Distributed Computing
2 - Port Numbering Model

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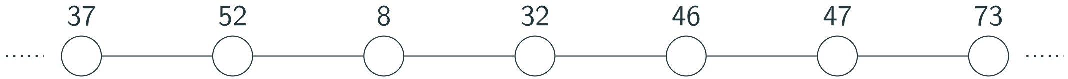
Model Definition

From LOCAL to Port-Numbering



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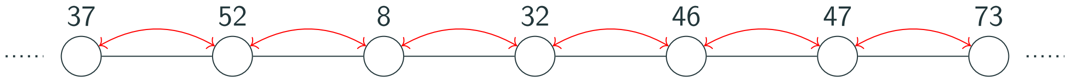

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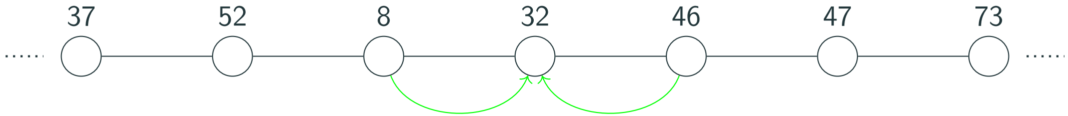

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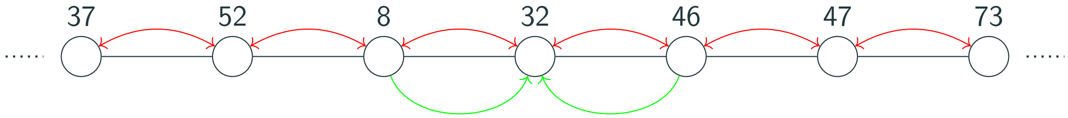

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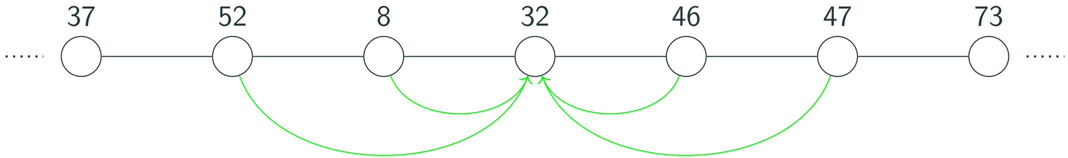

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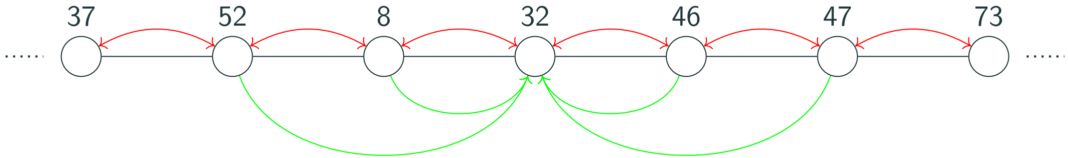

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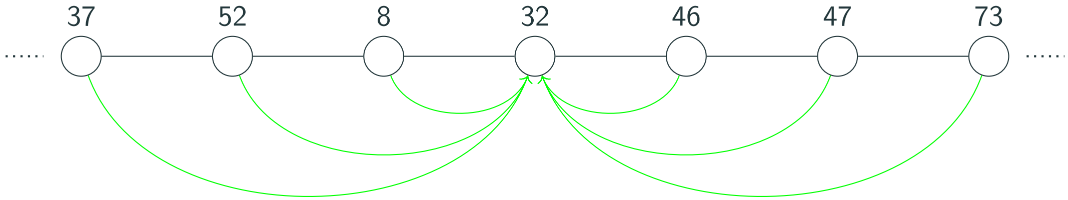

