Distributed Computing
11 - LOCAL Algorithms

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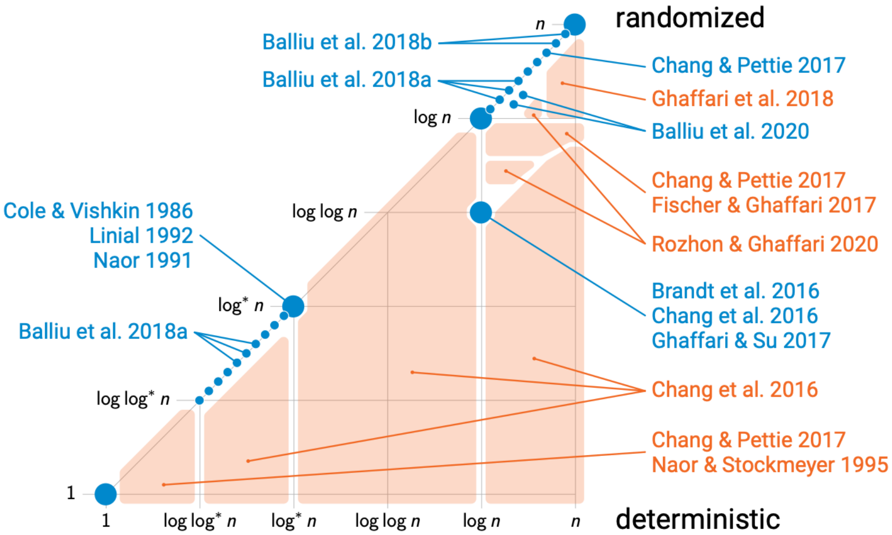
Creating Complexities

Complexities on Graphs with the LOCAL Model

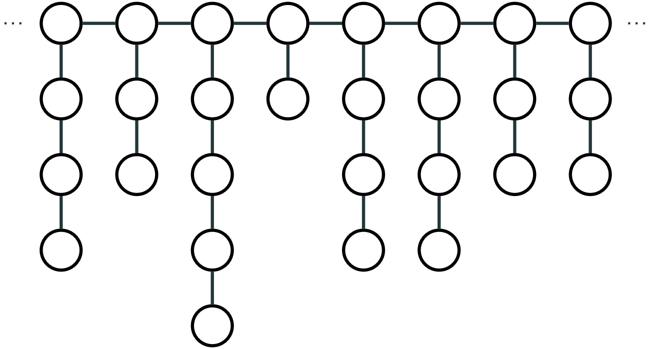


Figures from https ://jukkasuomela.fi/landscape-of-locality/

Infinite Number of Complexities

Chang, Pettie (2019)
For any $k > 0$, there exists a problem P_k that is solvable in time $\Theta(n^{1/k})$.

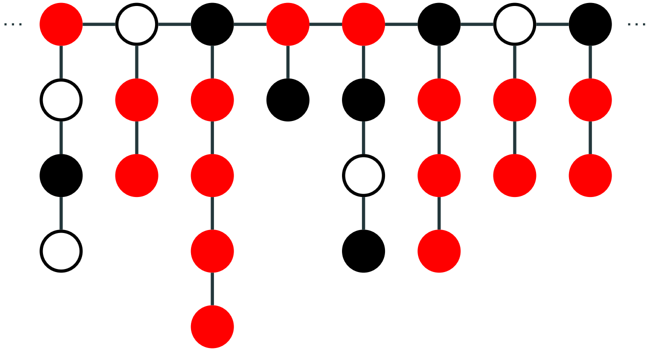
$2\frac{1}{2}$-coloring Example



Comb Tree : A path V_2 of degree 3 nodes. To each of those nodes there is a path appended.
For each node $v \in V_2$ either :

- Its appended path is 2-colored ($v \in D_2$)
- It is 2-colored with its $V_2 \setminus D_2$ neighbors

$2\frac{1}{2}$-coloring Example



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Question : Find a $O(\sqrt{n})$ algorithm.

