Probabilistic Stable Functions on Discrete Cones are Power Series.

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Abstract

We study the category $\text{Cstab}_m$ of measurable cones and measurable stable functions—a denotational model of an higher-order language with continuous probabilities and full recursion [8]. We look at $\text{Cstab}_m$ as a model for discrete probabilities, by showing the existence of a full and faithful functor preserving cartesian closed structure which embeds probabilistic coherence spaces—a fully abstract denotational model of an higher language with full recursion and discrete probabilities [7]—into $\text{Cstab}_m$. The proof is based on a generalization of Bernstein’s theorem in real analysis allowing to see stable functions between discrete cones as generalized power series.

Keywords Lambda calculus, Probabilistic computation, Denotational semantics

1 Introduction

Probabilistic reasoning allows us to describe the behavior of systems with inherent uncertainty, or on which we have an incomplete knowledge. To handle statistical models, one can employ probabilistic programming languages: they give us tools to build, evaluate and transform them. While for some applications it is enough to consider discrete probabilities, we sometimes want to modelize systems where the underlying space of events has inherent continuous aspects: for instance in hybrid control systems [1], as used e.g. in flight management. In the machine learning community [10, 12], statistical models are also used to express our beliefs—bayesian inference—the ability to condition values of variables via observations.

As a consequence, several probabilistic continuous languages have been introduced and studied, such Church [11], Anglican [19], as well as formal operational semantics for them [2]. Giving a fully abstract denotational semantics to a higher-order probabilistic language with full recursion, however, has proved to be harder than in the non-probabilistic case. For discrete probabilities, there have been two such fully abstract models: in [5], Danos and Harmer introduced a fully abstract denotational semantics of a probabilistic extension of idealized Algol, based on game semantics; and in [4] Ehrhard, Pagani and Tasson showed that the category $\text{Pcoh}$ of probabilistic coherence spaces gives a fully abstract model for $\text{PCF}_\otimes$, a discrete probabilistic variant of Plotkin’s $\text{PCF}$.

While there is currently no known fully abstract denotational semantics for a higher-order language with full recursion and continuous probabilities, several denotational models have been introduced. The pioneering work of Kozen [13] gave a denotational semantics to a first-order while-language endowed with a random real number generator. In [18], Staton et al give a denotational semantics to an higher-order language: they first develop a distributive category based on measurable spaces as a model of the first-order fragment of their language, and then extend it into a cartesian closed category using a standard construction based on the functor category.

Recently, Ehrhard, Pagani and Tasson introduced in [8] the category $\text{Cstab}_m$, as a denotational model of an extension of $\text{PCF}$ with continuous probabilities. It is presented as a refinement with measurability constraints of the category $\text{Cstab}$ of abstract cones and so-called stable functions between cones, consisting in a generalization of absolutely monotonous functions from real analysis.

Here, we look at the category $\text{Cstab}_m$ from the point of view of discrete probabilities. It was noted in [8] that there is a natural way to see any probabilistic coherent space as an object of $\text{Cstab}$. In this work, we show that this connection leads to a full and faithful functor $\mathcal{F}$ from $\text{Pcoh}$—the Kleisli category of $\text{Pcoh}$—into $\text{Cstab}$. It is done by showing that every stable function between probabilistic coherent spaces can be seen as a power series, using the extension to an abstract setting of Bernstein’s theorem for absolutely monotonous functions shown by McMillan [15]. We then show that the functor $\mathcal{F}$ we have built is cartesian closed, i.e. respects the cartesian closed structure of $\text{Pcoh}$. In the last part, we turn $\mathcal{F}$ into a functor $\mathcal{F}^m : \text{Pcoh} \rightarrow \text{Cstab}_m$, and we show that $\mathcal{F}^m$ too is cartesian closed.

To sum up, the main contribution of this paper is to show that there is a cartesian closed full embedding from $\text{Pcoh}$ into $\text{Cstab}_m$. Since $\text{Pcoh}$ is known to be a fully abstract denotational model of $\text{PCF}_\otimes$, a corollary of this result is that $\text{Cstab}_m$ too is a fully abstract model of $\text{PCF}_\otimes$.

An extended version of this paper with more details is available online [3].

2 Discrete and Continuous Probabilistic Extension of PCF: an Overview.

A simple way to add probabilities to a (higher-order) programming language is to add a fair probabilistic choice operator to the syntax. Such an approach has been applied to various extensions of the $\lambda$-calculus [14]. To fix ideas, we give here the syntax of a (minimal) probabilistic variant of Plotkin’s $\text{PCF}$ [16], that we will call $\text{PCF}_\otimes$. It is a typed languages, whose types are given by: $A ::= N | A \rightarrow A$, where $N$ is the base type of naturals numbers. The programs are generated as follows:

$$M,N \in \text{PCF}_\otimes ::= x | \lambda A . M | (MN) | (YN) | \text{ifz } (M,N,L) | \text{let } (x,M,N) | M @ N | \text{pred } (M)$$

The operator $@$ is the fair probabilistic choice operator, $Y$ is a recursion operator, and $n$ ranges over natural numbers.