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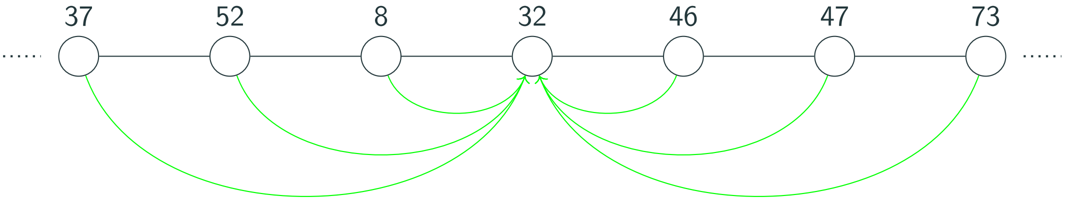
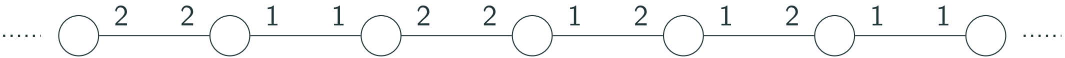
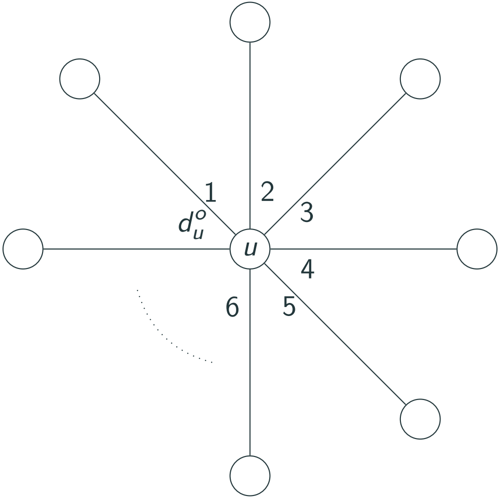




Communication

0 communication round



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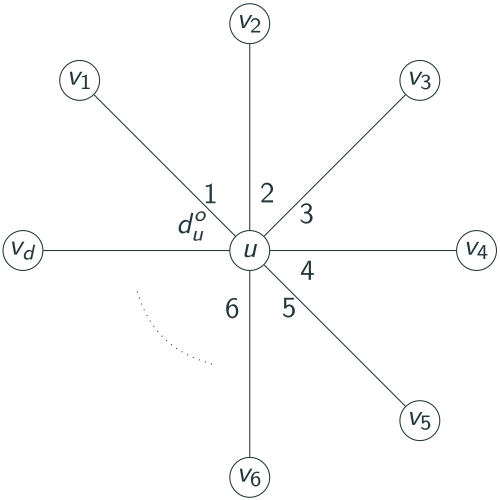
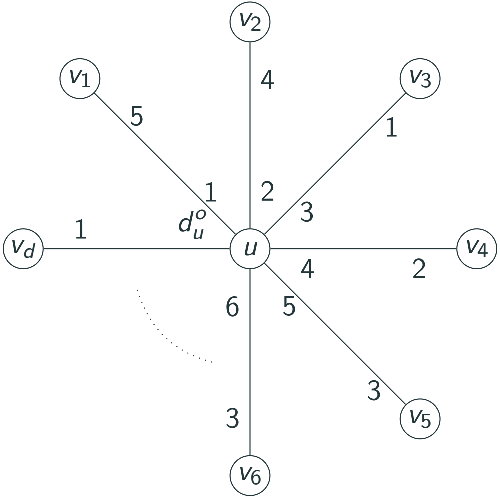


Communication

1 communication round



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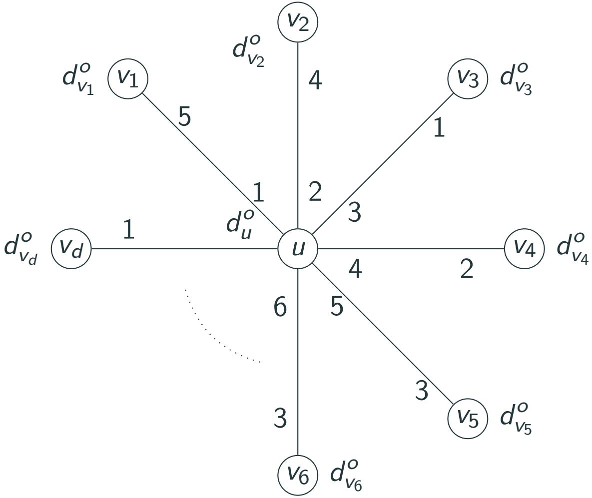

