi$. 
**Approximability (Heuristic)**

- Informally, see a derivation $\Pi$ as a set of symbols (type variables $o$ or $\to$ that we found inside each judgment of $P$).

- A **finite approximation** $^f\Pi$ of a derivation $\Pi$ is a finite subset of symbols of $\Pi$ which is itself a derivation. We write $^f\Pi \leq \Pi$.

- A derivation $\Pi$ is said to be **approximable** if for all finite subset $B$ of symbols of $\Pi$, there is an approximation $^f\Pi \leq \Pi$ that contains $B$. 
APPROXIMABILITY (FIGURE)
APPROXIMABILITY (FIGURE)
**APPROXIMABILITY (FIGURE)**
APPROXIMABILITY (FIGURE)
**Non-Determinism and Truncation**

\((\lambda x. r)s\)
NON-DETERMINISM AND TRUNCATION

\((\lambda x. r)s\)

Truncation possibly affects every type nested inside \(\Pi\).
**Non-Determinism and Truncation**

\[(\lambda x.r)s\]

Truncation possibly affects every type nested inside \(\Pi\).
NON-DETERMINISM AND TRUNCATION

$$(\lambda x. r)s$$

Assume $\sigma_1 = \sigma_2$. 

Diagram:

- $\lambda x$
- $\Pi_r$
- $\sigma_1 \rightarrow x_{\#1}$
- $\sigma_2 \rightarrow x_{\#2}$
- $\Pi_1$
- $\Pi_2$
NON-DETERMINISM AND TRUNCATION

\[(\lambda x. r)s\]

Assume \(\sigma_1 = \sigma_2\).

- Possible in \(\Pi\):
  \(#1 \mapsto \Pi_2, \#2 \mapsto \Pi_1\)

Assume \(\sigma_1 \neq \sigma_2\).

- Not possible in \(\Pi\):
  \(#1 \mapsto \Pi_2, \#2 \mapsto \Pi_1\)

- If \(f_{\sigma_1} = f_{\sigma_2}\), possible in \(f_\Pi\).
NON-DETERMINISM AND TRUNCATION

\[(\lambda x. r)s\]

Assume \(\sigma_1 = \sigma_2\).
- Possible in \(\Pi\):
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NON-DETERMINISM AND TRUNCATION

\((\lambda x.r)s\)

Assume \(\sigma_1 \neq \sigma_2\)

\[
\begin{array}{c}
\Pi_r \\
\sigma_1 \rightarrow x_{\#1} \\
\sigma_2 \rightarrow x_{\#2} \\
\lambda x \\
\sigma_1 \\
\sigma_2 \\
\Pi_1 \\
\Pi_2
\end{array}
\]
Non-Determinism and Truncation

$$(\lambda x.r)s$$

Assume $\sigma_1 \neq \sigma_2$

- Not possible in $\Pi$:
  $\#1 \leadsto \Pi_2$, $\#2 \leadsto \Pi_1$

If $f\sigma_1 \neq f\sigma_2$, not in $f\Pi$.

Assume $\sigma_1 \neq \sigma_2$

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**Non-Determinism and Truncation**

\[(\lambda x. r)s\]

Assume \(\sigma_1 \neq \sigma_2\)

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Representation Theorem
SEQUENTIAL INTERSECTION

▶ Types:

\[ S_k, T ::= 0 \in \mathcal{O} \mid (S_k)_{k \in K} \rightarrow T \]
SEQUENTIAL INTERSECTION

▶ Types:

\[ S_k, T \ ::= \ o \in \emptyset \mid (S_k)_{k \in K} \rightarrow T \]

▶ Sequence Type:

▶ Intersection type replacing multiset types.
Sequential Intersection

- **Types:**
  \[ S_k, T ::= o \in \mathcal{O} \mid (S_k)_{k \in K} \rightarrow T \]

- **Sequence Type:**
  - Intersection type replacing multiset types.
  - \( F = (T_k)_{k \in K} \) where \( T_k \) types and \( K \subset \mathbb{N} - \{0, 1\} \).
SEQUENTIAL INTERSECTION

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▶ The integer indexes \( k \) are called **tracks**.
**SEQUENTIAL INTERSECTION**

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  - We also write \( (S_k)_{k \in K} = (k \cdot S_k)_{k \in K} \).
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  - The integer indexes \( k \) are called **tracks**.
  - We also write \( (S_k)_{k \in K} = (k \cdot S_k)_{k \in K} \).

