Distributed Computing
13 - LOCAL Variants

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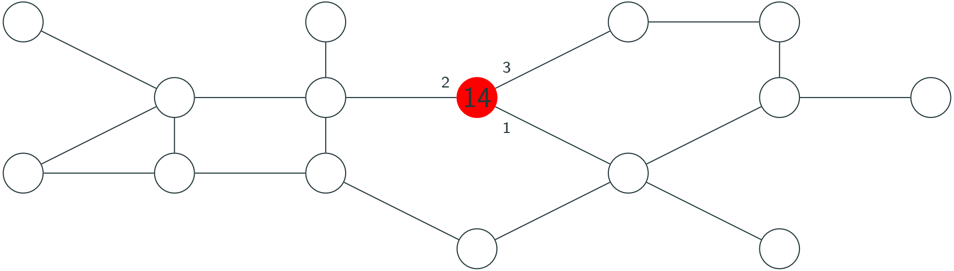
INSTITUT
DE RECHERCHE
EN INFORMATIQUE
FONDAMENTALE

Volume Complexity

CentLOCAL Model

- In parallel, each node v :
  - Knows its own Id_v and degree d_{Id_v}
  - At each step, they send a request (Id_u, k), with $k \leq d_{Id_u}$
  - They get (Id_w, d_{Id_w}, k') such that $(u, v) \in E$ are connected by port k from u and k' from w
- Complexity : maximal number of requests from a node



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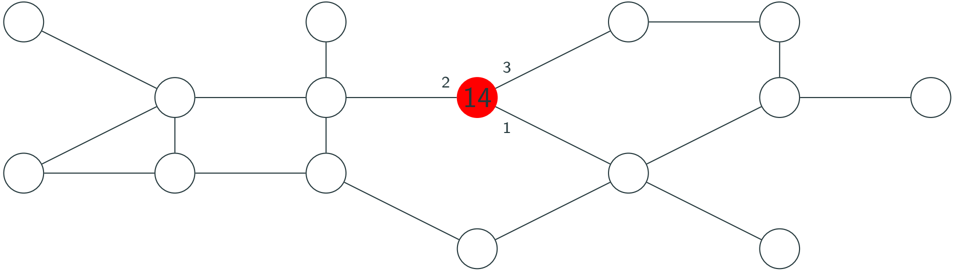

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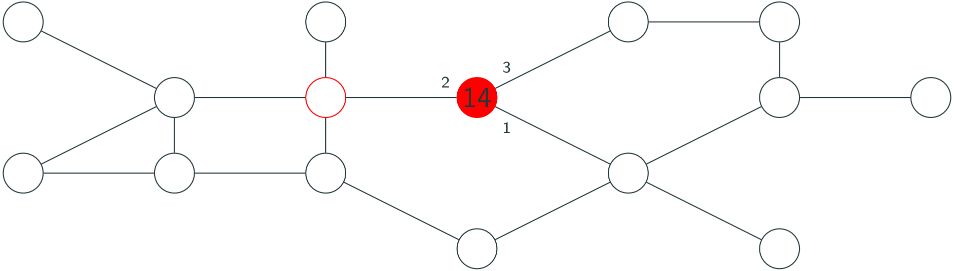

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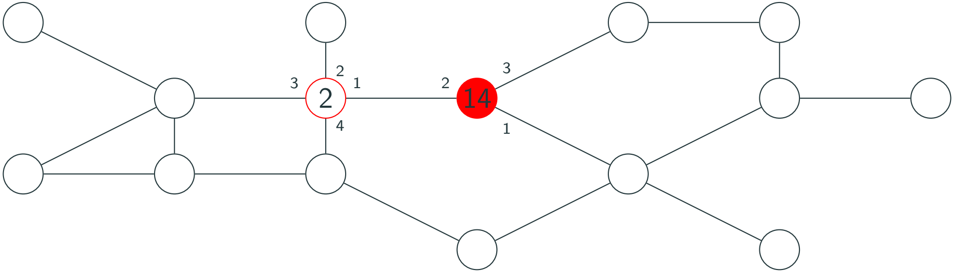
Request : (14,2) $\Rightarrow$ (2,4,1)



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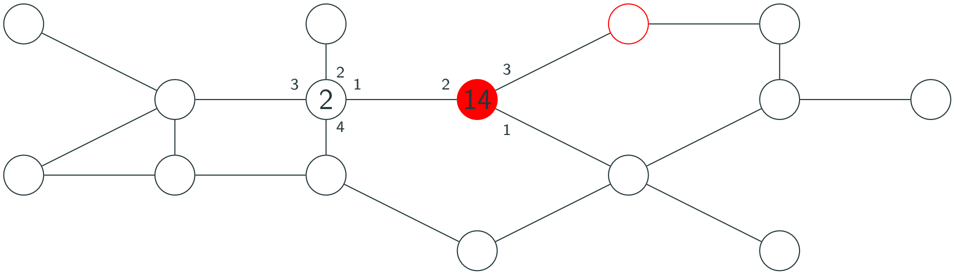