Formal Definition - PN

A **Port-Numbered Network** is a triple (V, P, p) :

- V is the set of nodes
- P is the set of ports. $P \subseteq V \times \mathbb{N}$
- $p : P \to P$ is the function that connects ports.
  $p(u, i)$ gives the node v to which u is connected, and to which port of v u is connected.
  We have $p(p(u, i)) = (u, i)$.

  Edges can be deduced from p.

Formal Definition - Algorithm

- **States** : S (not necessarily finite)
- **Input** : I (I might be a singleton if we do not provide inputs)
- **Output** : $O \subseteq S$
  - Can be directly vertex output.
  - Can be edge output (gives a mapping of outputs to each port of the vertex).
- **Messages** : M
- **Execution Functions** (depending on degree d of the node) :
  - $init_d : I \to S$
  - $send_d : S \to M^d$ (sends message i through port i)
  - $receive_d : S \times M^d \to S$ (receives the d messages from the ports and updates the state)
    $\forall o \in O$, we have $receive_d(o, m) = o$.
- **Algorithm** : $(I, S, O, M, (init_d)_{d\in\mathbb{N}}, (send_d)_{d\in\mathbb{N}}, (receive_d)_{d\in\mathbb{N}})$

Formal Definition - Distributed Graph Problems

From our network triple (V, P, p) :

- **Initialization** : $f : V \to I$ (pre-labelling of the nodes)
- **Configuration** : $x : V \to S$
- **Transition** : $x_k \to x_{k+1}$
  For each $v \in V$ of degree d, if $p(v, i) = (u, j)$,
  m_i corresponds to the jth element of $send_{d'}(x_k(u))$,
  we have $x_{k+1}(v) = receive_d(x_k(v), m_1, \ldots, m_d)$.
- **Execution** : $x_0, x_1, \ldots$ such that $x_0(v) = init_d(f(v))$ and for each k, $x_k \to x_{k+1}$
- **End** of execution : first k such that $\forall v \in V$, $x_k(v) \in O$

Formal Definition - Problem Solving

- **Distributed Graph Problems** : Π such as, for a PN Network $N = (V, P, p)$,
  $\Pi(N)$ is the set of accepted labellings.
- **Solution** : $f : V \to O$ such as $f \in \Pi(N)$
- Algorithm $\mathcal{A}$ **Solves** Π from input Π' on N in time k if
  For any $f_{init} \in \Pi'$, we have a sequence $x_0, \ldots, x'_k$
  starting from f_{init} with $k' \leq k$ and $x'_k \in \Pi$.
- Algorithm $\mathcal{A}$ **Solves** Π from input Π' on family $\mathfrak{F}$ of graphs in time $T : \mathbb{N} \to \mathbb{N}$ if
  For any PN Network $N = (V, P, p)$ representing a graph $G \in \mathfrak{F}$,
  $\mathcal{A}$ solves Π from input Π' on N in time $T(|V|)$

3 Algorithms

3-coloring a path

- $\mathfrak{F}$: Paths.
- Π : 3-colored paths.
- Π' : proper coloring of the path.

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Idea : Local maximums choose the smallest possible color.