- **Example:** \( (7 \cdot o_1, 3 \cdot o_2, 2 \cdot o_1) \rightarrow o \)
**Derivations of $S$**

The set $\text{Deriv}$ of rigid derivations is *coinductively* generated by:

\[
\begin{align*}
& x : (k \cdot T) \vdash x : T \quad \text{ax} \\
& C; x : (S_k)_{k \in K} \vdash t : T \\
& (S_k)_{k \in K} \vdash \lambda x. t : C(x) \to T \quad \text{abs} \\
& C ; t : (S_k)_{k \in K} \to T \\
& (D_k \vdash u : S_k)_{k \in K} \quad \text{app} \\
& C \cup_{k \in K} D_k \vdash t u : T
\end{align*}
\]
The set $\text{Deriv}$ of rigid derivations is \textit{coinductively} generated by:

$x : (k \cdot T) \vdash x : T$ \hspace{5em} $\text{ax}$

$(S_k)_{k \in K} \vdash \lambda x.t : C(x) \to T$ \hspace{5em} $\text{abs}$

$C \vdash t : (S_k)_{k \in K} \to T$ \hspace{5em} $(D_k \vdash u : S_k)_{k \in K}$

$\text{app}$

$C \cup_{k \in K} D_k \vdash tu : T$

- If $\text{Rt}(C)$ and the $\text{Rt}(D_k)$ are not pairwise disjoint, contexts are incompatible.
The set \( \text{Deriv} \) of rigid derivations is \textit{coinductively} generated by:

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\]

\[
\frac{C; x : (S_k)_{k \in K} \vdash t : T}{(S_k)_{k \in K} \vdash \lambda x. t : C(x) \rightarrow T}{\text{abs}}
\]

\[
\frac{C \vdash t : (S_k)_{k \in K} \rightarrow T}{C \cup_{k \in K} D_k \vdash t u : T}{\text{app}}
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- If \( \text{Rt}(C) \) and the \( \text{Rt}(D_k) \) are not pairwise disjoint, contexts are incompatible.
- Forget about the indexes: \( S \) collapses onto \( D \).
MAIN FEATURES

- **Trackability:** $S$ features **pointers** called **bipositions** (every symbol used inside a derivation $P$ can be pointed at).
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▶ **Subject reduction is deterministic:**
  
  ▶ Assume $P$ types $(\lambda x.r)s$. If there is an axiom rule typing $x$ on track 5 (#5-ax), by typing constraint, there will also be an argument derivation $P_5$ typing $s$ on track 5, concluded by exactly the same type $S_5$
**Main Features**

- **Trackability:** $S$ features **pointers** called **bipositions** (every symbol used inside a derivation $P$ can be pointed at).

- **Subject reduction is deterministic:**
  - Assume $P$ types $(\lambda x.r)s$. If there is an axiom rule typing $x$ on track 5 ($\#5$-ax), by typing constraint, there will also be an argument derivation $P_5$ typing $s$ on track 5, concluded by exactly the same type $S_5$.
  - During reduction, $\#5$-ax will be replaced by $P_5$, even if there are other $P_k$ concluded by $S = S_5$. 
**Pointers**

![Diagram of a triangle with a point labeled P]
**Pointers**

\[ P \]

\[(\text{pos. } a) \ C \vdash t : T \]
For instance
\[ a = 0 \cdot 1 \cdot 3 \cdot 0 \cdot 8 \cdot 1 \]
POINTERS

\[
P
\]

\[(\text{pos. } a) \ C \vdash t : T\]

Inside \(T\), nested pos. \(c\)
For instance
\[ c = 1 \cdot 5 \cdot 3 \cdot 1 \cdot 4 \]
POINTERS

\[ P \]

**(pos. \( a \))** \( C \vdash t : T \)

Inside \( T \), nested pos. \( c \)

Biposition (right h.s.): pair \((a, c)\)
**Pointers**

\( P \)

(pas. \( a \)) \( C \vdash t : T \)

Inside \( T \), nested pas. \( c \)

Biposition (right h.s.):
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Bisupport of \( P \): the set of (right or left) bipositions
Plan

Klop’s Question

Gardner/de Carvalho’s ITS $R_0$

The Infinitary Calculus $\Lambda^{001}$

Truncation and Approximability

Sequences as Intersection Types

Answer to Klop’s Problem

Complete Unsoundness of S

Surjectivity of Collapse

Representation Theorem
Approximability

- Every symbol inside a rigid derivation $P$ has a **biposition** (a pointer inside a type nested in a judgment of $P$).
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Approximability

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- A rigid derivation $P$ is said to be **approximable** if for all finite part $B$ of $P$, there is a finite approximation $^{f}P \leq P$ s.t. $^{f}P$ contains $B$. 
The Lattice of Approximations

Proposition:

- The set of $S$-derivations typing a same term $t$ is a c.p.o.
- The set of approximations of a derivation $P$ is a complete lattice.
- The set of finite approximations of a derivation $P$ is a lattice.
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Proposition:

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Order, meet and join are given by the set-theoretic operations $\subseteq$, $\cap$, $\cup$ on bisupports.
**Characterization of infinitary WN**

**Theorem**
A 001-term $t$ is WN iff $t$ is unforgetfully typable by means of an approximable derivation.
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Argument 1: If a term is typable by an approximable derivation, then it is head normalizing. Unforgetfulness makes HN hereditary.
CHARACTERIZATION OF INFINITARY WN

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**Argument 3:** Every NF can be typed by quantitative unforgetful derivations and every quantitative derivation typing a NF is approximable.
CHARACTERIZATION OF INFINITARY WN

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A $\mathcal{A}^0_0$-term $t$ is WN iff $t$ is unforgetfully typable by means of an approximable derivation.

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Argument 3: Every NF can be typed by quantitative unforgetful derivations and every quantitative derivation typing a NF is approximable.

Argument 4: Subject expansion property holds for s.c.r.s. (assuming approximability only).
Plan

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Representation Theorem
**Typable Terms in S**

- **Question:** let us drop approximability. What is the set of typable terms in S?
Typable Terms in $S$

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- We already know that $S$ is **unsound** ($S$ can type unproductive terms, like $\Omega$). Two possibilities:
**Typable Terms in S**

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- We already know that S is **unsound** (S can type unproductive terms, like \( \Omega \)). Two possibilities:

- Some terms are typable in System S, but some others are not: in that case, S will characterize a set of terms wider than the usual known sets of normalizable terms.
Typable Terms in $S$

- **Question:** let us drop approximability. What is the set of typable terms in $S$?

- We already know that $S$ is **unsound** ($S$ can type unproductive terms, like $\Omega$). Two possibilities:

- Some terms are typable in System $S$, but some others are not: in that case, $S$ will characterize a set of terms wider than the usual known sets of normalizable terms.

- Every term is typable in $S$. We say that $S$ is **completely unsound**. In that case, since $S$ enjoys SR and SE, $S$ will provide us with a new model for pure lambda-calculus.
We are actually in the second case:

- Every term has a non-empty denotation (including the mute terms).
- Terms are discriminated according to their order (the maximal number of abs that prefixes a reduct).
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- Terms are discriminated according to their **order** (the maximal number of abs that prefixes a reduct).

**Related works**

- Jacopini[75]: **easy** terms (*t* is easy if it can be consistently equated to any other term)
- Berarducci[96]: **mute** terms (“The most undefined terms”).
- Bucciarelli,Carraro,Favro,Salibra[15]: *Graph easy Sets of mute lambda terms*, TCS.
RELEVANCE VS IRRELEVANCE

- **Observation:** In system $\mathcal{R}$, $\lambda x. x$ (resp. $\lambda y. x$) can only be typed with a type of the form $[\tau] \rightarrow \tau$ (resp. $[] \rightarrow \tau$).
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  $\text{ax}$
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$$
\frac{x : [\sigma] \vdash x : \sigma}{ax}
$$

- If we replace $ax$ by $axw$:

$$
\frac{i_0 \in I}{\Gamma; x : [\sigma_i]_{i \in I} \vdash x : \sigma_{i_0}}
$$

... we obtain an irrelevant system, called $\mathcal{R}_w$. 
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... we obtain an irrelevant system, called $\mathcal{R}_w$.

▶ In $\mathcal{R}_w$, we may derive:

$$
\frac{x : [\tau, \tau_1, \tau_1] \vdash x : \tau}{\text{axw}} \quad \frac{x : [\sigma], y : [\tau] \vdash x : [\tau]}{\text{axw}}
$$

$$
\frac{\vdash \lambda x.x : [\tau, \tau_1, \tau_2] \rightarrow \tau}{\text{abs}} \quad \frac{x : [\tau] \vdash \lambda y.x : [\tau] \rightarrow \tau}{\text{abs}}
$$
IRRELEVANCY AND COMPLETE UNSOUNDNESS

- We have met the type $\rho$ satisfying $\rho = [\rho]_\omega \rightarrow \rho$. 
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- **Claim:** Let $t$ be a term. If $\Gamma(x) = [\rho]_\omega$ for all free variable $x$ of $t$, then $\Gamma \vdash t : \rho$ is derivable in $R_w$. 
IRRELEVANCY AND COMPLETE UNSOUNDNESS

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**Proof.**

\[
\begin{align*}
\Gamma; x : [\rho]_\omega & \vdash t : \rho \\
\Gamma & \vdash \lambda x. t : [\rho]_\omega \to \rho \ (= \rho) \quad \text{abs} \\
\Gamma & \vdash t : \rho \ (= [\rho]_\omega \to \rho) \quad (\Gamma \vdash u : \rho)_\omega \quad \text{app} \\
\Gamma & \vdash tu : \rho
\end{align*}
\]
RELEVANT COINDUCTIVE TYPES

- In $\mathcal{R}$, the typing rules constrain $[]$ to appear. Failure of the previous argument.
RELEVANT COINDUCTIVE TYPES

- In $R$, the typing rules constrain $[]$ to appear. Failure of the previous argument.