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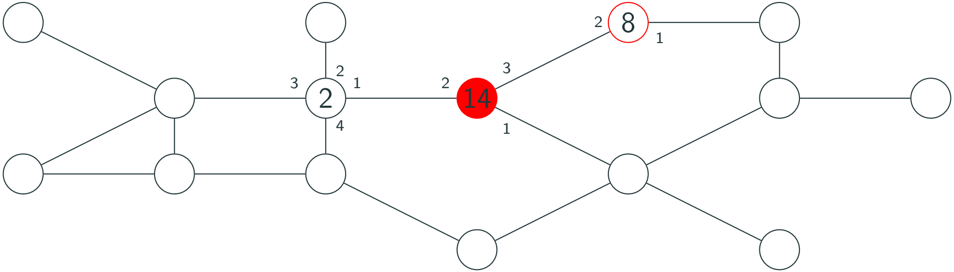
Request : (14,3) $\Rightarrow$ (8,2,2)



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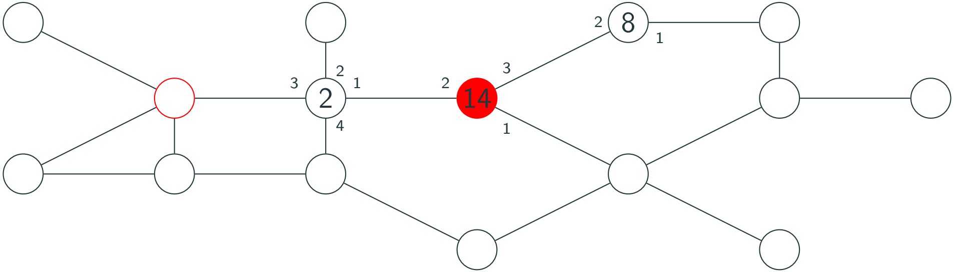

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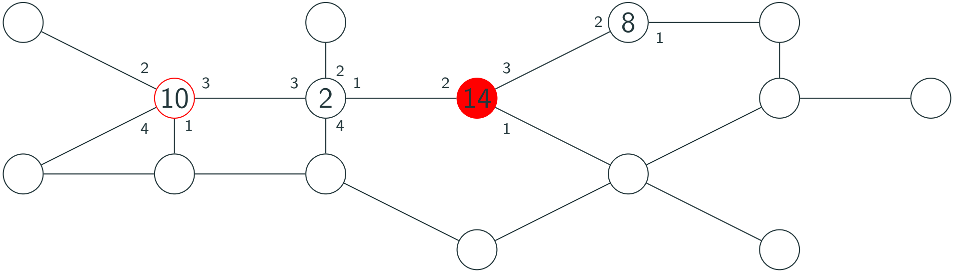
Request : (2,3) $\Rightarrow$ (10,4,3)



Greedy Problems

Problem A can be solved in time $\Theta(f(n))$ in the LOCAL model

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$$\Rightarrow A \text{ can be solved in time } \Omega(f(n)) \text{ and } O\left(\Delta^{f(n)}\right) \text{ in the CentLOCAL model}$$

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Problem A can be solved in time $\Theta(f(n))$ in the LOCAL model

$\Rightarrow$ A can be solved in time $\Omega(f(n))$ and $O\left(\Delta^{f(n)}\right)$ in the CentLOCAL model

Even *et. al* (2018)

There is a CentLOCAL algorithm in time $O(\Delta \times log^* n + \Delta^3)$ for $\leq \Delta^2$-coloring a graph.

There is a CentLOCAL algorithm in time $O(\Delta \times log^* n + \Delta^3)$ for orienting a graph where the longer oriented path is of length $\leq \Delta^2$.

Any greedy problem can be solved in time $O(f(\Delta) \times log^* n)$.

Complexity Gap

Rosenbaum and Suomela (2020)
In the CentLOCAL model, if n is not given in advance and identifiers do not require to be polynomial in n, there is no problem whose time complexity is in $\omega(log^* n) \cap o(n)$.

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- Take N such that $T(N) \ll N$
- Do a distance N-coloring
- Simulate the algorithm with the new identifiers

Mendability

Mendable Problems

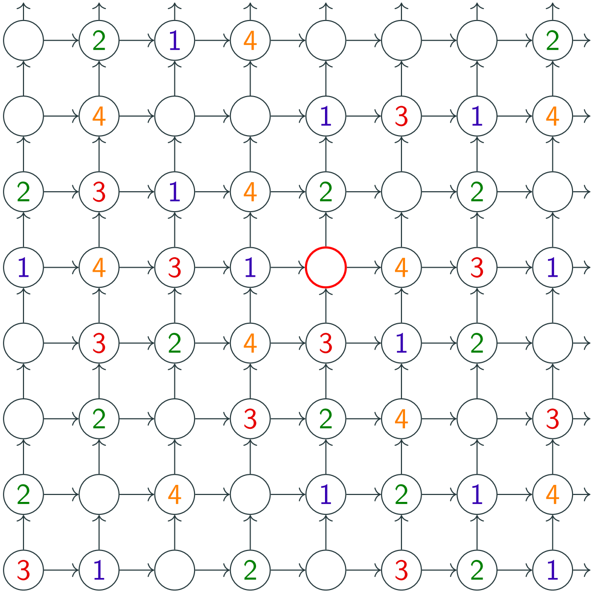
$\Gamma^* : V \to \mathcal{O} \cup \{\perp\}$ is a **Partial Solution** if :

- $\mathcal{O}$ is the Output Set,
- $\forall u \in V : \Gamma^*(u) \neq \perp \Rightarrow$ we can complete the labels of the neighbors of u.