The ifz construct
tests if its first argument (of type $N$) is 0, reduces to its second argument if it is the case, and to its third otherwise. We endow this language with a natural operational semantics [7], that we choose to be call-by-name. However, for expressiveness we need to be able to simulate a call-by-value discipline on terms of ground types $N$: it is enabled by the let-construct.

We can see that the kind of probabilistic behavior captured by PCF is *discrete*, in the sense that it manipulates distributions on countable sets. In [4], Ehrhard and Danos introduced a model of Linear Logic designed to lead to denotational models for discrete higher-order probabilistic computation: the category $\text{PCF}_{\text{coh}}$ of probabilistic coherence spaces. It was indeed shown in [7] that $\text{PCF}_{\text{coh}}$, the Kleisli category of $\text{PCF}$ is a fully abstract model of $\text{PCF}$, while the Eilenberg-Moore Category of $\text{PCF}$ is a fully abstract model of a probabilistic variant of Levy’s call-by-push-value calculus.

We are going to illustrate here on examples the ideas behind the denotational semantics of $\text{PCF}_{\text{coh}}$ in $\text{PCF}$. The basic idea is that the denotation of a program consists of a vector on $\mathbb{R}^N$, where $X$ is the countable sets of possible outcomes. In fact, the denotation of the program $\lambda \alpha \mathbb{N} : (0 \oplus i \mathbb{Q}(x, 1/2))$, $\lambda X$ is the usual encoding of a never terminating term using recursion operator. The denotation of $M$ consists of the following function $\mathbb{R}^N \to \mathbb{R}^N$.

$$M = \lambda x^N : \left(0 \oplus i \mathbb{Q}(x, 1/2) i \mathbb{Q}(x, 0, \Omega)\right),$$

where $\Omega$ is the usual encoding of a never terminating term using recursion operator. The denotation of $M$ consists of the following function $\mathbb{R}^N \to \mathbb{R}^N$.

$$f(x)_k = \begin{cases} \frac{1}{2} + \frac{1}{2} \sum_{i \not= 0} x_i : x_0 & \text{if } k = 0 \\ \frac{1}{2} x_0 & \text{if } k = 1 \\ 0 & \text{if } k \not= \{0, 1\} \end{cases}$$

We can see that $f(x)_k$ corresponds indeed to the probability of obtaining $k$ if we pass to $M$ a term $N$ with $x$ as denotation. Observe that $f$ here is a polynomial in $x$; however since we have recursion in our language, there are programs that do an unbounded number of calls to their arguments: then their denotations are not polynomials anymore, but they are still analytic functions. The analytic nature of $\text{PCF}_{\text{coh}}$ morphisms plays actually a key role in the proof of full abstraction for $\text{PCF}_{\text{coh}}$.

Observe that this way of building a model for $\text{PCF}_{\text{coh}}$ is utterly dependent on the fact that we consider discrete probabilities over a countable sets of values. In recent years, however, there has been much focus on continuous probabilities in higher-order languages.

The aim is to be able to handle classical mathematical distributions on reals, as for instance normal or Gaussian distributions, that are widely used to build generic physical or statistical models, as well as transformations over these distributions.

We illustrate the basic idea here by presenting the language $\text{PCF}_{\text{sample}}$, following [8], that can be seen as the continuous counterpart to the discrete language $\text{PCF}_{\text{coh}}$. It is a typed language, with types generated as $A ::= R \mid A \to A$, and terms generated as follows:

$$M \in \text{PCF}_{\text{sample}} ::= x \mid \lambda x^A : M \mid (MN) \mid (YN) \mid i \mathbb{Q}(x, M, N, L) \mid \mathbb{Q} \mid \mathbb{N} \mid \mathbb{M}(M_1, \ldots, M_n)$$

where $r$ is any real number, and $f$ is in a fixed countable set of measurable function $\mathbb{R}^N \to \mathbb{R}$. The constant $\mathbb{N}$ stands for the uniform distribution over $[0, 1]$. Observe that admitting every measurable functions as primitive in the language allows to encode every distribution that can be obtained in a measurable way from the uniform distribution, for instance Gaussian or normal distributions. This language is actually expressive enough to simulate other probabilistic features, as for instance Bayesian conditioning, as highlighted in [8]. Moreover, we can argue it is also more general than $\text{PCF}_{\text{coh}}$: first it allows to encode integers (since $\mathbb{N} \subseteq \mathbb{R}$) and basic arithmetic operations over them. Secondly, since the orders operator $\geq : \mathbb{R} \times \mathbb{R} \to [0, 1] \subseteq \mathbb{R}$ is measurable, we can construct in $\text{PCF}_{\text{sample}}$ terms like this one:

$$i \mathbb{Q}(\geq \mathbb{Q}(\hat{\mathbb{N}}, 1/2), M, N),$$

which encodes a fair choice between $M$ and $N$.