- **Question:** what is the set of typable terms in $R$?
Relevant Coinductive Types

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Relevant Coinductive Types

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- In that case, the type of $x$ must also be of the form $[\sigma'] \rightarrow [] \rightarrow \sigma'$. 
**Relevant Coinductive Types**

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- Difficulty to see the typing constraints on $x$. 
**Question:** what is the set of typable terms in $\mathbb{R}$?
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- *In the finite case:* type Normal Forms and proceed by expansion.
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We study then **typability** as a first order theory. For that, we will rather study typability in $\mathcal{S}$. 
Question: what is the set of typable terms in $R$?

- In the finite case: type Normal Forms and proceed by expansion.

- Problem for coinductive Types: no form of normalization is granted (e.g. $\Omega$ typable in $R$).

We study then **typability** as a first order theory. For that, we will rather study typability in $S$. System $S$ collapses on $R$. Thus, if every term is typable in $S$, then every term is typable in $R$. 
CANDIDATE SUPPORTS

What is a correct type?

Support:
\{\varepsilon, 1, 4, 4 \cdot 1, 4 \cdot 3, 4 \cdot 8\}
**Candidate Supports**

What is a correct type?

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Wrong Labels
**CANDIDATE SUPPORTS**

What is a correct type?

Correct Labels

Support:
{ε, 1, 4, 4 · 1, 4 · 3, 4 · 8}
CANDIDATE SUPPORTS

What is a correct type?

![Diagram showing two candidate supports with nodes labeled with numbers.]

Support:
\{\varepsilon, 1, 4, 4 \cdot 1, 4 \cdot 3, 4 \cdot 8\}

Support:
\{\varepsilon, 1, 4, 4 \cdot 3\}
**Candidate Supports**

What is a correct type?

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**Candidate Support:** a set of positions that is the support of a type

- \(c \rightarrow_{t_1} c \cdot k\) (a candidate supp is a tree)
- \(c \cdot 1 \rightarrow_{t_2} c \cdot k\) (if a node does not have a 1-son, it is a leaf)
CANDIDATE BISUPPORTS

- We want to show that every term $t$ is typable in $S$. 
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- Idea: we try to capture the notion of candidate bisupport: a set of pointers that is the bisupport of a $S$-derivation typing $t$. 
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- We must find suitable stability conditions.
We want to show that every term $t$ is typable in $S$.

Idea: we try to capture the notion of candidate bisupport: a set of pointers that is the bisupport of a $S$-derivation typing $t$.

We must find suitable stability conditions.

Then, we show that there is a non-empty set that satisfies them.
CANDIDATE BISUPPORTS

- \((a, c) \rightarrow_{\text{asc}} (a \cdot 1, 1 \cdot c)\) if \(t(a) = @\).
- \((a, c) \rightarrow_{t_1} (a, c \cdot k)\) if \(t(a) = \lambda x\).
- \((a, k \cdot c) \rightarrow_{\text{pi}} (\text{pos}(k), c)\) if \(t(a) = \lambda x\) and \(k \in \text{Tr}_1(a)\).
- \((a, k \cdot c) \rightarrow_{\text{pi}} b_\bot\) if \(t(a) = \lambda x\) and \(k \notin \text{Tr}_1(a), k \geq 2\).
- \((a \cdot 1, k \cdot c) \xrightarrow{a} (a \cdot k, c)\) if \(t(a) = @\).
- \((a, c) \rightarrow_{t_1} (a, c \cdot k)\).
- \((a, c \cdot 1) \rightarrow_{t_2} (a, c \cdot k)\) for any \(k \geq 2\).
- \((a, 1) \rightarrow_{\text{rt}} (a, \varepsilon)\) if \(t(a) = \lambda x\).
- \((a, \varepsilon) \rightarrow_{\text{up}} b_\bot\).
- \((a, \varepsilon) \rightarrow_{\text{up}} (a', c)\) if \(a \leq a'\)
CANDIDATE BISUPPORTS

