Distributed Computing
12 - Complexities, Volume and Exercises

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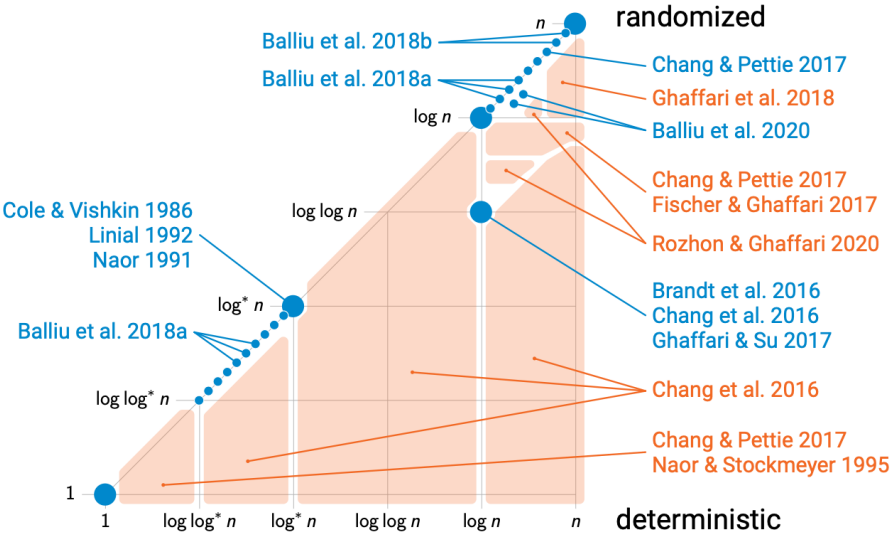
EN INFORMATIQUE
FONDAMENTALE

Université de Paris

Creating Complexities

Complexities on Graphs with the LOCAL Model



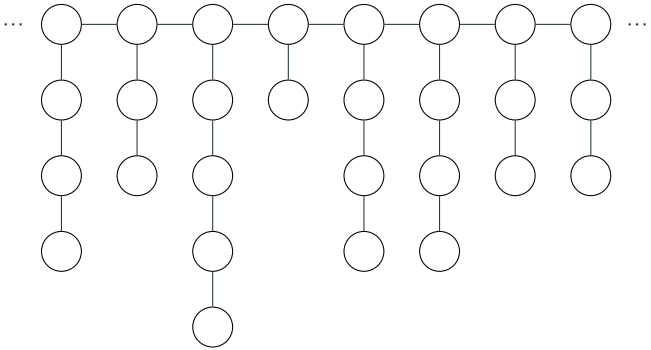
Figures from https ://jukkasuomela.fi/landscape-of-locality/

Infinite Number of Complexities

Chang, Pettie (2019)

For any $k > 0$, there exists a problem P_k that is solvable in time $\Theta(n^{1/k})$.

$2\frac{1}{2}$-coloring Example



Comb Tree : A path V_2 of degree 3 nodes. To each of those nodes there is a path appended.

For each node $v \in V_2$ either :

- Its appended path is 2-colored ($v \in D_2$)
- It is 2-colored with its $V_2 \setminus D_2$ neighbors

Question : Find a $O(\sqrt{n})$ algorithm.

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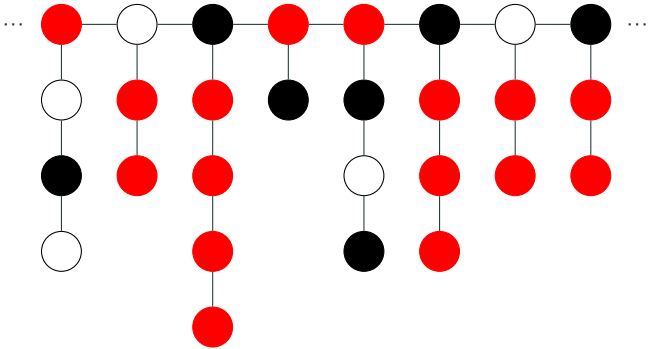

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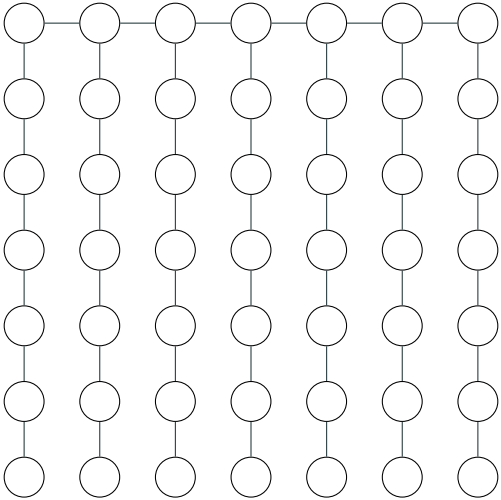

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Question : How to generalize this problem with the same results on any graphs ?