A problem is T-**Mendable** if, from any partial solution Γ^* and any $v \in V$ such that $\Gamma^*(v) = \perp$, there exists Γ' :

- $\Gamma'(v) \neq \perp$
- $\forall u \neq v,\ \Gamma'(u) = \perp \Leftrightarrow \Gamma^*(u) = \perp$
- $\forall u \in V,\ dist(u, v) > T \Rightarrow \Gamma'(u) = \Gamma^*(u)$

4-coloring the Grid



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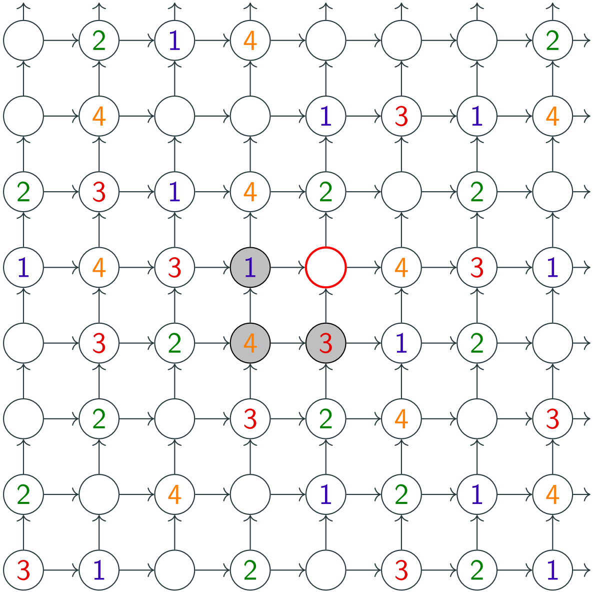

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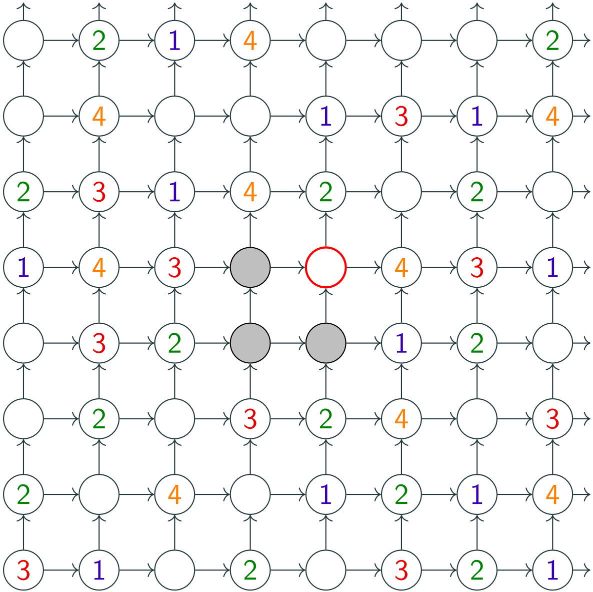

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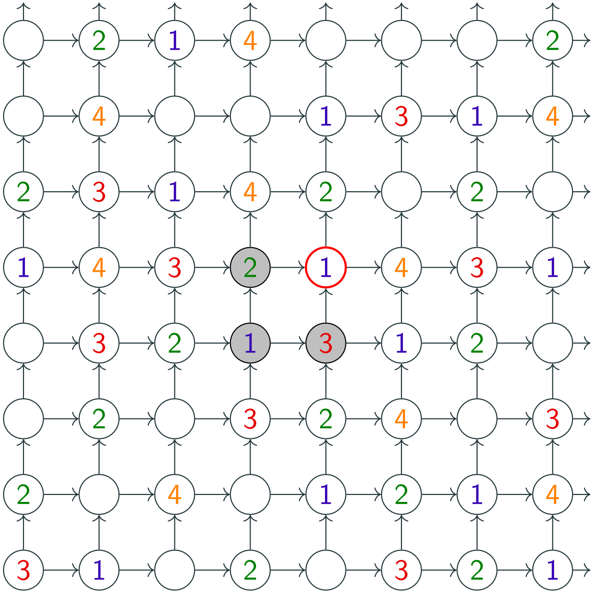

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Balliu *et. al* (2022)

Let Π be a T-mendable LCL problem. Π can be solved in rounds in the LOCAL model if we are given a distance-$(2T+1)$ coloring.

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Balliu *et. al* (2022)
Let Π be a $O(1)$-mendable LCL problem. Π can be solved in $O\left(\log^* n\right)$ rounds in the LOCAL model on bounded degree graphs.