Worst Case Graph

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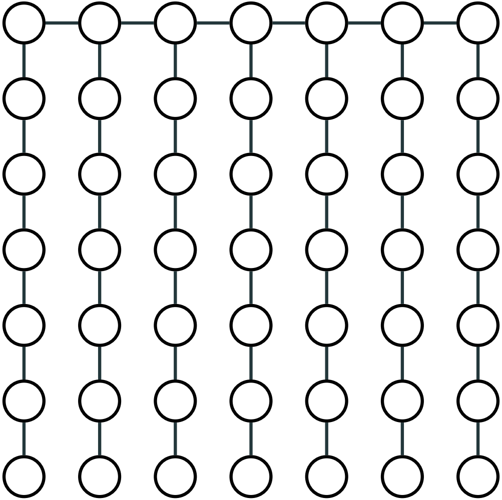

Generalization to any graphs

- G_1 : input graph
- $V_1 = \{v \in V(G_1) : degree(v) \leq 2\}$
- $G_2 = G_1 \setminus V_1$
- $V_2 = \{v \in V(G_2) : degree(v) \leq 2\}$
- V_1 nodes accept if, in regard to its neighbors in V_1 :
  - They are 2-colored with colors 1 and 2
  - They all are in color 3
  - They are on an oriented cycle
- $D_2 = \{v \in V_2 : \text{its } V_1 \text{ neighbors are 2-colored}\}$
  - D_2 nodes accept if they are in color 4
- $V_2 \setminus D_2$ nodes accept if, in regard to its neighbors in $V_2 \setminus D_2$:
  - They are 2-colored with colors 1 and 2
  - They have 2 neighbors and they all are in color 3
  - They are on an oriented cycle

Generalization to $n^{1/k}$

- G_1 : input graph
- $V_i = \{v \in V(G_i) : degree(v) \leq 2\}$
- $G_i = G_{i-1} \setminus V_{i-1}$
- $V_{k+1} = V(G_{k+1})$
- A vertex in $V(G)$ is **exempted** if
  - It has a lower level exempted neighbor
  - It has a lower level 2-colored neighbor
  - It is on an oriented cycle on its level
  - It is in V_{k+1}
- $D_i \subseteq V_i$ are the exempted nodes of level i
- $V_i \setminus D_i$ nodes ($i \leq k$) accept if, in regard to its neighbors in $V_i \setminus D_i$:
  - They are 2-colored with colors 1 and 2
  - They all are in color 3 (and have degree=2 if $i = k$)
  - They are on an oriented cycle

Maximal Matchings

Lower Bounds

Balliu *et. al* (2019)

Maximal Matching needs $\Omega(min\{\Delta, \log n / \log\log n\})$ rounds in the LOCAL Model.

Linial (1992)

An algorithm which colors the n-cycle with three colors requires at least $\frac{1}{2}(\log^* n - 3)$ communication rounds.

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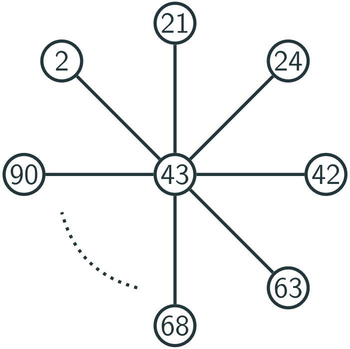
Panconesi, Rizzi (2001)
There exists a LCL algorithm to produce a Maximal Matching in $O(\log^* n + \Delta)$ communications.

Partition into Rooted Trees



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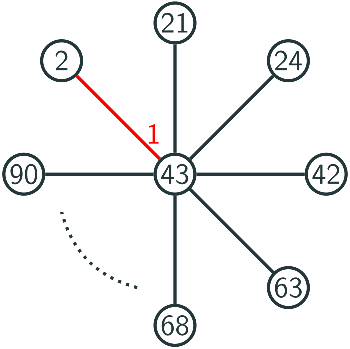

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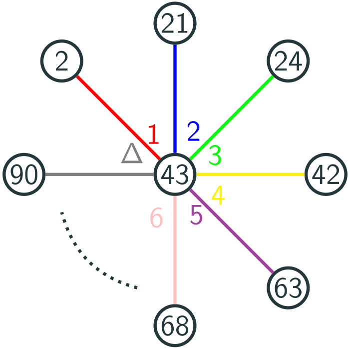

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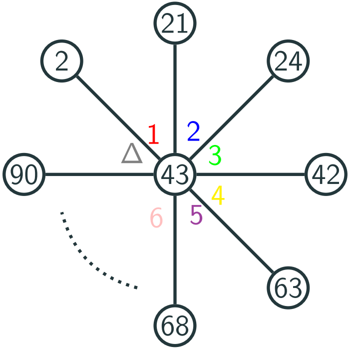