- G_1 : input graph
- $V_1 = \{v \in V(G_1) : degree(v) \leq 2\}$
- $G_2 = G_1 \setminus V_1$
- $V_2 = \{v \in V(G_2) : degree(v) \leq 2\}$
- V_1 nodes accept if, in regard to its neighbors in V_1 :
  - They are 2-colored with colors 1 and 2
  - They all are in color 3
- $D_2 = \{v \in V_2 :$ its V_1 neighbors are 2-colored$\}$
  - D_2 nodes accept if they are in color 4
- $V_2 \setminus D_2$ nodes accept if, in regard to its neighbors in $V_2 \setminus D_2$:
  - They are 2-colored with colors 1 and 2
  - They have 2 of those neighbors and are in color 3

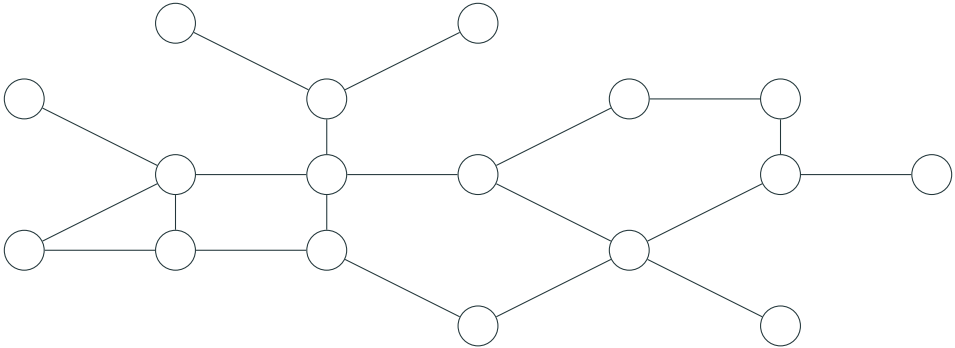
Generalization to $n^{1/k}$

- G_1 : input graph
- $V_i = \{v \in V(G_i) : degree(v) \leq 2\}$
- $G_i = G_{i-1} \setminus V_{i-1}$
- $V_{k+1} = V(G_{k+1})$
- A vertex in $V(G)$ is **exempted** if
  - it has a lower level exempted neighbor
  - it has a lower level 2-colored neighbor
  - it is in V_{k+1}
- $D_i \subseteq V_i$ are the exempted nodes of level i
- $V_i \setminus D_i$ nodes $(i \leq k)$ accept if, in regard to its neighbors in $V_i \setminus D_i$:
  - They are 2-colored with colors 1 and 2
  - They have 2 of those neighbors and are in color 3

The Centralized Local Model

SLOCAL Model

- Each node is activated one after another, to compute its own output
- A node has access to the outputs already computed to produce its own
- Complexity : maximal radius needed among nodes
- **Greedy** problems can be solved in radius $O(1)$



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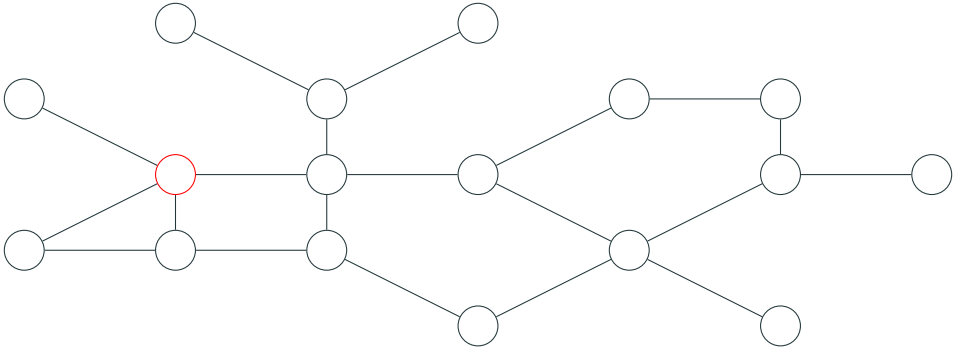