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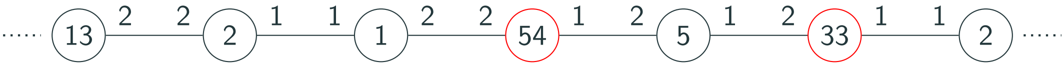

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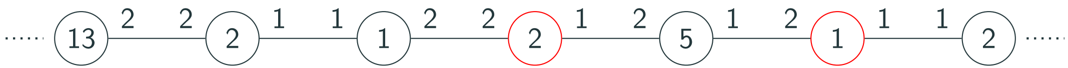

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3-coloring Algorithm

- $S = \mathbb{N}^+$
- $I = \mathbb{N}^+$
- $O = \{1, 2, 3\}$
- $M = \mathbb{N}^+$
- $init_d(x) = x$, $\forall d \leq 2$
- $send_d(x) = x^d$, $\forall d \leq 2$
- $receive_d(x, Y) = \min(\mathbb{N} \setminus (Y \cup \{x\}))$ if $x \geq 4$ and $x > Y$, $receive_d(x, Y) = x$ otherwise.

- Complexity ?

3-coloring Algorithm

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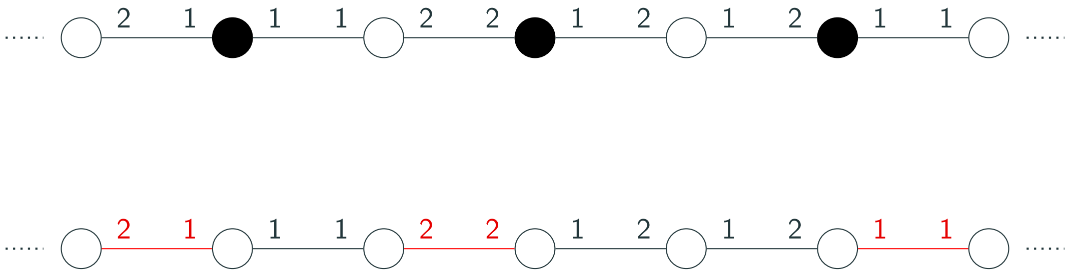
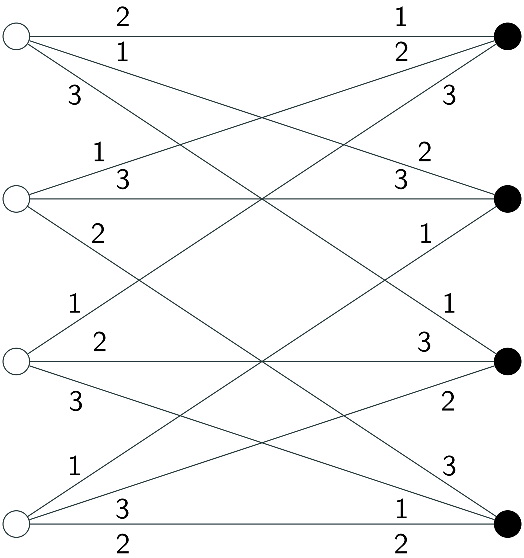
- Complexity : $O(n)$

Maximal Matching on Bipartite Graphs

- $\mathfrak{F}$: Bipartite Graphs.
- Π : Maximal Matching.
- Π' : Black/White-coloring.



Maximal Matching Algorithm



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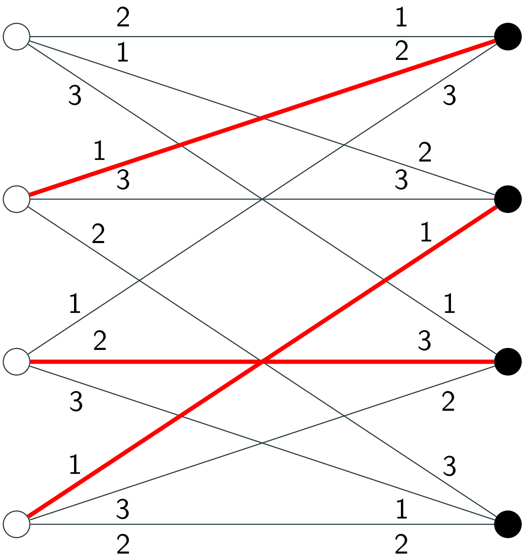