We see however, that $\text{PCF}_{\text{coh}}$ cannot be a model for $\text{PCF}_{\text{sample}}$: indeed it doesn’t even seem possible to write a probabilistic coherence space for the real type. In [8], Ehrhard, Pagani and Tasson introduced the cartesian closed category $\text{Cstab}_{\mathbb{N}}$ of measurable cones and measurable stable functions, and showed that it provides an adequate and sound denotational model for $\text{PCF}_{\text{sample}}$. The denotation of the base type $R$ is taken as the set of finite measures over reals, and the denotation of higher-order types is then built naturally using the cartesian closed structure. From there, it is natural to ask ourselves: *how good $\text{Cstab}_{\mathbb{N}}$ is as a model of probabilistic higher-order languages?*

The present paper is devoted to give a partial answer to this question: in the case where we restrict ourselves to a *discrete fragment* of $\text{PCF}_{\text{sample}}$. To make more precise what we mean, let us consider a continuous language with an *explicit* discrete fragment which has both $R$ and $\mathbb{N}$ as base types: we consider the language $\text{PCF}_{\text{coh}, \text{sample}}$ with all syntactic constructs of both $\text{PCF}_{\text{coh}}$ and $\text{PCF}_{\text{sample}}$, as well as an operator $\mathbb{real}$ with the typing rule:

$$\Gamma \vdash M : N$$

$$\Gamma \vdash \mathbb{real}(M) : R$$

designed to enable the continuous constructs to act on the discrete fragment, by giving a way to see any distribution on $\mathbb{N}$ as a distribution on $R$. We see that we can indeed extend in a natural way the denotational semantics of $\text{PCF}_{\text{sample}}$ given in [8] to $\text{PCF}_{\text{coh}, \text{sample}}$: in the same way that the denotational semantics of $R$ is taken as the set of all finite measures on $\mathbb{R}$, we take the denotational semantics of $\mathbb{N}$ as the set $\mathbb{M}(\mathbb{N})$ of all finite measures over $\mathbb{N}$. We take as denotational semantics of the operator $\mathbb{real}$ the function:

$$\mathbb{real}_{\text{Cstab}_{\mathbb{N}}} : \mu \in \mathbb{M}(\mathbb{N}) \mapsto \left( A \in \mathbb{M}(\mathbb{R}) \mapsto \sum_{n \in A} \mu(n) \right) \in \mathbb{M}(\mathbb{R}).$$

We will see later that this function is indeed a morphism in $\text{Cstab}_{\mathbb{N}} (\mathbb{M}(\mathbb{N}), \mathbb{M}(\mathbb{R})).$ What we would like to know is: what is the structure of the sub-category of $\text{Cstab}_{\mathbb{N}}$ given by the *discrete types* of $\text{PCF}_{\text{sample}, \text{coh}}$, i.e. the one generated inductively by $\mathbb{N}$ in $\text{Cstab}_{\mathbb{N}}$, and $\times$?

The starting point of our work is the connection highlighted in [8] between PCSS and complete cones: every PCSS can be seen as a complete cone, in such a way that the denotational semantics of $N$ in $\text{PCF}_{\text{coh}}$ becomes the set of finite measures over $\mathbb{N}$. We formalize this connection by a functor $\mathbb{PM} : \text{PCF}_{\text{coh}} \to \text{Cstab}_{\mathbb{N}}$. However, to
be able to use $\text{Pcoh}_n$ to obtain information about the discrete types sub-category of $\text{Cstab}_m$, we need to know whether this connection is preserved at higher-order types: does the $\Rightarrow$ construct in $\text{Cstab}_m$ make some wild functions not representable in $\text{Pcoh}_n$ to appear, e.g. not analytic? The main technical part of this paper consists in showing that this is not the case, meaning that the functor $F^m$ is full and faithful, and cartesian closed. It tells us that the discrete types sub-category of $\text{Cstab}_m$ has actually the same structure as the subcategory of $\text{Pcoh}_n$ generated by $\{\mathbb{N}\}_{\text{Pcoh}_n} \Rightarrow \text{x} \wedge \text{x}$. Since $\text{Pcoh}_n$ is a fully abstract model of $PCF_\omega$, it tells us that the discrete fragment of $PCF_{\text{sample}, \omega}$ is fully abstract in $\text{Cstab}_m$.