From $\log^* n$ to Mendability

On paths and cycles, are all $O(\log^* n)$ problems mendable?

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No : 3-color with $\{1, 2, 3\}$ or 2-color with $\{A, B\}$.

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Balliu *et. al* **(2022)**
Suppose Π is an LCL problem on directed cycles with no input. If Π is $O(\log^* n)$-solvable, we can define a new LCL problem Π' with the same round complexity, such that a solution for Π' is also a solution for Π, and Π' is $O(1)$-mendable.

The Case of Trees

Balliu *et. al* **(2022)**
In trees, there are exactly three classes : $O(1)$-mendable, $\Theta(\log n)$-mendable, and $\Theta(n)$-mendable problems.

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3-coloring the rooted tree is $O(n)$-mendable.
There exists a $O(1)$-mendable problem Π' that projects its solutions to a 3-coloring :

- A node is **monochromatic** if both its children have the same color.
- Otherwise, the node is **mixed**.
- Π' only accept connected components of mixed nodes of height $\leq k$.

Landscape of LOCAL Variants

LOCAL Variants landscape

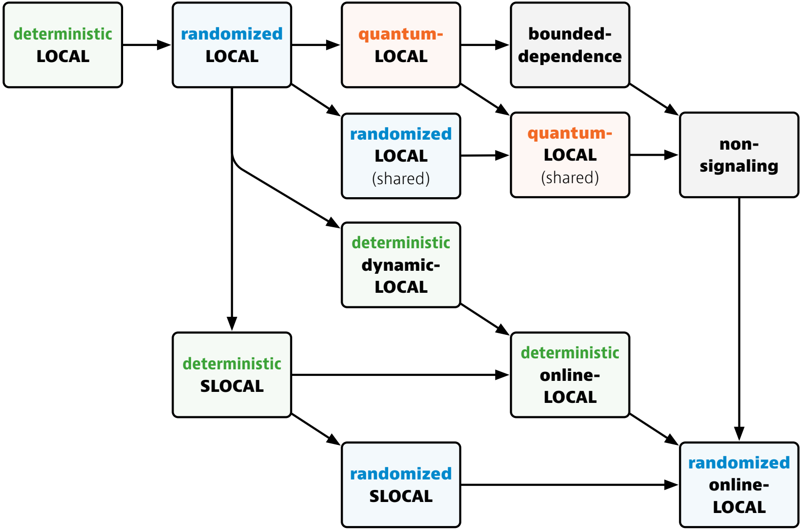


Figure from https ://jukkasuomela.fi/doc/meascomb-2024-06-27.pdf

LOCAL Variants landscape

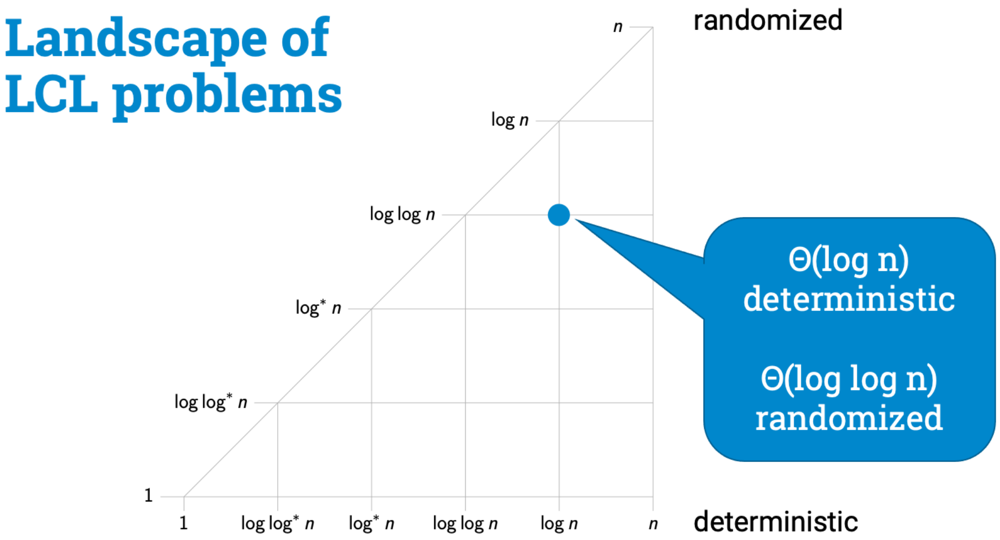
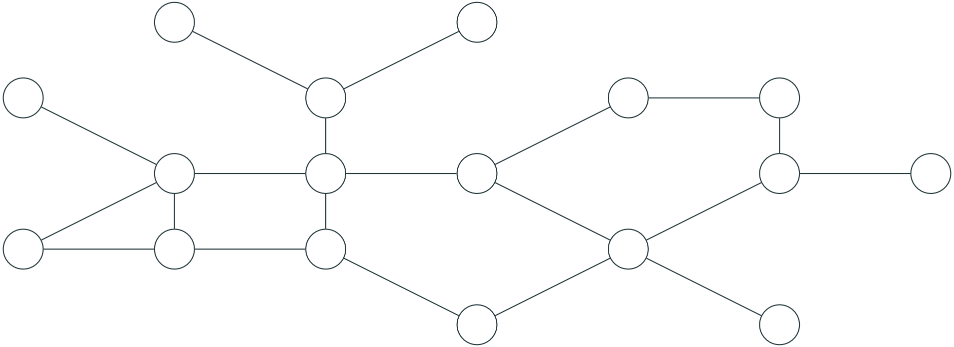