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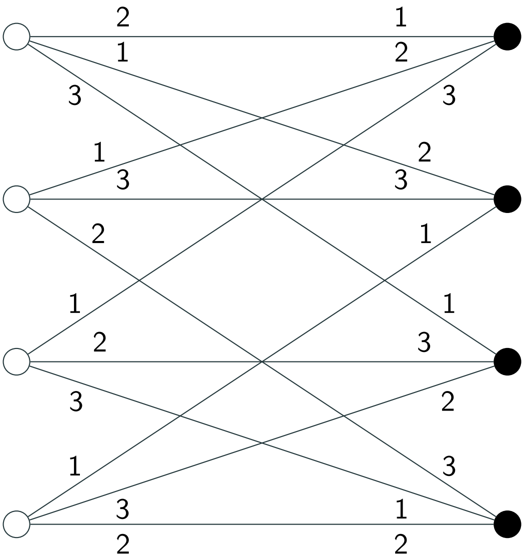

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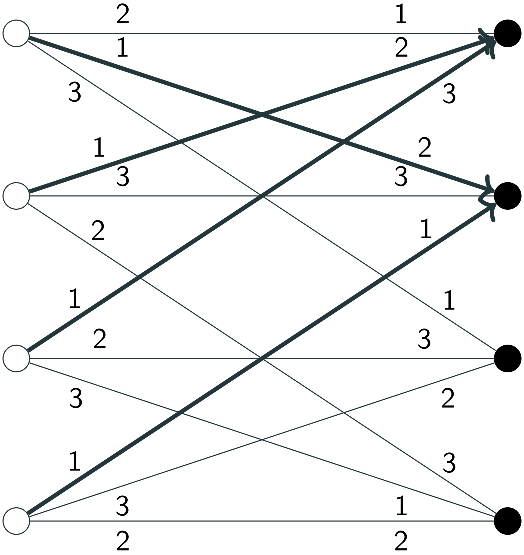

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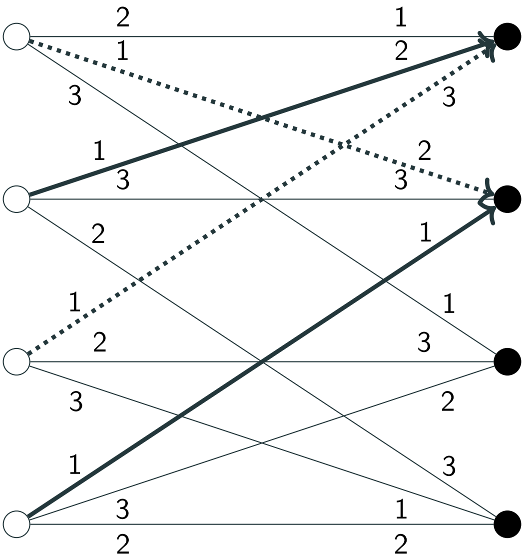

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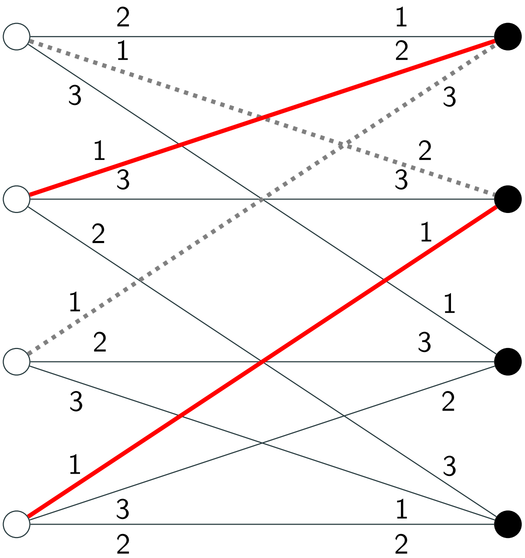