3 Cones ans Stable Functions

The category of measurable cones and measurable, stable functions ($\text{Cstab}_m$), was introduced by Ehrhard, Pagani, Tasson in [8] in the aim to give a model for $PCF_{\text{sample}}$.

They actually introduced it as a refinement of the category of complete cones and stable functions, denoted $\text{Cstab}$. Stable functions on cones are a generalization of well-known absolutely monotonic functions in real analysis: it is those functions $f: [0, \infty) \rightarrow \mathbb{R}$, which are infinitely differentiable, and such that moreover all their derivatives are non-negative. The relevance of such functions comes from a result due to Bernstein: every absolutely monotonic function coincides with a power series. Moreover, it is possible to characterize absolutely monotonic functions without explicitly asking for them to be differentiable: it is exactly those functions such that all the so-called higher-order differences, which are quantities defined only by sum and subtraction of terms of the forms $f(x)$, are non-negative. (see [20], chapter 4). The definition of pre-stable functions in [8] generalizes this characterization.

In this section, we are first going to recall basic facts about cones and stable functions, all extracted from [8]. Then we will prove a generalization of Bernstein’s theorem for pre-stable functions over a particular class of cones, which is the main technical contribution of this paper. We will do that following the work of McMillan on a generalization of Bernstein’s theorem for functions ranging over abstract domains endowed with partition systems, see [15].

3.1 Cones

The use of a notion of cones in denotational semantics to deal with probabilistic behavior goes back to Kozen in [13]. We take here the same definition of cone as in [8].

Definition 3.1. A cone $C$ is a $\mathbb{R}_+$-semimodule given together with an $\mathbb{R}_+$ valued function $\| \cdot \|_C$ called norm of $C$, and verifying:

$$
(x + y, x + y') \Rightarrow y = y' \quad \|ax\|_C = a\|x\|_C
$$

$$
\|x + x'\|_C \leq \|x\|_C + \|x'\|_C
$$

$$
\|x\|_C = 0 \Rightarrow x = 0
$$

The most immediate example of cone is the non-negative real half-line, when we take as norm the identity. Another example is the positive quadrants in a 2-dimensional plan. In a way, the notion of cones is the generalization of the idea of a space where all elements are non-negative. This analogy gives us a generic way to define a pre-order, using the $+$ of the cone structure.

Definition 3.2. Let $C$ be a cone. Then we define a partial order $\leq_C$ on $C$ by: $x \leq_C y$ if there exists $z \in C$, with $y = x + z$.

We define $\mathcal{B}C$ as the set of elements in $C$ of norm smaller or equal to 1. We will sometimes call it the unit ball of $C$. Moreover, we will also be interested in the open unit ball $\mathcal{B}^\circ C$, defined as the set of elements of $C$ of norm smaller than 1.

In [8], they restrict themselves to cones verifying a completeness criterion: it allows them to define the denotation of the recursion operator in $PCF_{\text{sample}}$, thus enforcing the existence of fixpoints.

Definition 3.3. A cone $C$ is said to be:

- sequentially complete if any non-decreasing sequence of elements of $\mathcal{B}C$ has a least upper bound $\sup_{\mathbb{N}} x_n \in \mathcal{B}C$.
- directed complete if for any directed subset $D$ of $\mathcal{B}C$, $D$ has a least upper bound $\sup D \in \mathcal{B}C$.

Observe that a directed-complete cone is always complete. We illustrate Definition 3.3 by giving the complete cone used in [8] as the denotational semantics of the base type $\text{R}$ in $PCF_{\text{sample}}$.

Example 3.4. We take $\text{Meas}(\mathbb{R})$ as the set of finite measures over $\mathbb{R}$, and the norm as $\|f\|_{\text{Meas}(\mathbb{R})} = \mu(\mathbb{R})$. $\text{Meas}(\mathbb{R})$ is a directed-complete cone. For every $r \in \mathbb{R}$, the denotational semantics of the term $r$ [8] is the Dirac measure with respect to $\delta_r$, defined by taking $\delta_r(U) = 1$ if $r \in U$, and $\delta_r(U) = 0$ otherwise.

In a similar way, we define $\text{Meas}(X)$ as the directed-complete cone of finite measures over $X$, for any measurable space $X$.

In [8], the authors ask for the cones they consider only to be sequentially complete. It is due to the fact they want to add measurability requirements to their cones, and as a rule, sequential completeness interacts better with measurability than directed completeness since measurable sets are closed under countable unions, but not general unions. We illustrate this point in the example below.

Example 3.5. Let be $A$ a measurable space, and $\mu$ a finite measure on $A$. We consider the cone of measurable functions $A \rightarrow \mathbb{R}_+$. We take $\|f\| = \int_A f d\mu$. Lebesgue’s Monotone Convergence Theorem shows that this cone is sequentially complete, but it is not directed complete.