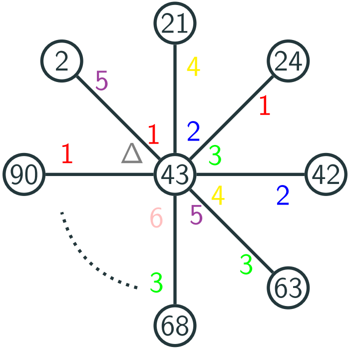
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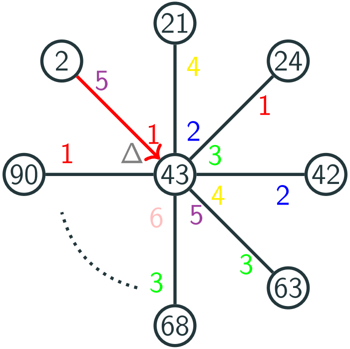
Partition into Rooted Trees



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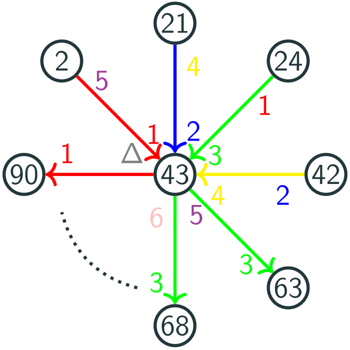

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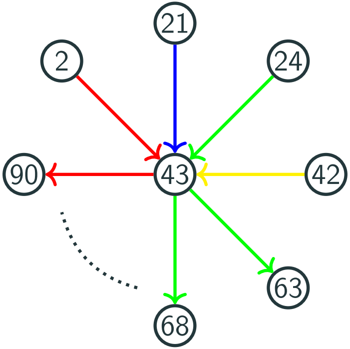

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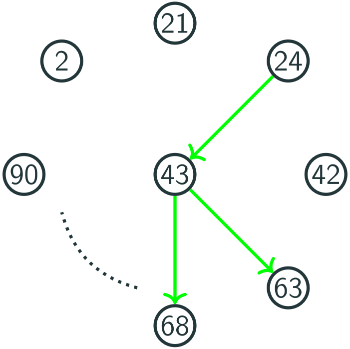

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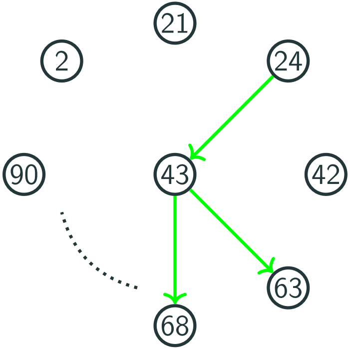
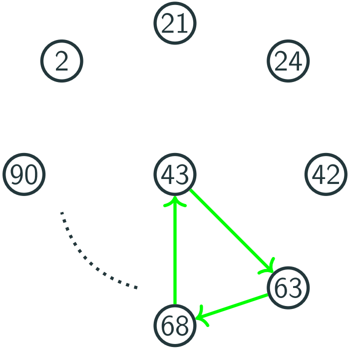


- At most one parent

Partition into Rooted Trees



- At most one parent
- No cycles

Maximal Matching Algorithm

1. Partition G into Δ Rooted Forests.

2. For each Forest, 3-color it in $O(\log^* n)$ communications (in parallel)

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1. Partition G into Δ Rooted Forests.

2. For each Forest, 3-color it in $O(\log^* n)$ communications (in parallel)

3. For Forest $f = 1$ to Δ do :
   - For Color $c = 1$ to 3 do :
     - For all node, add if possible an edge connecting it to one of its children of Color c in Forest f.

3-Coloring Trees

3-Coloring in $O(\log n)$

Bonamy *et. al* (2018)
There exists an algorithm that 3-colors trees in $O(\log n)$ rounds.

Tree Shattering
There exists an algorithm that computes an MIS in trees in $O(\log n)$ rounds. This MIS is such that, after its removal, connected components have size 1 or 2.

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- Build a $\mathcal{O}(\log n)$ light labeling of the nodes :
  1. Any node labeled i has at most two neighbors with label $\geq i$, at most one of which with label $\geq i+1$
  2. No two adjacent nodes labeled i both have a neighbor with label $\geq i+1$

3-Coloring in $O(\log n)$

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- Build a $\mathcal{O}(\log n)$ light labeling of the nodes :
  1. Any node labeled i has at most two neighbors with label $\geq i$, at most one of which with label $\geq i+1$
  2. No two adjacent nodes labeled i both have a neighbor with label $\geq i+1$
- Compute a 3-coloring for each paths of nodes with the same label
- Build the MIS by considering labels decreasingly.