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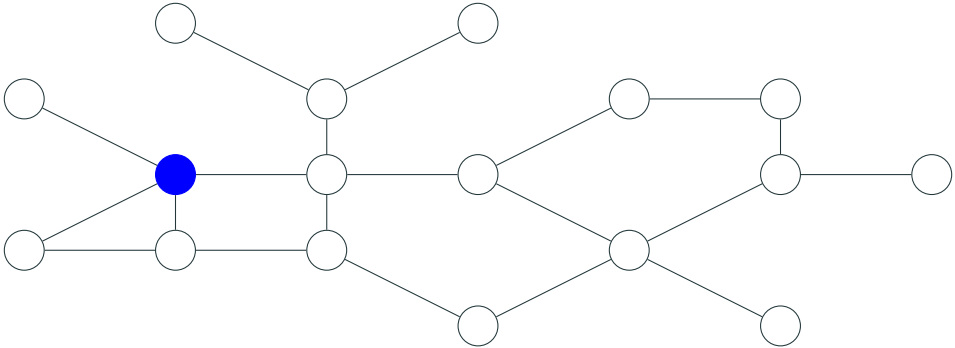

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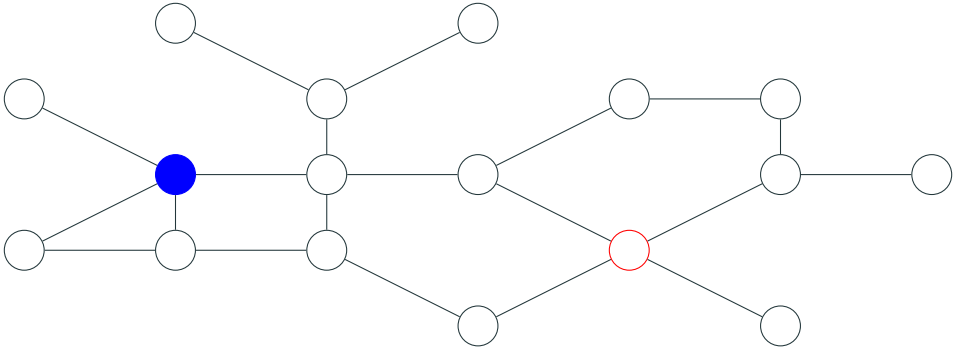

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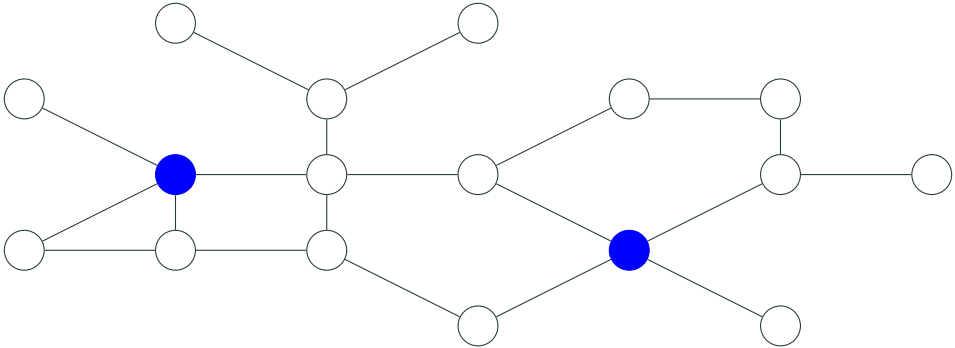

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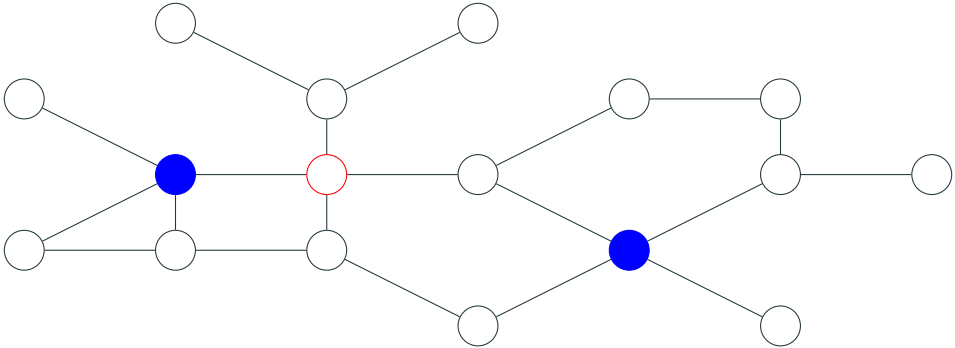