In this work, however, we are only interested in cones arising from probabilistic coherence spaces (PCS) in a way we will develop in Section 4. Since those cones have an underlying discrete structure, we will be able to show that they are actually directed complete. We will need this information, since we will apply McMillan’s results [15] obtained in the more general framework of abstract domains with partitions, in which he asks for directed completeness.

Our completeness conditions allow us to also enforce the existence of infimum, as stated in the following lemma:

Lemma 3.6. If a cone $C$ is:

- sequentially complete, then every non-increasing sequence $(x_n)_{n \in \mathbb{N}}$ has a greatest lower bound $\inf(x_n)_{n \in \mathbb{N}}$.
- directed complete, then for every $D \subseteq C$ directed for the reverse order, $D$ has a greatest lower bound $\inf D$.

Proof. We do the proof when $C$ is directed complete, but it is exactly the same in the sequentially complete case. Let be $D$ a reverse directed set. If all elements of $D$ are zero, then $\inf D = 0$. Otherwise, let be $x > 0 \in D$. We consider the subset $E = \{ \frac{x}{y} \mid y \leq x, y \in D \}$. It is easy to see that it is a directed subset of $\mathcal{B}C$, which means that, since $C$ is directed complete, it has a supremum. So we can take...
It is shown in [8] that the addition and multiplication by a scalar are Scott-continuous in complete cones, in a sequential sense. Here, we show that in directed complete cones, it holds also in a directed sense.

**Lemma 3.7.** The addition $+: C \times C \to C$ and the scalar multiplication $: \mathbb{R}_+ \times C \to C$ are Scott-continuous:

- for any directed subsets $D$ and $E$ of $C$, and $K$ of $\mathbb{R}_+$:
  
  \[
  \sup \{ x + y \mid x \in D, y \in E \} = \sup D + \sup E,
  \]
  and
  
  \[
  \sup \{ \lambda \cdot x \mid \lambda \in K, x \in E \} = \sup K \cdot \sup E.
  \]

- for any reverse directed subsets $D$, $E$ of $C$, and $K$ of $\mathbb{R}_+$:
  
  \[
  \inf \{ x + y \mid x \in D, y \in E \} = \inf D + \inf E,
  \]
  and
  
  \[
  \inf \{ \lambda \cdot x \mid \lambda \in K, x \in E \} = \inf K \cdot \inf E.
  \]

### 3.2 Pre-Stable Functions between Cones

As said before, the notion of pre-stable function is a generalization of the notion of absolutely monotonict real functions. More precisely, the idea is to define so-called higher-order differences, and to specify that they must be all non-negative.

First, we want to be able to talk about those $u = (u_1, \ldots, u_n)$, such that $|x + \sum_{i=1}^{n} u_i| \leq 1$ for a fixed $x \in \mathcal{B}C$, and $n \in \mathbb{N}$. To that end, we introduce a cone $C^n$ whose unit ball is exactly such elements. It is an adaptation of the definition given in [8] for the case where $n = 1$, and we show in the same way that it is indeed a cone.

**Definition 3.8 (Local Cone).** Let be a cone, $n \in \mathbb{N}$, and $x \in \mathcal{B}C$. We call $n$-local cone at $x$, and we denote $C^n$ the cone $\mathbb{R}^n$ endowed by:

\[
\|(u_1, \ldots, u_n)\|_{C^n} = \inf \left\{ \frac{1}{r} \mid x + r \cdot \sum_{1 \leq i \leq n} u_i \in \mathcal{B}C \wedge r > 0 \right\}.
\]

We can show that whenever $C$ is a directed-complete cone, $C^n$ is also directed-complete.

For $n \in \mathbb{N}$, we use $\mathcal{P}_n(x)$ (respectively $\mathcal{P}_n(x)$) for the set of all subsets $I$ of $\{1, \ldots, n\}$ such that $n - \text{card}(I)$ is even (respectively odd).

We are now ready to introduce higher-order differences. Since we have only explicit addition, not subtraction, we define separately the positive part $\Delta^+_n$ and the negative part $\Delta^-_n$ of those differences:

For $f : \mathcal{B}C \to D$, $x \in \mathcal{B}C$, $\bar{u} \in \mathcal{B}C^n$, and $c \in \{-, +\}$, we define:

\[
\Delta^n_c(f)(x, \bar{u}) = \sum_{I \in \mathcal{P}_n(x)} f(x + \sum_{i \in I} u_i).
\]

**Definition 3.9.** We say that $f$ is pre-stable if, for every $x \in \mathcal{B}C$, it holds that:

\[
\Delta^n_x(f)(x, \bar{u}) \leq \Delta^n_x(f)(x, \bar{u}).
\]

If $f$ is pre-stable, we will set $\Delta^n_x(f)(x, \bar{u}) = \Delta^n_x(f)(x, \bar{u}) - \Delta^n_x(f)(x, \bar{u})$. Observe that the quantity $\Delta^n_x(f)(x, \bar{u})$ is actually symmetric in $\bar{u}$.