- \((a, c) \rightarrow_{asc} (a \cdot 1, 1 \cdot c)\) if \(t(a) = @\).
- \((a, 1 \cdot c) \rightarrow (a \cdot 0, c)\) if \(t(a) = \lambda x\).
- \((a, k \cdot c) \rightarrow_{pi} (pos(k), c)\) if \(t(a) = \lambda x\) and \(k \in Tr_1(a)\).
- \((a, k \cdot c) \rightarrow_{pi} b_\perp\) if \(t(\overline{a}) = \lambda x\) and \(k \notin Tr_1(a), k \geq 2\).
- \((a \cdot 1, k \cdot c) \xrightarrow{a} (a \cdot k, c)\) if \(t(a) = @\).
- \((a, c) \rightarrow_{t_1} (a, c \cdot k)\).
- \((a, c \cdot 1) \rightarrow_{t_2} (a, c \cdot k)\) for any \(k \geq 2\).
- \((a, 1) \rightarrow_{rt} (a, \varepsilon)\) if \(t(a) = \lambda x\).
- \((a, \varepsilon) \rightarrow_{up} b_\perp\).
- \((a, \varepsilon) \rightarrow_{up} (a', c)\) if \(a \leq a'\)
**GUIDELINES OF THE PROOF**

**Goal:** checking that the former conditions cannot prove that the type of $t$ must be empty.  
In that case, we can build a derivation whose bisupport is minimal.
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Guidelines of the proof

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- In \( \mathcal{P}' \), commutations and nice interactions occur. Considering a minimal case, we show that \( \mathcal{P}' \) cannot prove that \( t \) has an empty type. *Contradiction.*
GUIDELINES OF THE PROOF

**Goal:** checking that the former conditions cannot prove that the type of $t$ must be empty.
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- In $P'$, commutations and nice interactions occur. Considering a minimal case, we show that $P'$ cannot prove that $t$ has an empty type. *Contradiction*.

This works for the infinitary $\lambda$-calculus.
**Order**

Theorem (complete unsoundness): in \( \mathcal{R} \), every term is typable.
**Theorem (complete unsoundness):** in $\mathcal{R}$, every term is typable.

**Definition:** The order of a $\lambda$-term $t$ is the maximal $n \in \mathbb{N} \cup \{\infty\}$ s.t.

$$t \rightarrow^* t' = \lambda x_1 \ldots \lambda x_n.t'_0.$$ 

A zero term is a term of order 0.
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**Proposition:** if $t$ is a zero-term, then, $t$ is typable with $o$. 
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Definition (relational model): For all closed $\lambda$-term $t$, we set

$$[t] = \{\tau \mid \vdash t : \tau \text{ is derivable}\}$$
**Order**

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**Proposition:** if $t$ is a zero-term, then, $t$ is typable with $o$.

**Definition (relational model):** For all closed $\lambda$-term $t$, we set

$$\llbracket t \rrbracket = \{\tau \mid \vdash t : \tau \text{ is derivable}\}$$

**Theorem:** This yields a non-sensible model that discriminates terms according to their order.
Plan

Klop’s Question

Gardner/de Carvalho’s ITS $\mathcal{R}_0$

The Infinitary Calculus $\Lambda^{001}$

Truncation and Approximability

Sequences as Intersection Types

Answer to Klop’s Problem

Complete Unsoundness of $S$

Surjectivity of Collapse

Representation Theorem
**THE PROBLEM OF COLLAPSE**

- **Question:** Any derivation of $S$ collapses on a derivation of $R$. Is this collapse surjective? Is every derivation of $R$ the collapse of a derivation of $S$?
THE PROBLEM OF COLLAPSE

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> The app-rule can be restated as follows:

$$C \vdash t : (S_k)_{k \in K} \rightarrow T \quad (D_k \vdash u : S'_k)_{k \in K'} \quad (S_k)_{k \in K} = (S'_k)_{k \in K'}$$

$$\frac{C \uplus \bigcup_{k \in K} D_k \vdash t u : T}{\text{app}}$$
THE PROBLEM OF COLLAPSE

▶ **Question:** Any derivation of $S$ collapses on a derivation of $R$. Is this collapse surjective? Is every derivation of $R$ the collapse of a derivation of $S$?

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\[
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\]

▶ Thus, the choice of types in axiom rules must ensure that we have a **syntactic equality** for every **app**-rule.
THE PROBLEM OF COLLAPSE

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$$

$$
C \uplus \bigcup_{k \in K} D_k \vdash t u : T \quad \text{app}
$$

▶ Thus, the choice of types in axiom rules must ensure that we have a **syntactic equality** for every app-rule.