Figure from https ://jukkasuomela.fi/landscape-of-locality/

SLOCAL Model

- Each node is activated one after another, to compute its own output
- A node has access to the outputs already computed to produce its own
- Complexity : maximal radius needed among nodes



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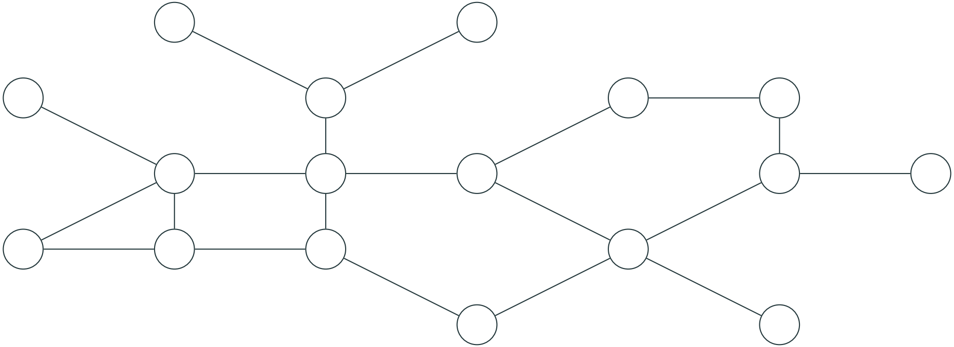

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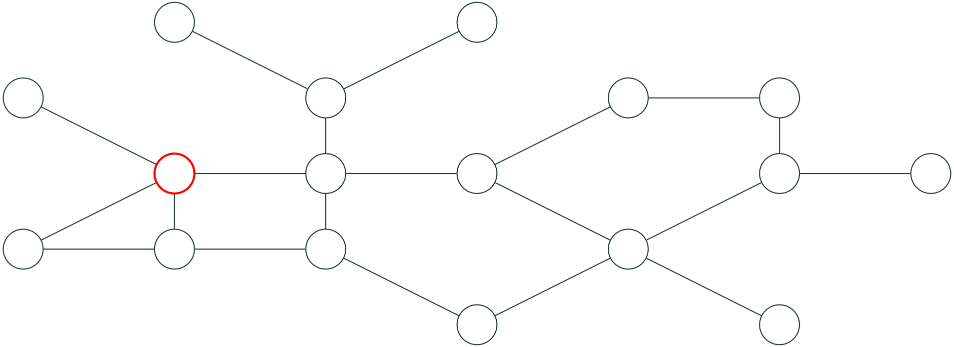

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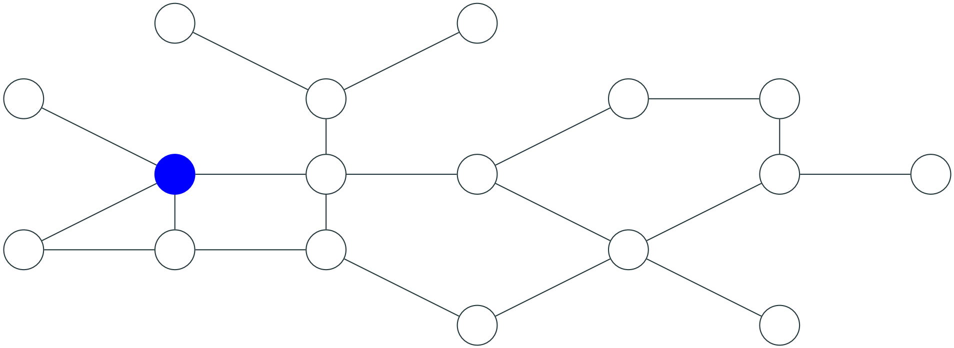

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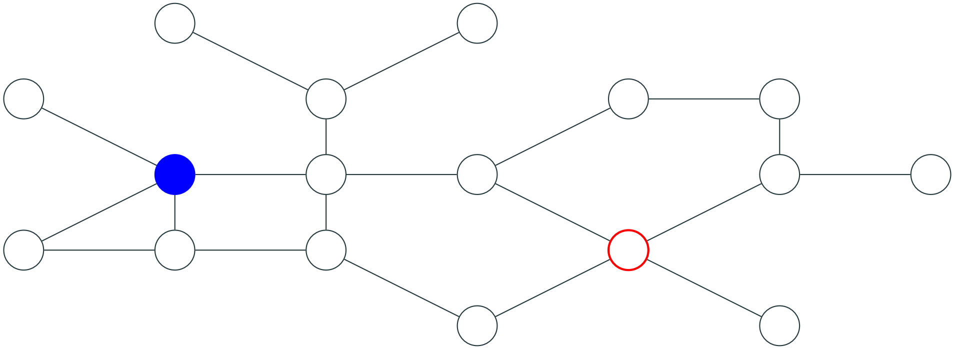