**Definition 3.10.** A function $f : \mathcal{B}C \to D$ is called a stable function from $C$ to $D$ if it is pre-stable, sequentially Scott-continuous, and moreover there exists $\lambda \in \mathbb{R}_+$ such that $f(\mathcal{B}C) \subseteq \lambda \cdot \mathcal{B}D$.

**Definition 3.11.** Cstab is the category whose objects are sequentially complete cones, and morphisms from $C$ to $D$ are the stable functions $f$ from $C$ to $D$ such that $f(\mathcal{B}C) \subseteq \mathcal{B}D$.

It was shown in [8] that it is possible to endow Cstab with a cartesian closed structure. The product cone is defined as $\prod_{i \in I} C_i = \{ (x_i)_{i \in I} \mid \forall i \in I, x_i \in C_i \}$, and $\|\|_{I \in \mathcal{I}} C_i = \sup_{i \in \mathcal{I}} \|\|_{i \in \mathcal{I}} C_i$. The function cone $\mathcal{B}C \to D$ is the set of all stable functions, with $\|f\|_{C \to D} = \sup_{x \in \mathcal{B}C} \|f(x)\|_{D}$. It was shown in [8] that these cones are indeed sequentially complete, and that the lab in $C \Rightarrow D$ is computed pointwise. We will use also the cone of pre-stables from $C$ to $D$, which is also sequentially complete.

### 3.3 A generalization of Bernstein’s theorem for pre-stable functions

We are now going to show an analogue of Bernstein’s Theorem for pre-stable functions on directed-complete cones. The idea is to first define an analogue to derivatives for pre-stable functions, and to show that pre-stable functions can be written as the infinite sum generated by an analogue to Taylor expansion on $\mathcal{B}C$. This result is actually an application of McMillan’s work [15] in a more general setting. Here, we are going to give the main steps of the construction directly on cones, as well as highlighting some properties of the Taylor series which are true for cones, but not in the general framework McMillan considered.

**3.3.1 Derivatives of a pre-stable function**

We are now going, following McMillan [15], to construct derivatives for pre-stable functions on directed-complete cones. This construction is based on the use of a notion of partition: a partition of $x \in \mathcal{B}C$ is a multiset $\pi = \{u_1, \ldots, u_n\} \in M_f(C)$ such that $x = \sum_{i \leq n} u_i \circ \pi$. We write $x \sim x$ when the multiset $\pi$ is a partition of $x$. We will denote by $+$ the usual union on multiset: $[y_1, \ldots, y_n] + [z_1, \ldots, z_m] = [y_1, \ldots, y_n, z_1, \ldots, z_m]$. We call $\mathcal{P}(x)$ the set of partitions of $x$.

**Definition 3.12 (Refinement Preorder).** If $\pi_1, \pi_2$ are in $\mathcal{P}(x)$, we say that $\pi_1 \leq \pi_2$ if $\pi_1 = \{u_1, \ldots, u_n\}$, and $\pi_2 = \{a_1 + \ldots + a_n\}$ with each of the $a_i$ a partition of $u_i$.

Observe that when $\pi_1$ and $\pi_2$ are partition of $x$, $\pi_2 \leq \pi_1$ means that $\pi_1$ is a more finely grained decomposition of $x$. If $\bar{u}$ is a $n$-tuple in $\mathcal{B}C$, we extend the refinement order to $\mathcal{P}(\bar{u}) = \mathcal{P}(u_1) \times \ldots \times \mathcal{P}(u_n)$. Observe that the refinement preorder turns $\mathcal{P}(\bar{u})$ into a directed set.

**Definition 3.13 (from [15]).** Let $C, D$ be two cones, and let $f : \mathcal{B}C \to D$ be a pre-stable function. Let be $\bar{u} = (u_1, \ldots, u_n) \in \mathcal{B}C^n$. Then we define $\Phi^f_{n}(\bar{u}) : \mathcal{P}(\bar{u}) \to D$ as:

\[
\Phi^f_{n}(\pi_1, \ldots, \pi_n) = \sum_{y_1 \in \pi_1} \ldots \sum_{y_n \in \pi_n} \Delta^n f(x \mid y_1, \ldots, y_n).
\]

It holds (see [15] for more details) that $\Phi^f_{n}$ is a non-increasing function (the fact that $\Phi^f_{n}$ is indeed a non-increasing function when $f$ is a pre-stable function is shown by Lemma 3.2 of [15]). It is obtained by looking at the definition of higher-order differences).

Since $\mathcal{P}(\bar{u})$ is a directed set, $\Phi^f_{n}$ has a greatest lower bound whenever $D$ is a directed-complete cone.