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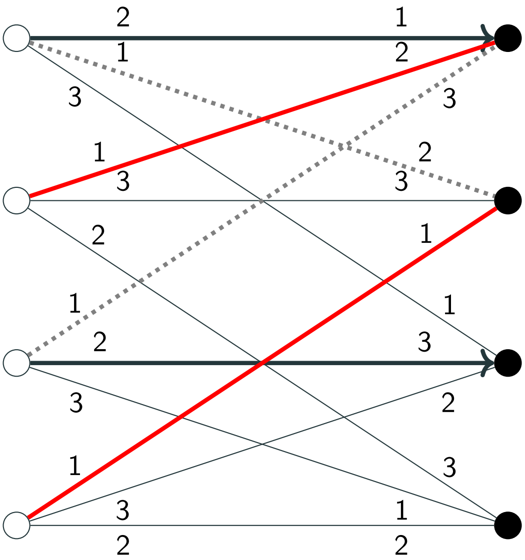

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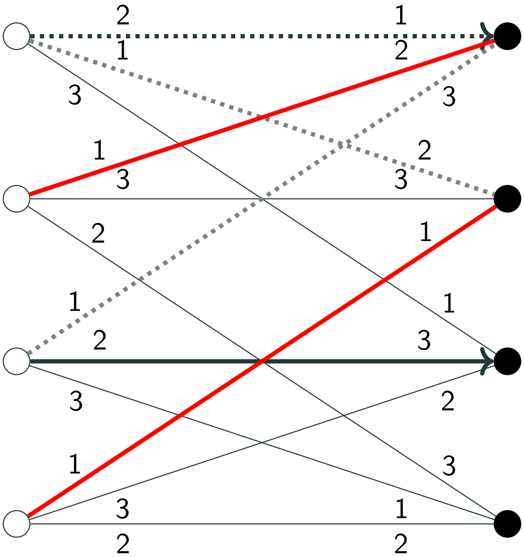

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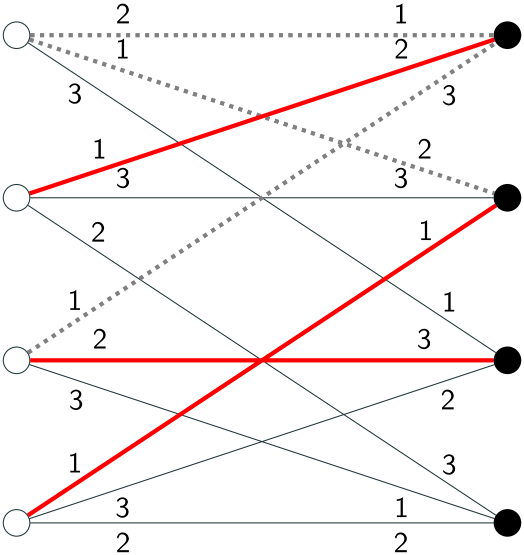

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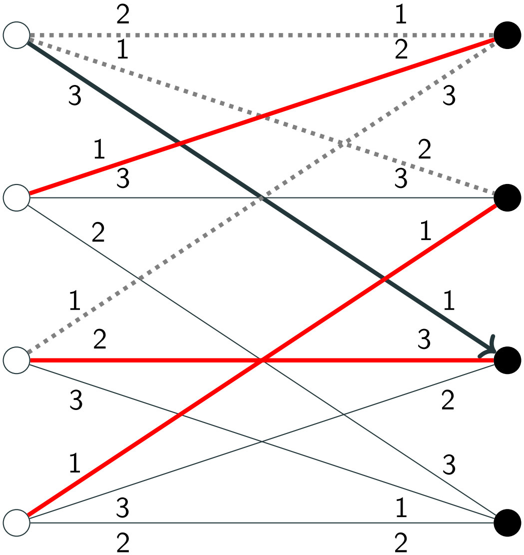

