Model Definition
From LOCAL to Port-Numbering

...... 37  52  8  32  46  47  73  ......
From LOCAL to Port-Numbering
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......

37 - 52 - 8 - 32 - 46 - 47 - 73 - ......
From LOCAL to Port-Numbering
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…… 37 52 8 32 46 47 73 ……
From LOCAL to Port-Numbering
Communication

0 communication round
0 communication round
1 communication round
1 communication round
A **Port-Numbered Network** is a triple \((V, P, p)\):

- \(V\) is the set of nodes
- \(P\) is the set of ports. \(P \subseteq V \times \mathbb{N}\)
- \(p : P \to P\) is the function that connects ports.
  \(p(u, i)\) gives the node \(v\) to which \(u\) is connected, and to which port of \(v\) \(u\) is connected.
  We have \(p(p(u, i)) = (u, i)\).
  
  *Edges can be deduced from \(p\).*
Formal Definition - Algorithm

- **States**: \( S \) (not necessarily finite)
- **Input**: \( I \) (\( I \) might be a singleton if we do not provide inputs)
- **Output**: \( O \subseteq S \)
  - Can be directly vertex output.
  - Can be edge output (gives a mapping of outputs to each port of the vertex).
- **Messages**: \( M \)
- **Execution Functions** (depending on degree \( d \) of the node):
  - \( init_d : I \rightarrow S \)
  - \( send_d : S \rightarrow M^d \) (sends message \( i \) through port \( i \))
  - \( receive_d : S \times M^d \rightarrow S \) (receives the \( d \) messages from the ports and updates the state)
    \( \forall o \in O, \text{ we have } receive_d(o, m) = o. \)
- **Algorithm**: \( (I, S, O, M, (init_d)_{d \in \mathbb{N}}, (send_d)_{d \in \mathbb{N}}, (receive_d)_{d \in \mathbb{N}}) \)
Formal Definition - Distributed Graph Problems

From our network triple \((V, P, p)\):

- **Initialization**: \(f : V \rightarrow I\) (pre-labelling of the nodes)
- **Configuration**: \(x : V \rightarrow S\)
- **Transition**: \(x_k \rightarrow x_{k+1}\)
  
  For each \(v \in V\) of degree \(d\), if \(p(v, i) = (u, j)\),
  
  \(m_i\) corresponds to the \(j^{th}\) element of \(send_d(x_k(u))\),
  
  we have \(x_{k+1}(v) = receive_d(x_k(v), m_1, \ldots, m_d)\).

- **Execution**: \(x_0, x_1, \ldots\) such that \(x_0(v) = init_d(f(v))\) and for each \(k\), \(x_k \rightarrow x_{k+1}\)

- **End** of execution: first \(k\) such that \(\forall v \in V, x_k(v) \in O\)
Formal Definition - Problem Solving

- **Distributed Graph Problems**: \( \Pi \) such as, for a PN Network \( N = (V, P, p) \), \( \Pi(N) \) is the set of accepted labellings.

- **Solution**: \( f : V \to O \) such as \( f \in \Pi(N) \)

- **Algorithm A Solves** \( \Pi \) from input \( \Pi' \) on \( N \) in time \( k \) if
  
  For any \( f_{init} \in \Pi' \), we have a sequence \( x_0, \ldots, x'_k \)

  starting from \( f_{init} \) with \( k' \leq k \) and \( x'_k \in \Pi \).

- **Algorithm A Solves** \( \Pi \) from input \( \Pi' \) on family \( \mathcal{F} \) of graphs in time \( T : \mathbb{N} \to \mathbb{N} \) if
  
  For any PN Network \( N = (V, P, p) \) representing a graph \( G \in \mathcal{F} \),

  \( A \) solves \( \Pi \) from input \( \Pi' \) on \( N \) in time \( T(|V|) \)
3 Algorithms
3-coloring a path

- $\tilde{\mathcal{F}}$: Paths.
- $\Pi$: 3-colored paths.
- $\Pi'$: proper coloring of the path.