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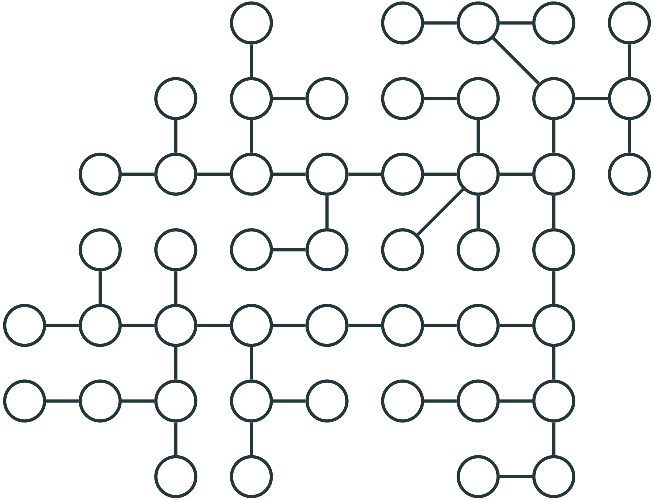

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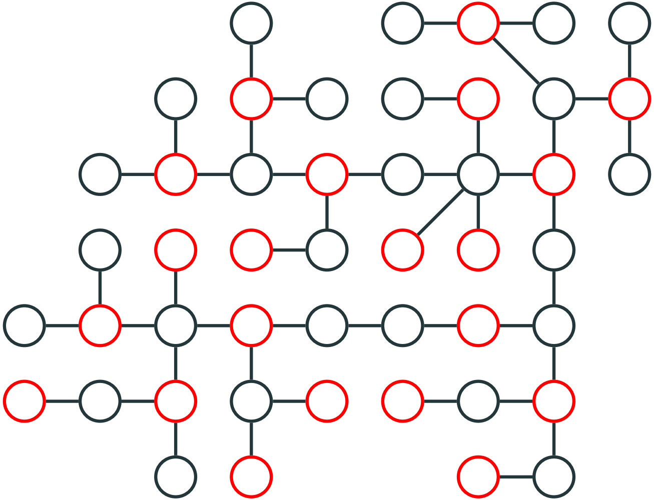

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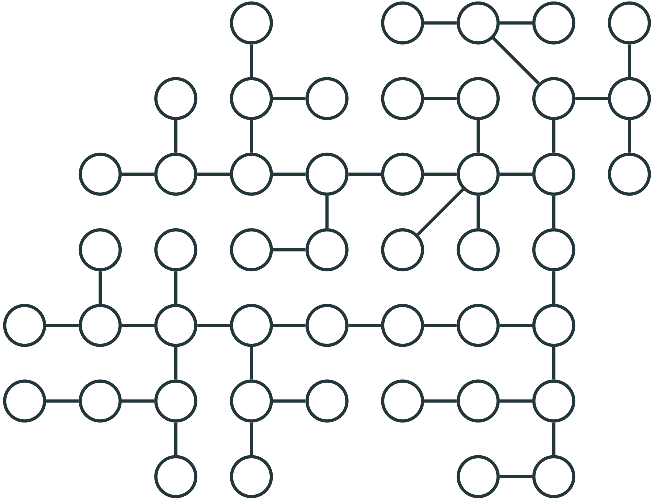

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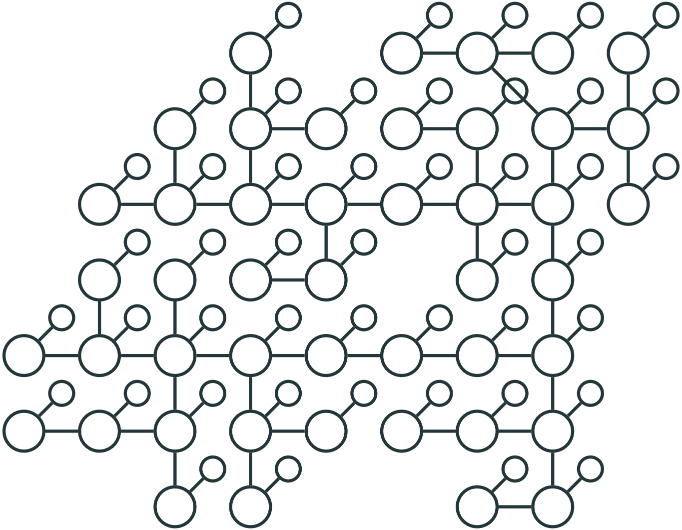