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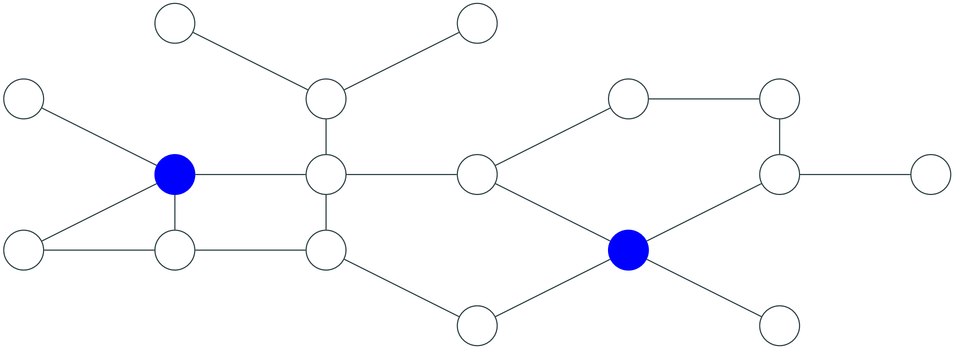

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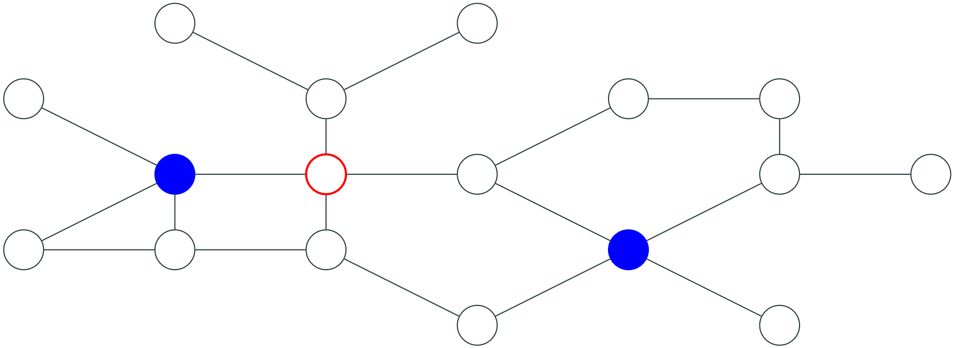

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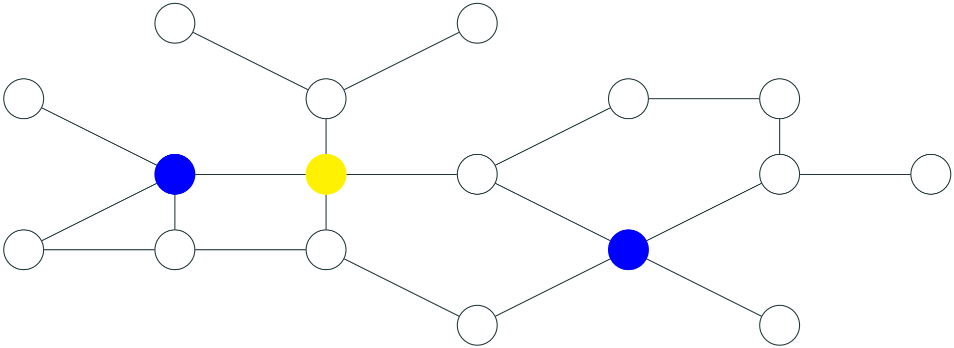

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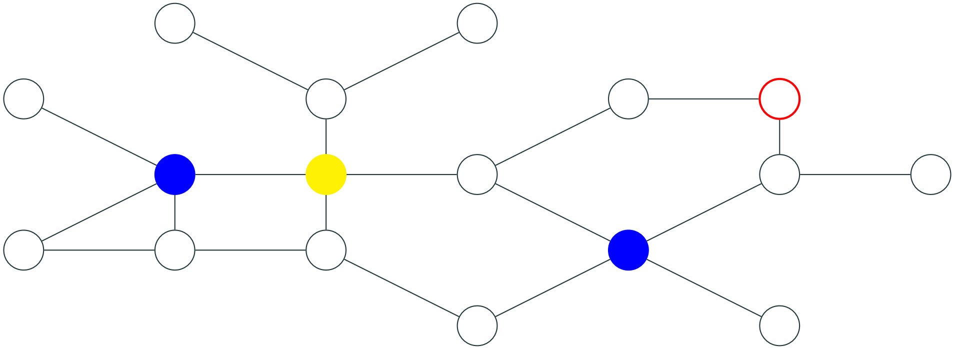