**Definition 3.14 (from [15]).** Let be $\mathcal{B}C, D$ a directed-complete cone, and $f : \mathcal{B}C \to D$ a pre-stable function. Let be $\bar{u} \in \mathcal{B}C^n$. We say $f_{\bar{u}}$ is:

\[
\Phi^f_{n}(\bar{u}) : \mathcal{P}(\bar{u}) \to D
\]

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Then the derivative of \( f \) in \( x \) at rank \( n \) towards the direction \( \vec{u} \) is the function \( D^n f(x | \vec{u}) : \mathcal{B} C^n \to D \) defined as

\[
D^n f(x | \vec{u}) = \inf_{x \in \mathcal{B} C} \phi^x_\vec{u}(\vec{x}).
\]

We are now going to illustrate Definition 3.14 on a basic case where we take \( f : \mathbb{R} \to \mathbb{R}_+ \), in order to highlight the link with differentiation in real analysis.

**Example 3.15.** We take \( C \) and \( D \) as the positive real half-line, and \( x \in [0, 1] \). Let be \( h \) such that \( x + h \leq 1 \). Then:

\[
D^1 f(x | h) = \inf_{y \in \mathcal{B} C} \sum_{n \in \pi} f(x + y) - f(x)
\]

We know already, since \( f \) is pre-stable hence absolutely monotone as function on reals, that \( f \) is convex, and moreover differentiable (see [20]). From there, by considering a particular family of partitions, we can show that \( D^1 f(x | h) = \lambda \cdot f'(x) \).

**Proof.** First, let \( \pi \) be any partition of \( y \). Since \( f \) is differentiable and convex, it holds that:

\[
\forall z, f(x + z) - f(x) \geq f'(x) \cdot z.
\]

As a consequence, we see that for any partition \( \pi \) of \( h \), it holds that

\[
\sum_{y \in \pi} f(x + y) - f(x) \geq f'(x) \cdot h,
\]

and it implies that \( D^1 f(x | h) \geq f'(x) \cdot h \). To show the reverse inequality, it is enough to consider the particular family of partition \( \pi_n = \left[ \frac{h}{n}, \ldots, \frac{h}{n} \right] \) of \( h \) we see that:

\[
\sum_{y \in \pi_n} f(x + y) - f(x) = n \cdot f(x + \frac{h}{n}) - f(x)
\]

\[
= \frac{h}{n} \cdot f(x + \frac{h}{n}) - f(x) \xrightarrow{n \to \infty} \lambda \cdot f'(x)
\]

\( \square \)

**Lemma 3.16.** Let \( C \) be a cone, \( D \) a directed complete cone, \( f \) a pre-stable function from \( C \) to \( D \). Let be \( x \in \mathcal{B} C \). Then \( D^n f(x | \cdot) \) is a symmetric function \( \mathcal{B} C^n \to D \) such that moreover:

- \( 0 \leq D^n f(x | \vec{u}) \leq \Delta^n f(x | \vec{u}) \)
- Both \( \vec{u} \mapsto D^n f(x | \vec{u}) \) and \( \vec{u} \mapsto \Delta^n f(x | \vec{u}) = D^n f(x | \vec{u}) \) are pre-stable function from \( \mathcal{B} C^n \) to 0.

**Proof.** The proof is given in Lemma 3.31 in [15]. It comes almost directly from Definition 3.14.

\( \square \)

We have seen in Example 3.15 that our so-called derivatives of pre-stable functions play the same role as the derivative of a differentiable function, which are actually linear operators \( d f^x : \mathbb{R} \to \mathbb{R} \). While the abstract domains considered in [15] do not have to be \( \mathbb{R}_+ \) semi-module, so have no notion of linearity, we are able to show in the complete cone case that the \( D^n f \) are linear in the sense of Lemma 3.17 below.

**Lemma 3.17.** Let \( C, D \) be two directed complete cones, \( x \in \mathcal{B} C \).

- Let be \( f : \mathcal{B} C \to D \) be a pre-stable function. Then \( D^n f(x | \cdot) : \mathcal{B} C^n \to D \) is \( n \)-linear, in the sense that, for each of its arguments, it commutes with the sum and multiplication by a scalar:
  - \( f \) is \( n \)-linear, in the sense that for each argument, it commutes with the sum and multiplication by a scalar.
- For any \( \vec{u} \in \mathcal{B} C^n \), the function \( f \in \text{Cstab}(C, D) \mapsto D^n f(x | \vec{u}) \in D \) is linear and directed Scott-continuous.