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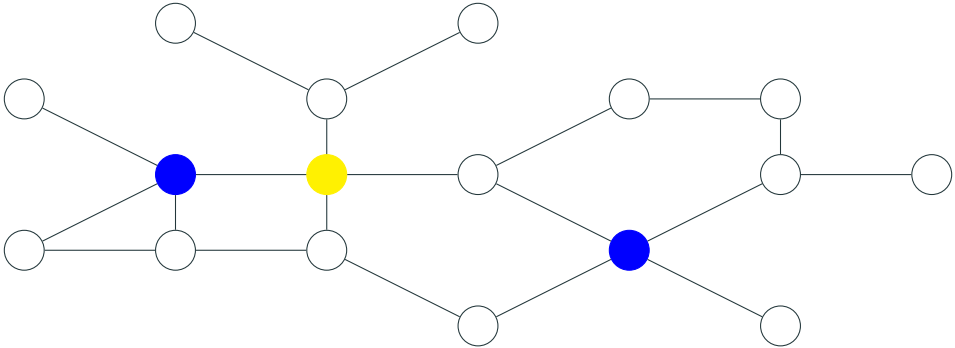

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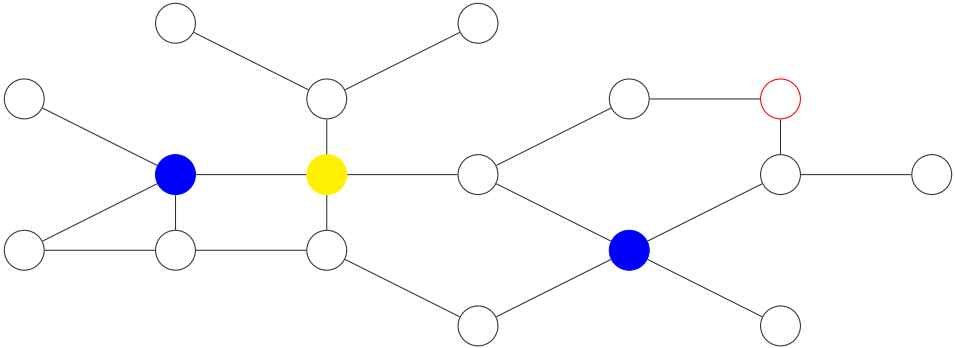

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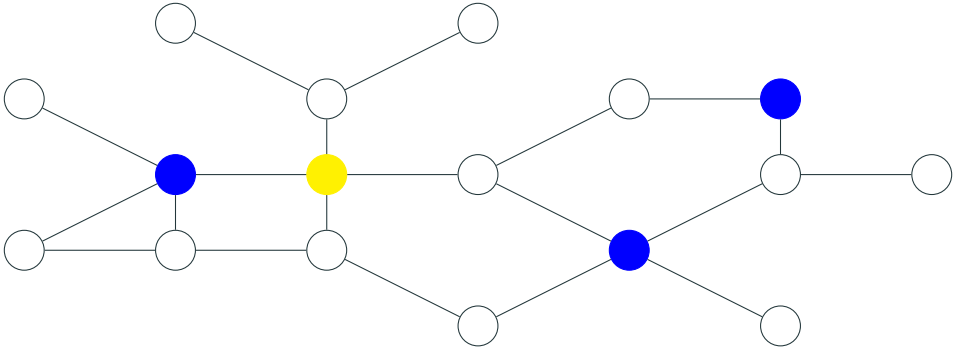

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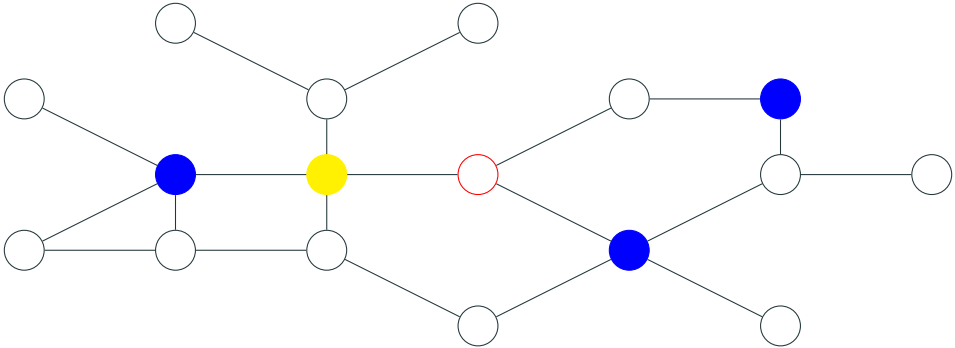