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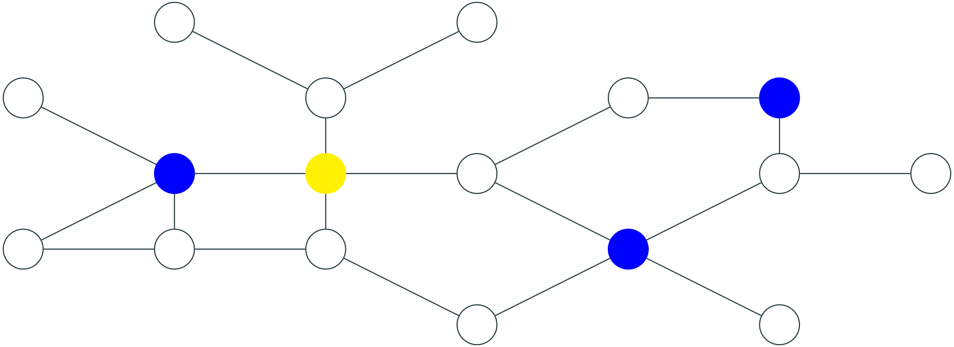

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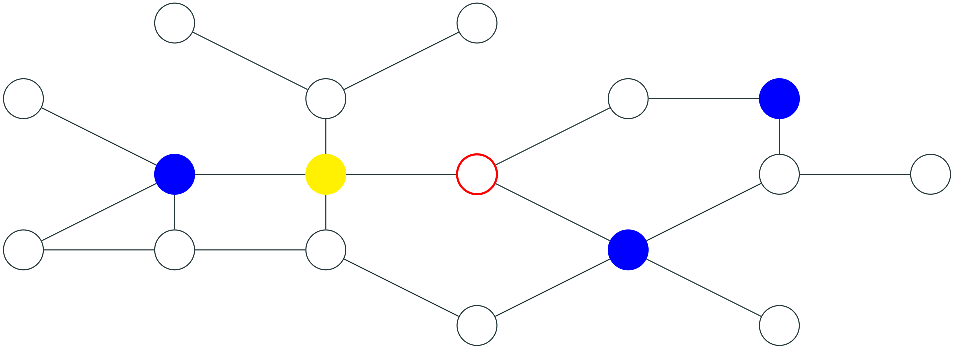

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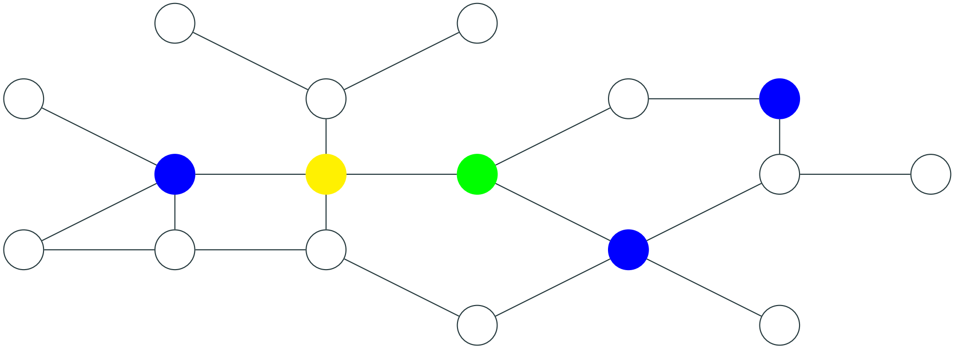

The Complexities of Sinkless Orientation

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Sinkless Orientation has complexity in the **deterministic** LOCAL model.

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Idea of the algorithm :

- At distance at most $\log n$, we have :
  1. a node of degree ≤ 2
  2. a cycle of length $\leq 2 \log n$
- Case (1) : orient your edge towards the closest one
- Case (2) : pick the smallest one (in size, and then according to identifiers)
  Orient this cycle from smallest identifier to smaller neighbor
- Otherwise, orient your edge towards the closest cycle
- All nodes are now satisfied, orient other edges arbitrarily

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Brandt *et. al* **(2016), Mohsen Ghaffari, Hsin-Hao Su (2017)**
Sinkless Orientation has complexity $\Theta(\log_\Delta \log n)$ in the **randomized** LOCAL model.

Balliu *et. al* **(2023)**
Sinkless Orientation has complexity $O(\log \log n)$ in the **deterministic** SLOCAL model.

Randomized Sinkless Orientation

General idea :

- Compute a random solution
- Remove outputs that do not work
- Shattering of the graph (i.e. small connected components of unsatisfied nodes)
- Run deteministic algorithm on small components

Randomized Sinkless Orientation

We assume a Δ-regular graph with $\Delta > 500$

1. Each edge, with probability $1/4$, decides to orient itself (or not)
2. Identify bad nodes :
   - Type I : More than $\Delta/2$ incident oriented edges
   - Type II : Not type I and neighbors of a Type I node
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3. Unorient edges incident to Type I nodes

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Observations :

- Good nodes have an outgoing edge
- (Admitted) WHP, each connected component of bad nodes have diameter $\operatorname{poly}\log n$

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4. Use the deterministic algorithm on the bad nodes in $O(\log \log n)$

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Composition in the SLOCAL model

Balliu *et. al* **(2023)**
Let $\mathcal{A}$ and $\mathcal{B}$ be SLOCAL algorithms with respective localities $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$, and let $\mathcal{B}$ depends on the output of $\mathcal{A}$. Then there exists an algorithm $\mathcal{C}$ with locality $T_{\mathcal{A}} + 2T_{\mathcal{B}}$ that solves the same problem as $T_{\mathcal{B}}$ without dependency on the output of $T_{\mathcal{A}}$.