▶ Moreover, we must avoid track conflict in the contexts.
Hybrid Derivations

- Type system $S_h$ is obtained from $S$ by replacing the $app$-rule by:

$$
C ⊢ t : (S_k)_{k ∈ K} → T \quad (D_k ⊢ u : S'_k)_{k ∈ K'} \quad (S_k)_{k ∈ K} ≡ (S'_k)_{k ∈ K'}
$$

$$
C, \bigcup_{k ∈ K} D_k \vdash t u : T
$$

where $(S_k)_{k ∈ K} ≡ (S'_k)_{k' ∈ K'}$ means that $(S_k)_{k ∈ K}$ and $(S'_k)_{k' ∈ K'}$ collapse on the same type of $\mathcal{R}$. 

- Easy to show that every $\mathcal{R}$-derivation is the collapse of a $S_h$-derivation.
HYBRID DERIVATIONS

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$$

$$
\begin{array}{c}
\frac{C \uplus \bigcup_{k \in K} D_k \vdash tu : T}{\text{app}} \\
\end{array}
$$

where $(S_k)_{k \in K} \equiv (S'_{k})_{k' \in K'}$ means that $(S_k)_{k \in K}$ and $(S'_{k})_{k' \in K'}$ collapse on the same type of $\mathcal{R}$.

▶ Easy to show that every $\mathcal{R}$-derivation is the collapse of a $S_h$-derivation.
Operable Derivation

- Let $P$ be a hybrid derivation typing $t$. 
Operable Derivation

- Let $P$ be a hybrid derivation typing $t$.
  - If $a$ is the position of a judgment typing a redex $(\lambda x. r)s$ inside $t$, a root isomorphism $\rho_a : (S_k)_{k \in K}(a) \rightarrow (S'_k)_{k \in K'}(a)$ tells us how to perform subject reduction.
**Operable Derivation**

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  - Say $\rho_a(5) = 7$. Then, above $a$, there is an $x$-axiom rule on track 5 (#5-ax) and argument derivation $P|a.7$ on track 7.
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  - Then, during reduction, #5-$ax$ must be replaced by $P|_{a.7}$
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Interfaces:

- A sequence type isomorphism $\phi_a : (S_k)_{k \in K}(a) \to (S'_k)_{k \in K'}(a)$
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- Interfaces:
  - A sequence type isomorphism $\phi_a : (S_k)_{k \in K}(a) \rightarrow (S'_k)_{k \in K'}(a)$
  - A complete interface is given by a family of full sequence type isomorphisms $\phi_a : (S_k)_{k \in K}(a) \rightarrow (S'_k)_{k \in K'}(a)$ when $a$ ranges over the app-nodes of $P$. 
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  - If $b$ is the pos. of a redex, notion of residuals (of positions, bipositions and interfaces) after firing the redex $a$.

- An operable derivation is a hybrid derivation endowed with a complete interface (for each app-rule).
**Representation Lemma**

**Lemma**
Let $\Pi$ a $R$-derivation typing $t$ and a reduction sequence $R$ (of length $\leq \omega$) and $P$ a hybrid representative of $\Pi$. Any reduction choice sequence along $R$ can be built-in inside a complete interface for $P$. 

Intuition of the Proof:
▶ Consider a reduction sequence $t_0 b_0 \rightarrow t_1 b_1 \rightarrow t_2 b_2 \rightarrow ...$
▶ Reduction step by reduction step, choose an interface $I_i$ representing the reduction choice (w.r.t. the derivation $P_i$ typing $t_i$ the $i$-th of the sequence).
▶ It produces a reduced derivation $P_i^+$ typing $t_{i+1}$.
▶ Since each interface isomorphism of the reduced derivation is a residual an interface isomorphism, interface $I_i$ can be lifted to $P_i$. 


**Representation Lemma**

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- Consider a reduction sequence \( t_0 \xrightarrow{b_0} t_1 \xrightarrow{b_1} t_2 \xrightarrow{b_2} \ldots \).
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Representation Theorem
Theorem: For all $\mathcal{R}$-derivation $\Pi$, there is a trivial $s$-derivation $P$ that collapses into $\Pi$. 
**RESTATEMENT**

**Theorem:**
For all $\mathcal{R}$-derivation $\Pi$, there is a trivial $s$-derivation $P$ that collapses into $\Pi$.

**Claim**
Every operable derivation $P$ is isomorphic to a trivial derivation.
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*Question:* what is a isomorphism of o.d. $\Psi : P_1 \rightarrow P_2$?
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- A well-behaved bijection from $\text{supp}(P_1)$ to $\text{supp}(P_2)$. 
Restatement

Theorem:
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Every operable derivation $P$ is isomorphic to a trivial derivation.