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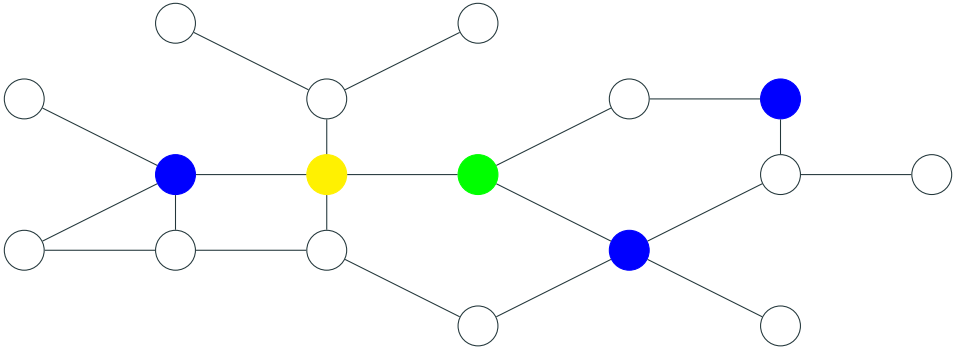

CentLOCAL Model

- In parallel, each node v :
  - Knows its own Id_v and degree d_{Id_v}
  - At each step, they send a request (Id_u, k), with $k \leq d_{Id_u}$
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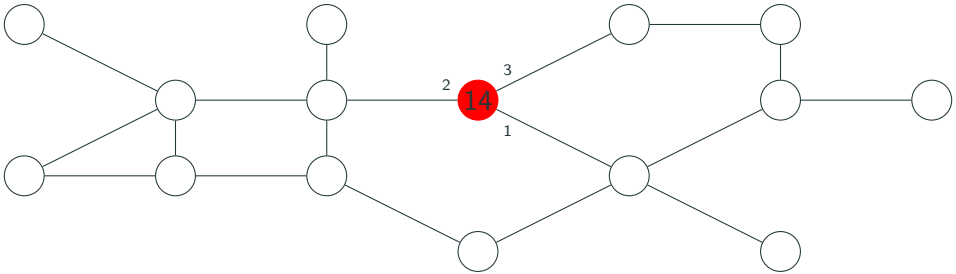

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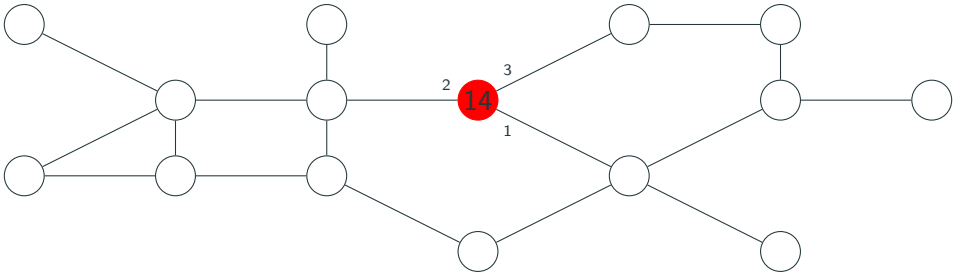

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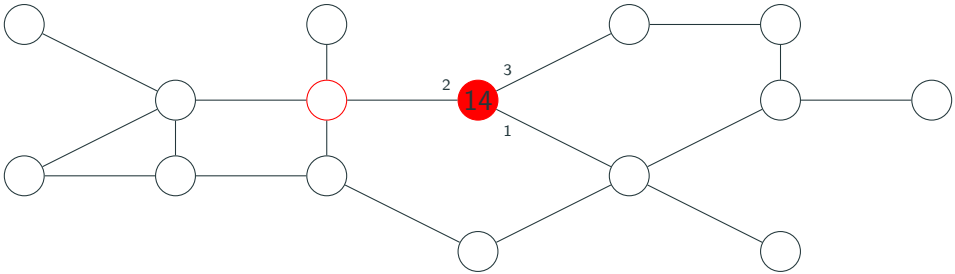

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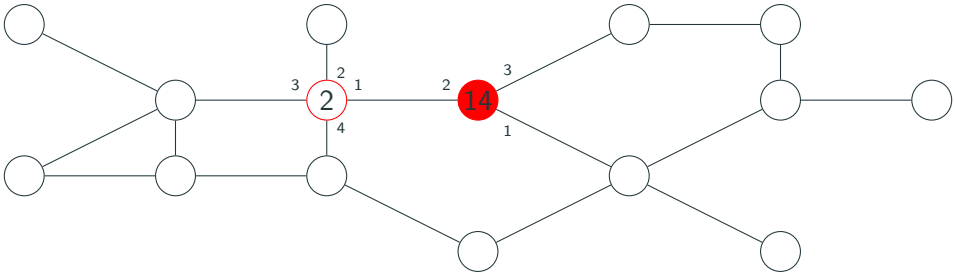
Request : (14,2) $\Rightarrow$ (2,4,1)