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Proof :

- To compute $\mathcal{B}(v)$, we need output of $\mathcal{A}$ on $N_{T_{\mathcal{A}}}(v)$
- $\mathcal{C}$ stores the outputs of $\mathcal{A}$ pre-computed by each u for their neighborhood on u
- $\mathcal{C}$ needs to look at distance $T_{\mathcal{B}}$ of $N_{T_{\mathcal{B}}}(v)$ to check for pre-computed outputs

SLOCAL Sinkless Orientation of High Degree Nodes

Algorithm that processes edges sequentially :

1. u is satisfied if it has an outgoing edge or has degree $\leq \log n + 1$
2. For $uv \in E$:
   - If u or v is satisfied, orient edge to the satisfied node
   - Otherwise, orient towards the node with the fewest processed edges
     This edge gets marked
3. At the end, all nodes are satisfied

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Observations :

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- An unsatisfied node with b marked edges towards itself is in a component of size $\geq 2^b$

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When edge $v \to u$ is added, indegree of u is $b + 1$ and indegree of v is $\geq b$

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When edge $v \to u$ is added, indegree of u is $b + 1$ and indegree of v is $\geq b$
If u and v where in the same component, there is a cycle of marked edges

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3. At the end, all nodes are satisfied

Observations :

- At each step, we consider the connected components using marked edges
- An unsatisfied node with b marked edges towards itself is in a component of size $\geq 2^b$

Proof : By induction on b. For $b = 0$, it is true.
When edge $v \to u$ is added, indegree of u is $b + 1$ and indegree of v is $\geq b$
If u and v where in the same component, there is a cycle of marked edges
In that cycle, u has two ingoing edges
$\Rightarrow$ The cycle has a node with two ongoing edges
$\Rightarrow$ One of the edges of the cycle could not have been marked

SLOCAL Sinkless Orientation

Three steps algorithm :

1. Compute a MIS I of G^{2T+1} with $T = \log(\log n + 1)$
   - Partition into clusters : each node u selects closest element $v \in I$ and join C_v
   - uv inter-cluster edge if u and v in different clusters
   - uv intra-cluster edge if u and v in same cluster
   - G_C cluster graph, with $C_u C_v \in E_C$ if inter-cluster edge between C_u and C_v
   - One round in G_C is $O(T)$ rounds in G

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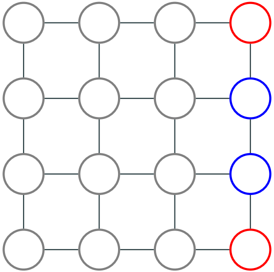
2. Compute SLOCAL Sinkless Orientation of High Degree Nodes on G_C
   - High degree clusters (at least $\log n + 1$ neighbors) has an outgoing edge
   - Low degree clusters have a node of degree ≤ 2 or a cycle
3. Depending on the cluster :
   - High degree clusters compute a spanning tree toward a non-sink node of the cluster.
   - Low degree clusters orient toward low degree node or cycle

Shared Randomness Separation

Carrying input problem :

- Nodes form an oriented grid of size $\sqrt{n} \times \sqrt{n}$
- Nodes on the right have input 0 or 1
- Each node must copy the output on its right
- At least one of the right nodes must have input=output



The complexities :

- Deterministic :
- Randomized :
- Shared Randomness :

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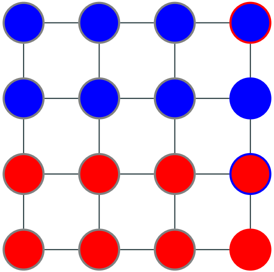

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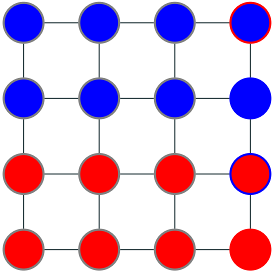

The complexities :

- Deterministic : $\Theta(\sqrt{n})$
- Randomized : $\Theta(\sqrt{n})$
- Shared Randomness : $\Theta(\sqrt{n})$

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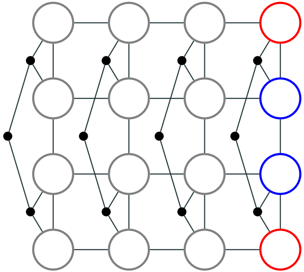
- Binary tree on each column



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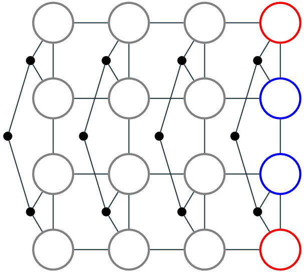

The complexities :

- Deterministic : $\Theta(\sqrt{n})$
- Randomized : $\Theta(\sqrt{n})$
- Shared Randomness : $\Theta(\log n)$

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