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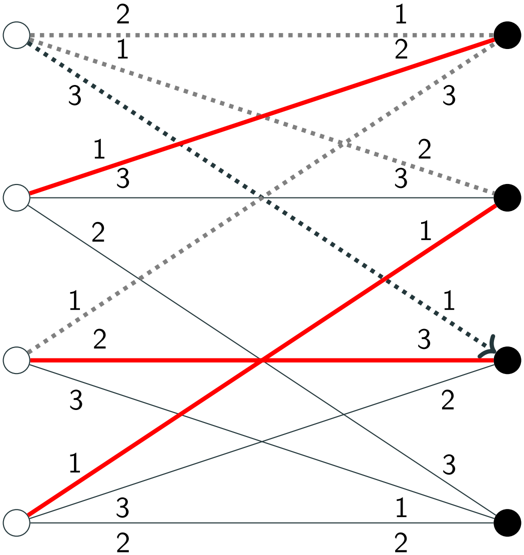

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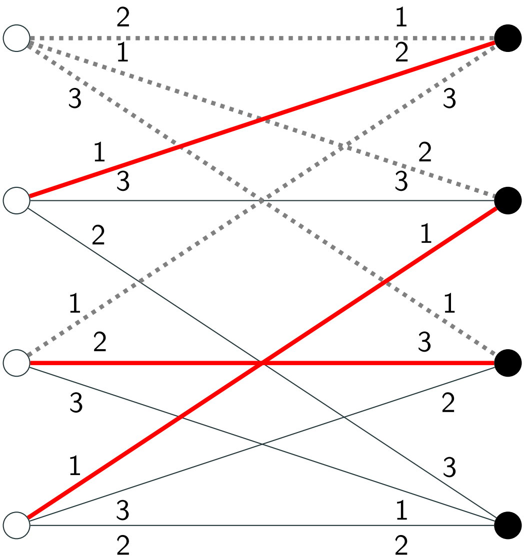
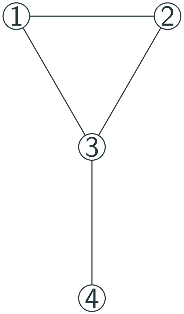


Vertex Cover 3-approximation

Vertex Cover : $C \subseteq V$ such that each edge $e = \{u, v\} \in V^2$ has at least one endpoint in C.

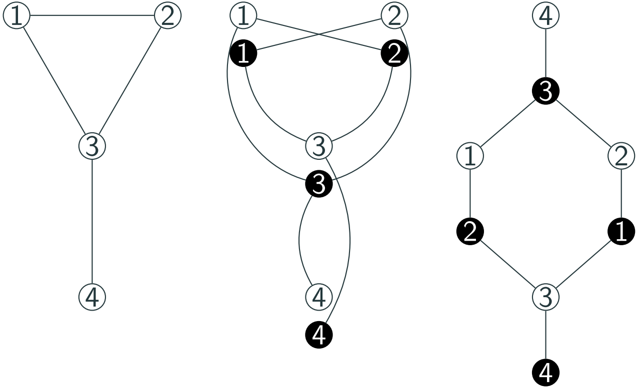
- $\mathfrak{F}$: Any graphs.
- Π : 3-approximation of Vertex-Cover.
- Π' : Nothing.