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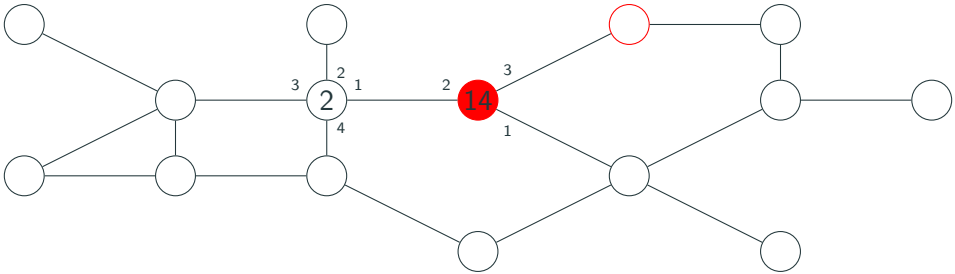

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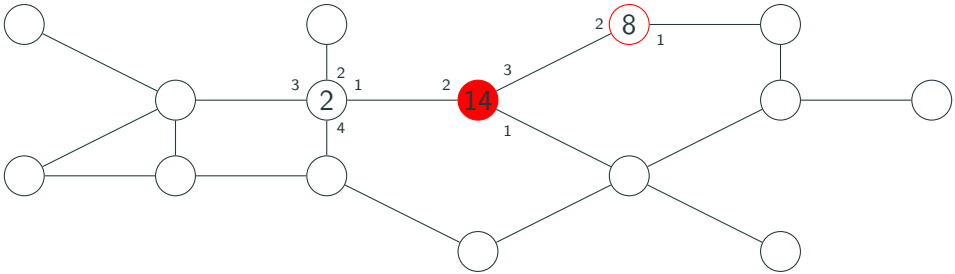

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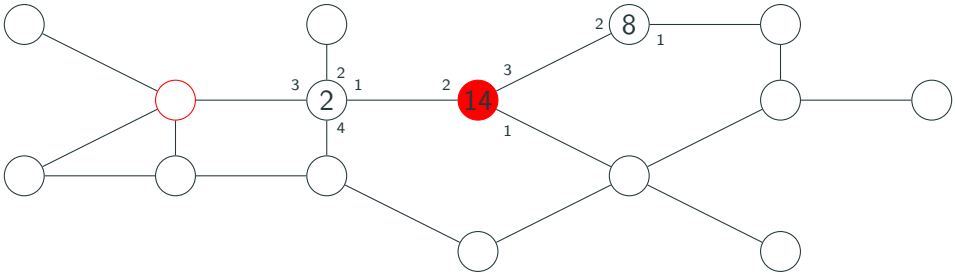

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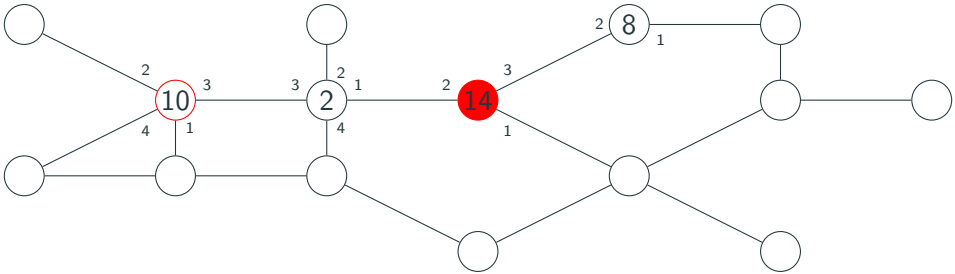
Request : (2,3) $\Rightarrow$ (10,4,3)



Greedy Problems

Problem A can be solved in time $\Theta(f(n))$ in the LOCAL model

$\Rightarrow$ A can be solved in time in the CentLOCAL model

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$\Rightarrow$ A can be solved in time $\Omega(f(n))$ and $O\left(\Delta^{f(n)}\right)$ in the CentLOCAL model

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Even *et. al* **(2018)**

There is a CentLOCAL algorithm in time $O(\Delta \times log^* n + \Delta^3)$ for $\leq \Delta^2$-coloring a graph.

There is a CentLOCAL algorithm in time $O(\Delta \times log^* n + \Delta^3)$ for orienting a graph where the longer oriented path is of length $\leq \Delta^2$.

Any greedy problem can be solved in time $O(f(\Delta) \times log^* n)$.

Complexity Gap

Rosenbaum and Suomela (2020)

In the CentLOCAL model, if n is not given in advance and identifiers do not require to be polynomial in n, there is no problem whose time complexity is in $\omega(log^* n) \cap o(n)$.