![Diagram of a path with colors]
3-coloring a path

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- $\Pi'$: proper coloring of the path.
3-coloring a path

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Idea: Local maximums choose the smallest possible color.
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3-coloring Algorithm

- \( S = \mathbb{N}^+ \)
- \( I = \mathbb{N}^+ \)
- \( O = \{1, 2, 3\} \)
- \( M = \mathbb{N}^+ \)
- \( init_d(x) = x, \ \forall d \leq 2 \)
- \( send_d(x) = x^d, \ \forall d \leq 2 \)
- \( receive_d(x, Y) = \min(\mathbb{N} \setminus (Y \cup \{x\})) \) if \( x \geq 4 \) and \( x > Y \), \( receive_d(x, Y) = x \) otherwise.
- Complexity?
3-coloring Algorithm

- $S = \mathbb{N}^+$
- $I = \mathbb{N}^+$
- $O = \{1, 2, 3\}$
- $M = \mathbb{N}^+$
- $init_d(x) = x, \forall d \leq 2$
- $send_d(x) = x^d, \forall d \leq 2$
- $receive_d(x, Y) = \min(\mathbb{N} \setminus (Y \cup \{x\}))$ if $x \geq 4$ and $x > Y$, $receive_d(x, Y) = x$ otherwise.
- Complexity: $O(n)$
Maximal Matching on Bipartite Graphs

- $\mathcal{G}$: Bipartite Graphs.
- $\Pi$: Maximal Matching.
- $\Pi'$: Black/White-coloring.

![Diagram of Bipartite Graphs with Maximal Matching and Black/White-coloring]
Maximal Matching Algorithm
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Vertex Cover 3-approximation

Vertex Cover: \( C \subseteq V \) such that each edge \( e = \{u, v\} \in V^2 \) has at least one endpoint in \( C \).

- \( \mathcal{G} \): Any graphs.
- \( \Pi \): 3-approximation of Vertex-Cover.
- \( \Pi' \): Nothing.
Virtual Bipartite Black/White Graph

Graph
Virtual Bipartite Black/White Graph

Graph \Rightarrow Duplicate Nodes
Virtual Bipartite Black/White Graph

Graph ⇒ Duplicate Nodes ⇒ Find a Maximal Matching
Virtual Bipartite Black/White Graph

Graph ⇒ Duplicate Nodes ⇒ Find a Maximal Matching ⇒ Take Matched Nodes
Analysis

- **C is a Vertex Cover**:
  - In the virtual graph, each edge touches an edge of the Maximal Matching
  - In the virtual graph, each edge touches a node in C
  - In the graph, each edge touches a node in C

- **It is a 3-approximation of any Vertex Cover C̄**:
  - The Maximal Matching in the virtual graph forms cycles and paths in the graph
  - C̄ is a Vertex Cover of those cycles and paths
  - Any Vertex Cover of a cycle uses at least half of the nodes
  - Any Vertex Cover of a path uses at least a third of the nodes
Impossibility
k-Coloring a path

- \( \mathcal{F} \): Paths.
- \( \Pi \): \( k \)-colored paths.
- \( \Pi' \): Nothing.
k-Coloring a path

- $\emptyset$ : Paths.
- $\Pi$ : $k$-colored paths.
- $\Pi'$ : Nothing.
Maximal Matching a cycle

- \( \mathcal{F} \) : Cycles.
- \( \Pi \) : Maximal Matching.
- \( \Pi' \) : Nothing.
Maximal Matching a cycle

- $\mathcal{F}$: Cycles.
- $\Pi$: Maximal Matching.
- $\Pi'$: Nothing.

![Diagram](attachment://diagram.png)
Tools to break symmetry:

- Identifiers (LOCAL Model)
- Randomness (Simulation of LOCAL model)
- Inputs
- ...
Design an algorithm for the Maximal Matching in a $k$-colored graph.
Bibliography

- Jukka Suomela’s courses (https://jukkasuomela.fi/da2020/da2020-03.pdf)