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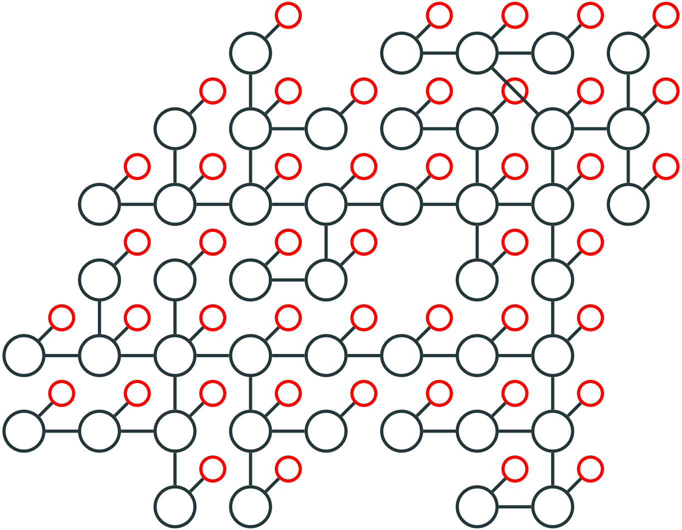

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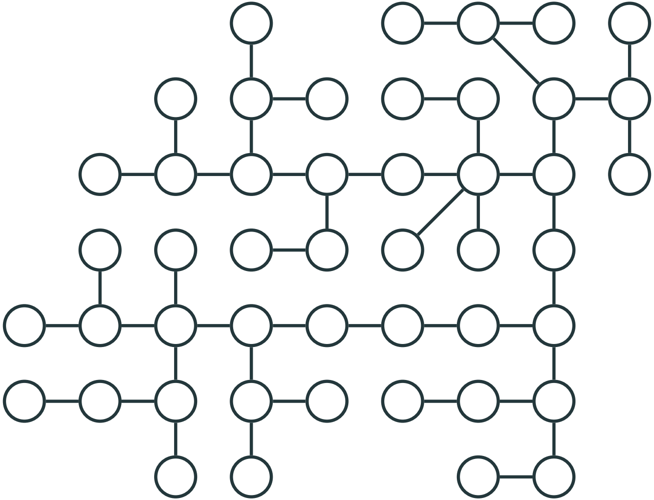

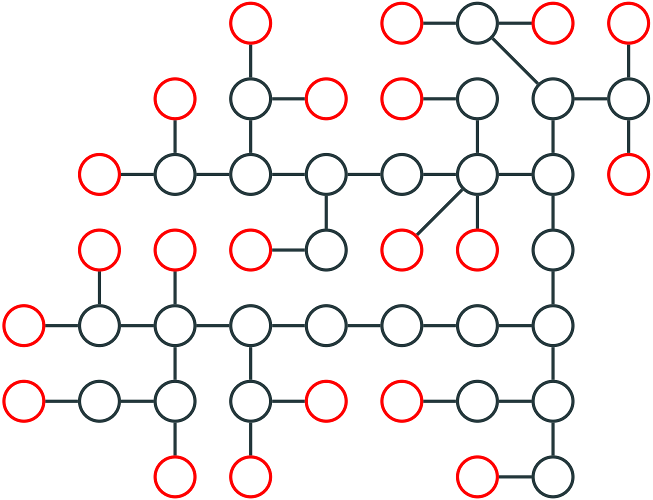

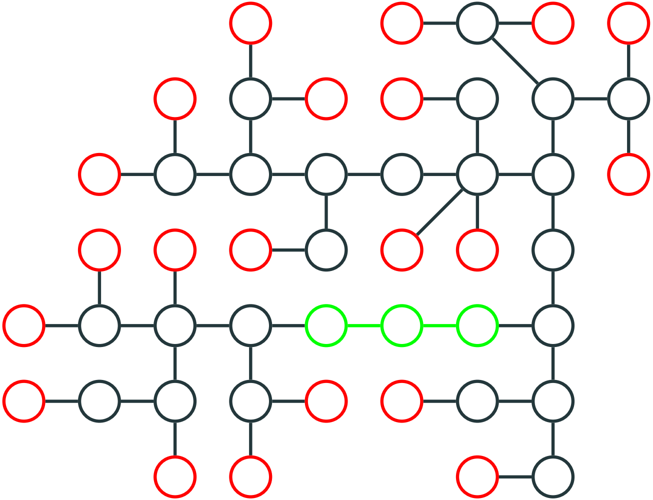

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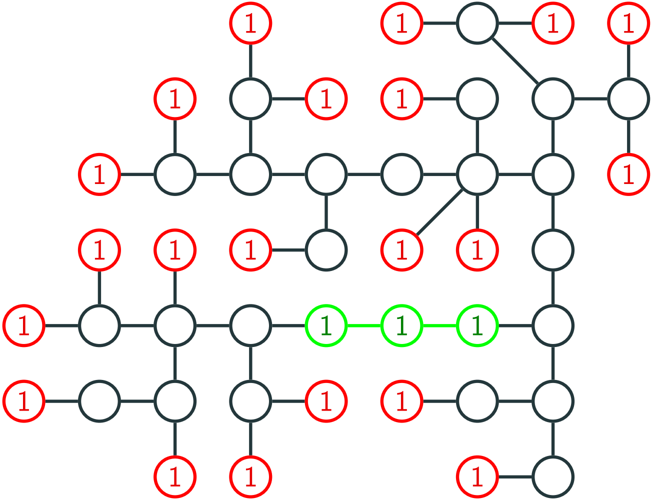