Virtual Bipartite Black/White Graph



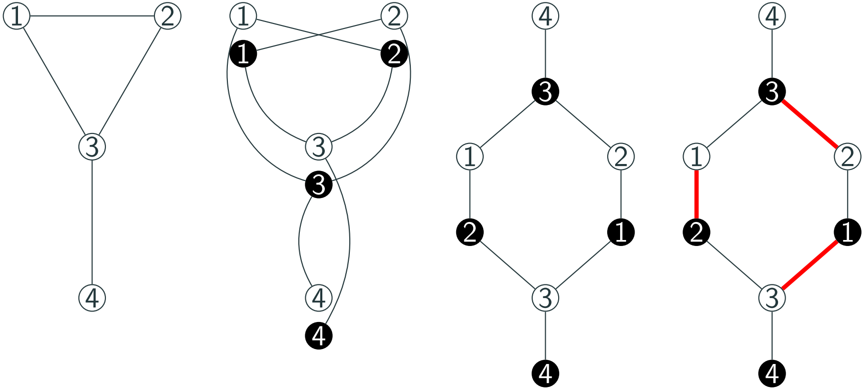
Graph

Virtual Bipartite Black/White Graph



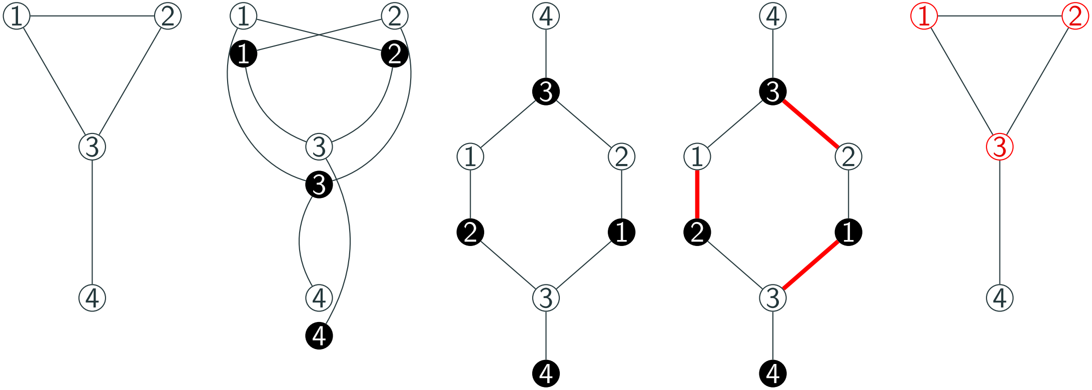
Graph ⇒ Duplicate Nodes

Virtual Bipartite Black/White Graph



Graph ⇒ Duplicate Nodes ⇒ Find a Maximal Matching

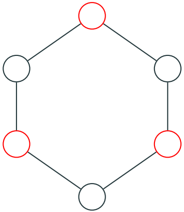
Virtual Bipartite Black/White Graph

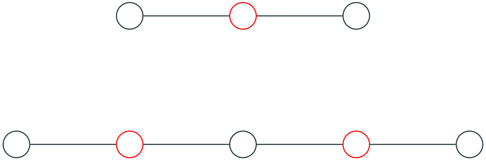


Graph ⇒ Duplicate Nodes ⇒ Find a Maximal Matching ⇒ Take Matched Nodes

Analysis

- C is a Vertex Cover :
  - In the virtual graph, each edge touches an edge of the Maximal Matching
  - In the virtual graph, each edge touches a node in C
  - In the graph, each edge touches a node in C
- It is a 3-approximation of any Vertex Cover C^* :
  - The Maximal Matching in the virtual graph forms cycles and paths in the graph
  - C^* is a Vertex Cover of those cycles and paths
  - Any Vertex Cover of a cycle uses at least half of the nodes
  - Any Vertex Cover of a path uses at least a third of the nodes





Impossibility

k-Coloring a path

- $\mathfrak{F}$: Paths.
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Maximal Matching a cycle

- $\mathfrak{F}$: Cycles.
- Π : Maximal Matching.
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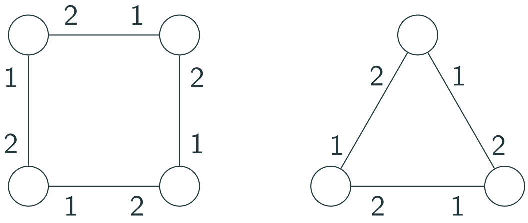

Break Symmetry

Tools to break symmetry :

- Identifiers (LOCAL Model)
- Randomness (Simulation of LOCAL model)
- Inputs
- ...

Exercise

Design an algorithm for the Maximal Matching in a k-colored graph.

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