Question: what is a isomorphism of o.d. $\Psi : P_1 \rightarrow P_2$?

- A well-behaved bijection from $\text{supp}(P_1)$ to $\text{supp}(P_2)$.
- Between each associated axioms rules of $P_1$ and $P_2$, a type isomorphism (w.r.t. the former bijection).
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- A well-behaved bijection from $\text{supp}(P_1)$ to $\text{supp}(P_2)$.
- Between each associated axioms rules of $P_1$ and $P_2$, a type isomorphism (w.r.t. the former bijection).
- Commutation with interface isomorphisms of $P_1$ and $P_2$. 
**Related and Future Work**

- Quantitative types for \( \lambda \mu \) (ongoing work with Delia Kesner) and an explicit classical calculus.

- Can infinitary Strong Normalization be characterized?

- *Categorical Adaptation* of this framework (ongoing work with D. Mazza and L. Pellisier).

- Equational theory of the Model.

- Is the collapse of \( R \) onto \( D \) (idempotent intersection) also surjective?
QUESTIONS

Thank you for your attention!
ASCENDANCE

Some bipositions can be intuitively identified in a derivation.
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\[
\begin{align*}
C \vdash t : (S_k)_{k \in K} \rightarrow T & \quad (\text{pos. } a \cdot 1) \\
(D_k \vdash u : S_k (\text{pos. } a \cdot k) )_{k \in K} & \\
C \cup_{k \in K} D_k \vdash t u : T & \quad (\text{pos. } a)
\end{align*}
\]
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\end{align*}
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\[ C \cup_{k \in K} D_k \vdash tu : T \]

(pos. \( a \))

Two occurrences of the same type
**Ascendance**

Some bipositions can be intuitively identified in a derivation.

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C \vdash t : (S_k)_{k \in K} \rightarrow T \quad (\text{pos. } a \cdot 1) \quad (D_k \vdash u : S_k (\text{pos. } a \cdot k))_{k \in K}
\]

\[
C \cup_{k \in K} D_k \vdash tu : T \quad (\text{pos. } a)
\]
ASCENDANCE

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C \cup_{k \in K} D_k \vdash tu : T \quad \text{(pos. } a) \\
\text{Nested position } c \text{ here corresponds to...}
\]
ASCENDANCE

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\[
\begin{align*}
C \vdash t : (S_k)_{k \in K} \rightarrow T & \quad (\text{pos. } a \cdot 1) \\
C \cup_{k \in K} D_k \vdash tu : T & \quad (\text{pos. } a)
\end{align*}
\]

Nested position 1 \cdot c there.

Nested position c here corresponds to...
ASCENDANCE

Some bipositions can be intuitively identified in a derivation.

\[ \text{We then set:} \quad (a, c) \rightarrow_{\text{asc}} (a \cdot 1, 1 \cdot c) \quad \text{when} \quad t(a) = @ \]
ASCENDANCE

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\[
\begin{align*}
C; \ x : (S_k)_{k \in K} \vdash t : T & \quad \text{(pos. } a \cdot 0) \\
C \vdash \lambda x. t : (S_k)_{k \in K} \rightarrow T & \quad \text{(pos. } a) 
\end{align*}
\]
**ASCENDANCE**

Some bipositions can be intuitively identified in a derivation.

\[
\frac{C; \, x : (S_k)_{k \in K} \vdash t : T \quad (\text{pos. } a \cdot 0)}{C \vdash \lambda x.t : (S_k)_{k \in K} \to T \quad (\text{pos. } a)}
\]

We then set:

\[
(a, 1 \cdot c) \rightarrow_{\text{asc}} (a \cdot 0, 1 \cdot c) \text{ when } t(a) = \lambda x
\]
Polar Inversion

Let us remind rules \( \text{ax} \) and \( \text{abs} \):

\[
\begin{align*}
\text{ax} & : x : (k \cdot T) \vdash x : T \\
\text{abs} & : C \vdash t : T \\
C ; (S_k)_{k \in K} & \vdash \lambda x . t : C(x) \rightarrow T
\end{align*}
\]
**Polar Inversion**

Let us remind rules $\text{ax}$ and $\text{abs}$:

$\text{ax}$

$x : (k \cdot T) \vdash x : T$

$\text{abs}$

$C \vdash t : T$

$C; (S_k)_{k \in K} \vdash \lambda x.t : C(x) \to T$

In a derivation:

$C; x : (S_k)_{k \in K} \vdash t : T$  \hspace{1cm} (pos. $a \cdot 0$)