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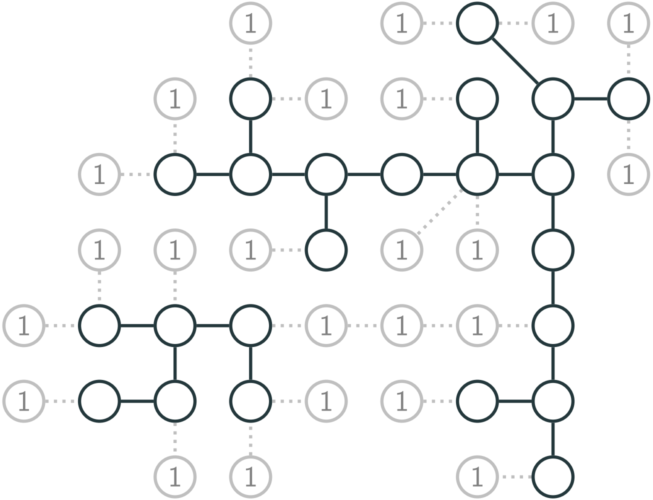

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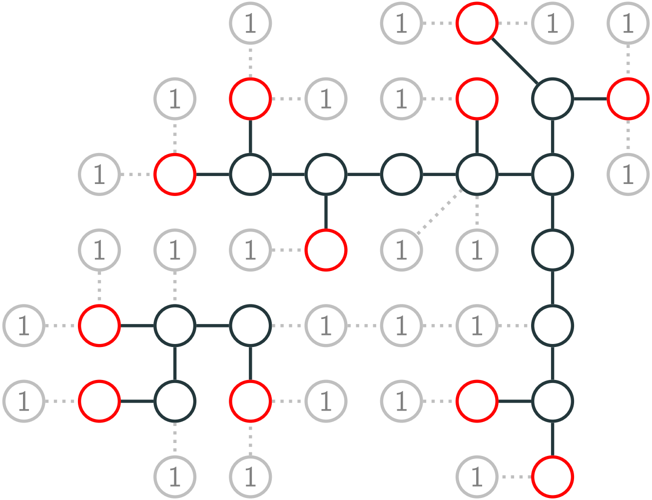

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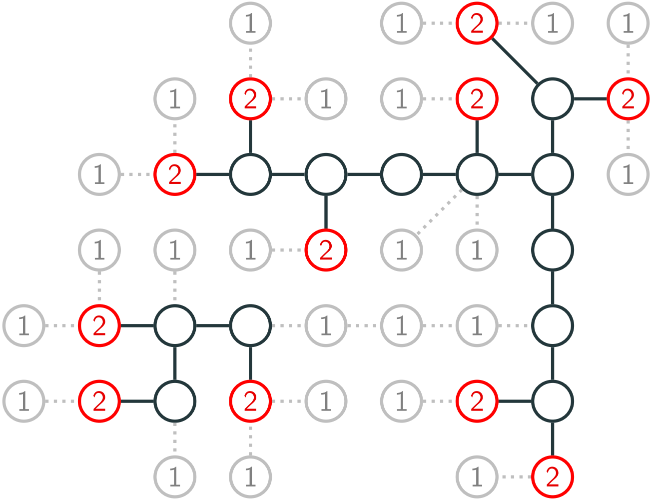

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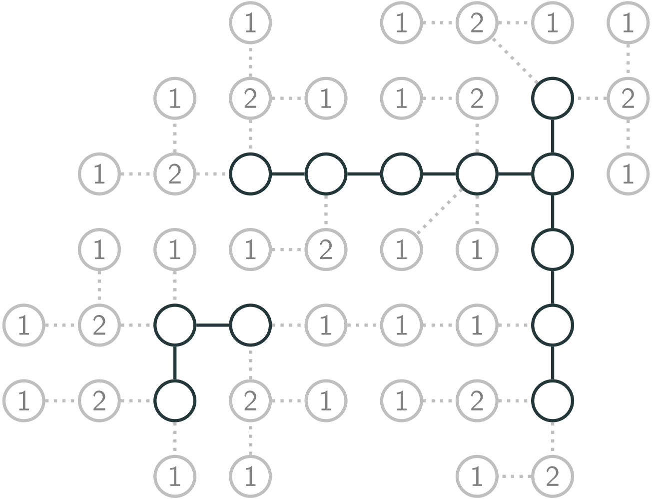