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- Take N such that $T(N) \ll N$
- Do a distance N-coloring
- Simulate the algorithm with the new identifiers

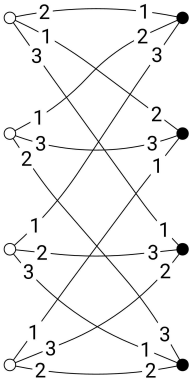
Maximal Matching Lower Bound

Maximal Matching Algorithm

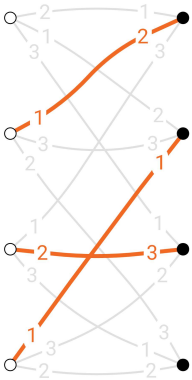
computer network with port numbering

bipartite, 2-colored graph

Δ-regular (here Δ = 3)



output: ***maximal matching***



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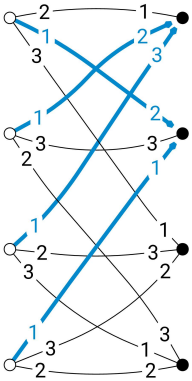

Very simple algorithm

unmatched white nodes:
send ***proposal*** to port 1

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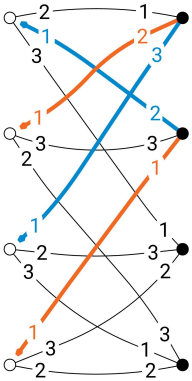

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black nodes:
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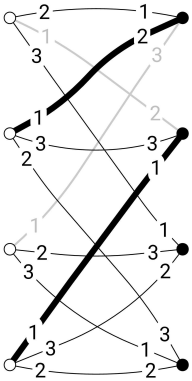

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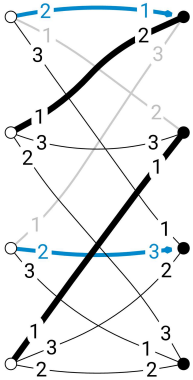

Very simple algorithm

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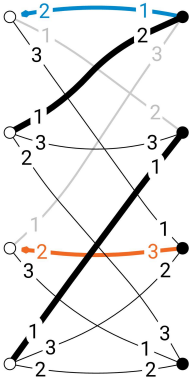

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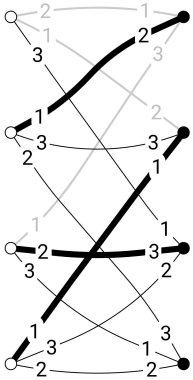

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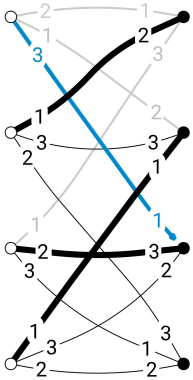

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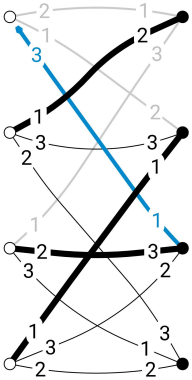

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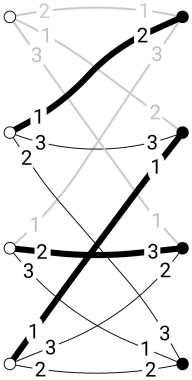

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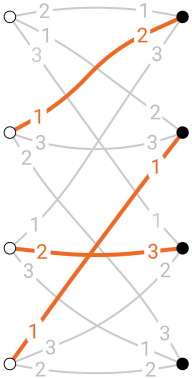