$C \vdash \lambda x.t : (S_k)_{k \in K} \to T$  \hspace{1cm} (pos. $a$)
Polar Inversion

Let us remind rules $\text{ax}$ and $\text{abs}$:

$\text{ax}$

\[
x : (k \cdot T) \vdash x : T
\]

$\text{abs}$

\[
C \vdash t : T \\
C ; (S_k)_{k \in K} \vdash \lambda x . t : C(x) \rightarrow T
\]

Let $k \geq 2$. We have two cases:

\[
C ; x : (S_k)_{k \in K} \vdash t : T
\]

(\text{pos. } a \cdot 0)

\[
C \vdash \lambda x . t : (S_k)_{k \in K} \rightarrow T
\]

(\text{pos. } a)

Look at $S_7$

inside this seq. type.
**Polar Inversion**

Let us remind rules $\text{ax}$ and $\text{abs}$:

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\begin{align*}
\text{ax} & \quad x : (k \cdot T) \vdash x : T \\
\text{abs} & \quad C \vdash t : T \\
& \quad C; (S_k)_{k \in K} \vdash \lambda x.t : C(x) \rightarrow T
\end{align*}
\]

Let $k \geq 2$. We have two cases:

- First case:

\[
C; x : (S_k)_{k \in K} \vdash t : T \quad \text{(pos. } a \cdot 0) \\
C \vdash \lambda x.t : (S_k)_{k \in K} \rightarrow T \quad \text{(pos. } a)
\]

Look at $S_7$ inside this seq. type.
**Polar Inversion**

Let us remind rules $\text{ax}$ and $\text{abs}$:

$\text{ax}$

\[
\frac{x : (k \cdot T) \vdash x : T}{C \vdash t : T}
\]

$\text{abs}$

\[
\frac{C \vdash t : T}{C; (S_k)_{k \in K} \vdash \lambda x.t : C(x) \to T}
\]

Let $k \geq 2$. We have two cases:

- **First case:**

\[
\frac{C; x : (S_k)_{k \in K} \vdash t : T}{C \vdash \lambda x.t : (S_k)_{k \in K} \to T}
\]  

(pos. $a$)

We then set: $(a, 7 \cdot c) \to_{pi} (a', c)$ when $t(a) = \lambda x$
**Polar Inversion**

Let us remind rules \( \text{ax} \) and \( \text{abs} \):

\[
\frac{x : (k \cdot T) \vdash x : T}{x : (k \cdot T) \vdash x : T} \quad \text{ax}
\]

\[
\frac{C \vdash t : T}{C; (S_k)_{k \in K} \vdash \lambda x. t : C(x) \rightarrow T} \quad \text{abs}
\]

Let \( k \geq 2 \). We have two cases:

**Second case:**

\[
\frac{C; x : (S_k)_{k \in K} \vdash t : T}{C \vdash \lambda x. t : (S_k)_{k \in K} \rightarrow T} \quad \text{(pos. } a \cdot 0 \text{)}
\]

Look at \( S_7 \) inside this seq. type.
Polar Inversion

Let us remind rules \( \text{ax} \) and \( \text{abs} \):

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\begin{align*}
\text{ax} & \quad x : (k \cdot T) \vdash x : T \\
\text{abs} & \quad C \vdash t : T \\
C; (S_k)_{k \in K} & \vdash \lambda x. t : C(x) \rightarrow T
\end{align*}
\]

Let \( k \geq 2 \). We have two cases:

Second case:

No \( \text{ax} \)-rule typing \( x \) with track 7.

\[
\begin{align*}
C; x : (S_k)_{k \in K} & \vdash t : T \quad \text{(pos.} a \cdot 0) \\
C & \vdash \lambda x. t : (S_k)_{k \in K} \rightarrow T \quad \text{(pos.} a)
\end{align*}
\]

Look at \( S_7 \) inside this seq. type.
POLAR INVERSION

Let us remind rules \textit{ax} and \textit{abs}:

\[
\frac{x : (k \cdot T) \vdash x : T}{\text{ax}}
\]

\[
\frac{C \vdash t : T}{C; (S_k)_{k \in K} \vdash \lambda x. t : C(x) \rightarrow T}{\text{abs}}
\]

Let \( k \geq 2 \). We have two cases:

\[
\frac{C; x : (S_k)_{k \in K} \vdash t : T}{\text{(pos. } a \cdot 0)}
\]

\[
\frac{C \vdash \lambda x. t : (S_k)_{k \in K} \rightarrow T}{\text{(pos. } a)}
\]

Look at \( S_7 \)
inside this seq. type.

We then set: \((a, 7 \cdot c) \rightarrow_{\pi} b \perp\) when \( t(a) = \lambda x \)