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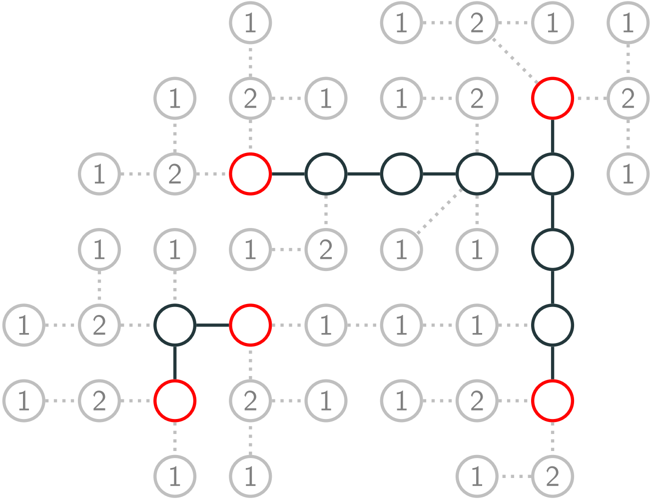

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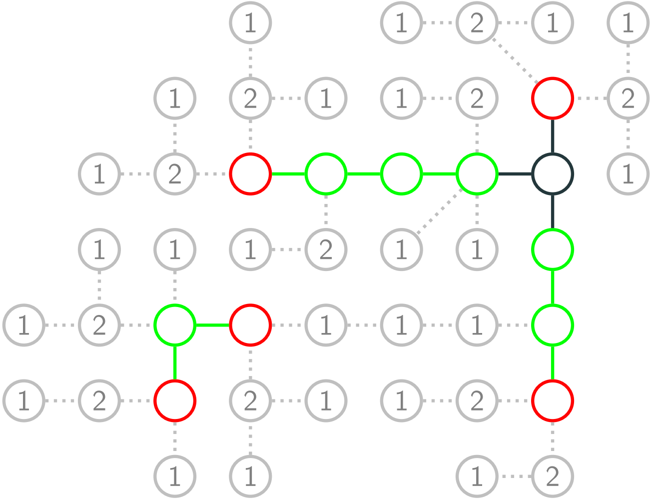

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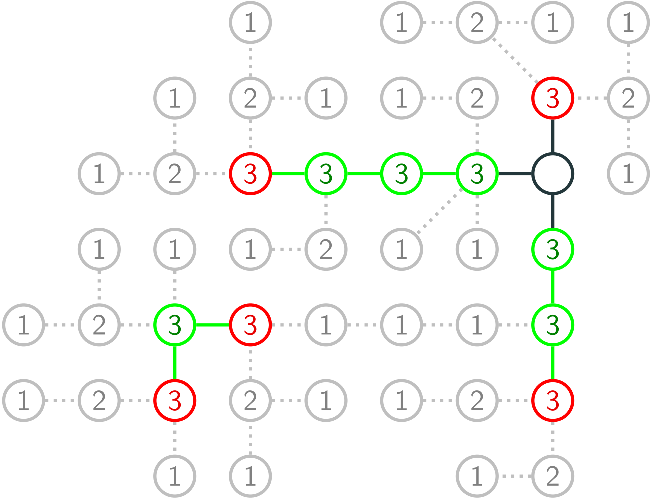