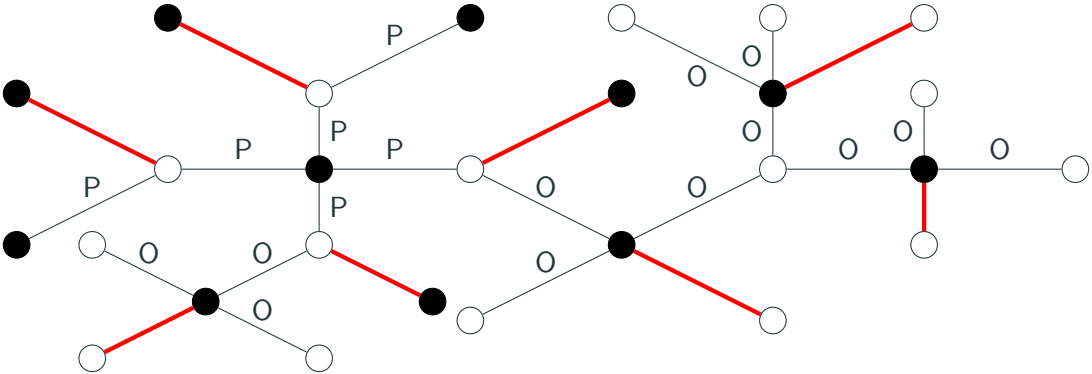
Very simple algorithm

Finds a ***maximal matching*** in $O(\Delta)$ communication rounds

Note: running time does not depend on n

Maximal Matching in Δ-regular graphs

- Maximal Matching, $\Sigma = \{\mathsf{M}, \mathsf{O}, \mathsf{P}\}$
- $B = \left(\mathsf{M}\mathsf{O}^{\Delta-1} \,\middle|\, \mathsf{P}^{\Delta}\right)$, $W = \left(\mathsf{M}[\mathsf{PO}]^{\Delta-1} \,\middle|\, \mathsf{O}^{\Delta}\right)$



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Question : Do one-round reduction on the W-algorithm.

Parametrized Matching

Parametrized problem $\Pi_\Delta(x, y)$

- Addition of joker output X.
- $B_\Delta(x, y) = \left(\mathrm{MO}^{d-1} \,\middle|\, \mathrm{P}^d\right) \mathrm{O}^y \mathrm{X}^x$
- $W_\Delta(x, y) = \left([\mathrm{MX}][\mathrm{POX}]^{d-1} \,\middle|\, [\mathrm{OX}]^d\right) [\mathrm{POX}]^y [\mathrm{MPOX}]^x$

$d = \Delta - x - y$

Questions :

1. Prove that : Maximal Matching in T rounds, $\Rightarrow \Pi_\Delta(0, 0)$ in T rounds.
2. (At home) Prove that : $\Pi_\Delta(0, 0)$ on W in T rounds $\Rightarrow \Pi_\Delta(1, 0)$ on B in $T - 1$ rounds.

We have : $\Pi_\Delta(x, y)$ in T rounds $\Rightarrow \Pi_\Delta(x + 1, y + x)$ in $T - 1$ rounds.

3. What lower bound on the complexity can you deduce ?

Weak k-matching

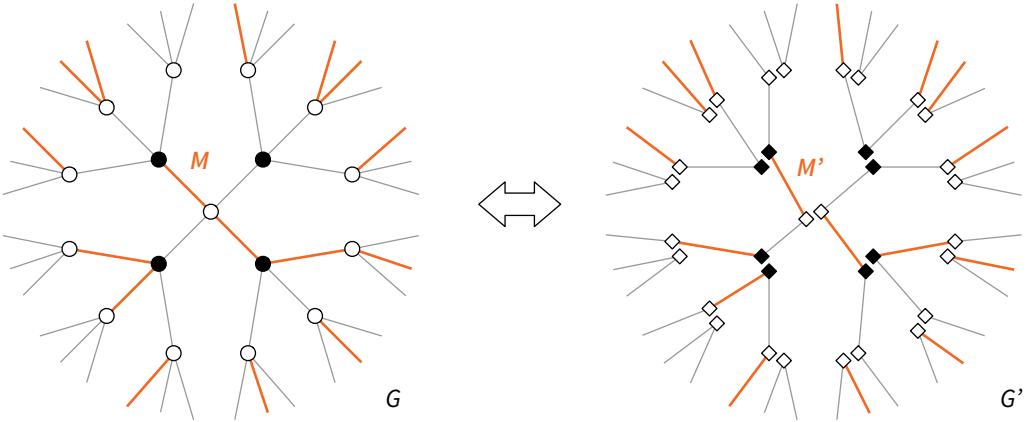
Weak k-matching : For each node v,

- v is matched to $l \in [1, k]$ nodes
- v is unmatched and all the neighbors of v are matched

Questions : Suppose that we can solve Maximal Matching in $o(\Delta)$ rounds

1. Prove that we solve weak "$\Delta^{1/2}$-Matching" in $o(\Delta^{1/2})$ rounds.
2. Deduce that we solve $\Pi_\Delta(\Delta^{1/2} - 1, 0)$ in $o(\Delta^{1/2})$ rounds.
3. Deduce that we solve $\Pi_\Delta(\Delta^{1/2} + o(\sqrt{\Delta}), o(\Delta))$ in 0 rounds.
4. Conclude.

Weak $\sqrt{\Delta}$-matching



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Reminder :

- $B_\Delta(x, y) = \left(\mathrm{M}\mathrm{O}^{d-1} \,\middle|\, \mathrm{P}^d\right)\mathrm{O}^y\mathrm{X}^x$
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Balliu *et. al* **(2019)**
There is no deterministic distributed algorithm that solves Maximal Matching in $o(\Delta)$ rounds in the LOCAL model.

Bibliography

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