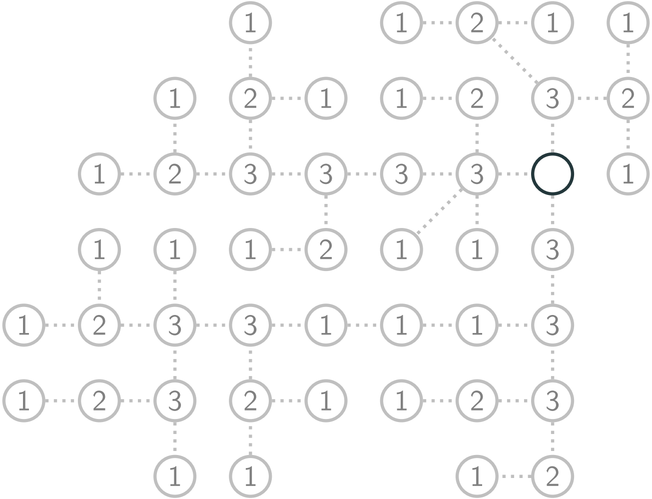
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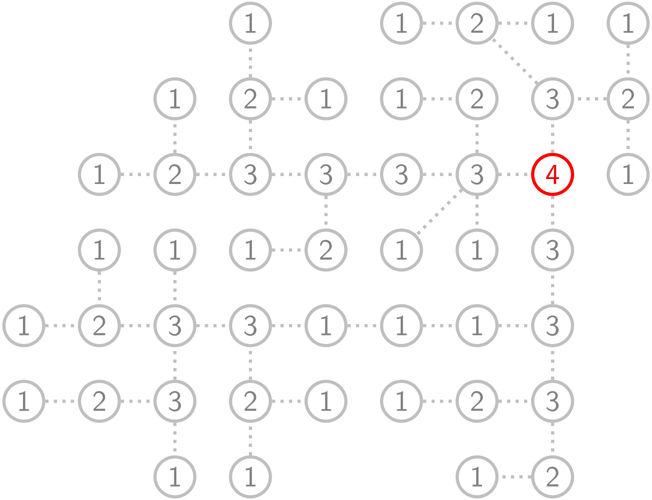
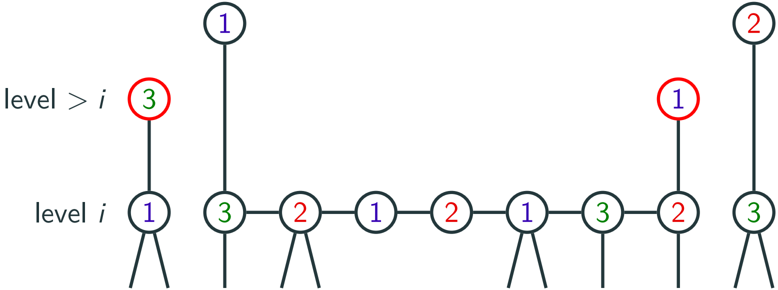


Constructing the Maximal Independent Set

- In parallel, compute a 3-coloring on each level
- From higher to lower level
- Add, if possible, neighbors to higher level nodes
- Greedily complete the MIS



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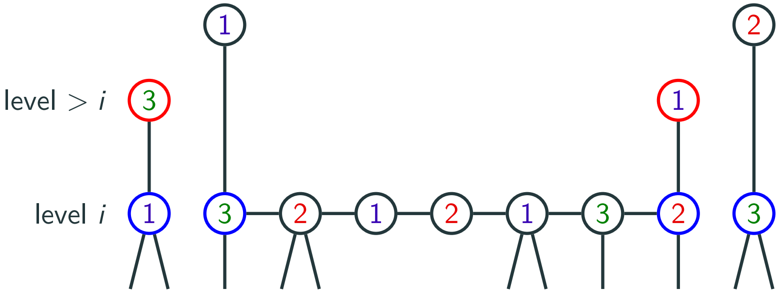

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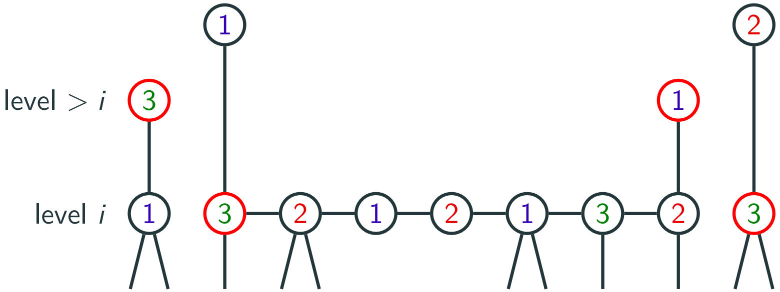

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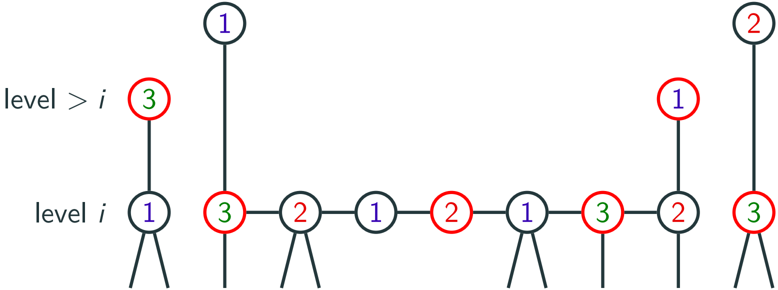

Bibliography

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