**Proof.** We are going to use the following auxiliary lemma:
We show now that \( f \in \text{Cstab}(C, D) \mapsto D^n f(x \mid \mathbf{u}) \in D \) is linear and Scott-continuous. It is immediate that it is linear, since every one of the \( f \mapsto \Delta^n_k(x \mid \mathbf{u}) \) is. We are now going to use 3.18 to show the Scott-continuity; it tells us that we have only to check that for every \( E \subseteq C \rightrightarrows E \) directed for the reverse order, \( D^n f(\inf E \mid \mathbf{u}) = \inf D^n f(E \mid \mathbf{u}) \). Observe that:

\[
D^n(\inf E)(x \mid \mathbf{u}) = \inf_{\pi \in \mathcal{P}(\mathbf{u})} \sum_{y_1 \in P} \ldots \sum_{y_n \in P} \Delta^n(\inf E)(x \mid y_1, \ldots, y_n)
\]

\[
= \inf_{\pi \in \mathcal{P}(\mathbf{u})} \sum_{y_1 \in P} \ldots \sum_{y_n \in P} \inf \{ \Delta^n f(x \mid y_1, \ldots, y_n) \}
\]

\[
= \inf_{\pi \in \mathcal{P}(\mathbf{u})} \{ \sum_{y_1 \in P} \ldots \sum_{y_n \in P} \Delta^n f(x \mid y_1, \ldots, y_n) \}
\]

\[
= \inf_{\pi \in \mathcal{P}(\mathbf{u})} D^n f(x \mid \mathbf{u}) \text{ since the infs can be exchanged.}
\]

The linearity of the derivatives means that for every \( x \in \mathcal{P} C \), we can extend \( D^n f(x \mid \cdot) \) to a function \( C^n \to D \). We will use implicitly this extension in the following, especially in Definition 3.19.

### 3.3.2 Taylor Series for pre-stable functions

We have seen above that the \( D^n f \) are a notion of differential for pre-stable functions. Following further this idea, and the work of McMillan [15], we define an analogue to the Taylor expansion. In all this section \( C, D \) and \( E \) are going to be directed complete cones, and \( f : \mathcal{BC} \to D \) a pre-stable function.

**Definition 3.19.** Let be \( x \in \mathcal{P} C \). We call **Taylor partial sum of** \( f \) **in** \( x \) **at the rank** \( N \) **the function** \( T_f^N(x \mid \cdot) : C^N \to D \) **defined as:**

\[
T_f^N(x \mid y) = f(x) + \sum_{k=1}^N \frac{1}{k!} D^k f(x \mid y, \ldots, y).
\]

The next step consists in establishing that the \( T_f^N \) are actually a non-increasing bounded sequence in the cone of pre-stable functions from \( C \) to \( D \), which will allow us to define the **Taylor series of** \( f \), as the supremum of the \( T_f^N f \).

To that end, we are first going to establish an alternative characterization of the Taylor series, which is one used in [15], in the framework of abstract domains. It consists in substituting each of the \( D^n f(x \mid y, \ldots, y) \) with its expression given by Lemma 3.20 below. The validity of Lemma 3.14, and thus the equivalence of the two definitions, depends on the fact we work with directed-complete cones.

**Lemma 3.20 (Alternative Caractérisation of Derivatives).** Let \( x \in \mathcal{P} C, y \in \mathcal{B} C \), and \( k \in \mathbb{N} \). Then it holds that \( D^k f(x \mid y, \ldots, y) \) is equal to:

\[
\sup_{\pi = (u_1, \ldots, u_n) \in \mathcal{P}(y)} \sum_{\alpha(1), \ldots, \alpha(n)} D^k f(x \mid u_\alpha(1), \ldots, u_\alpha(n))
\]

**Proof.** We first introduce the following notation: if \( \pi = (u_1, \ldots, u_n) \) is a partition of \( x \), and \( \sigma : [1, k] \rightrightarrows [1, n] \) an injective function, we denote \( \sigma(\pi) = (u_{\sigma(1)}, \ldots, u_{\sigma(k)}) \). We denote by \( A = \sup_{\pi \in \mathcal{P}(y)} \sum_{\sigma : [1, k] \rightrightarrows [1, n]} D^k f(x \mid \sigma(\pi)) \)

for every \( n \in \mathbb{N} \):

\[
A \geq \sum_{\sigma : [1, k] \rightrightarrows [1, n]} D^k f(x \mid \frac{y}{n}, \ldots, \frac{y}{n})
\]

\[
= \frac{n!}{(n-k)!} D^k f(x \mid \frac{y}{n}, \ldots, \frac{y}{n})
\]

\[
= \frac{n!}{n!(n-k)!} n^k D^k f(x \mid y, \ldots, y)
\]

The sequence \( \frac{n!}{n!(n-k)!} \) tends to 1 when \( n \) tends to infinity (see in the long version). By Scott-continuity, it means that \( A \geq D^k f(x \mid y, \ldots, y) \).

Let us show now that \( A \leq D^k f(x \mid y, \ldots, y) \). Let be \( \pi = (u_1, \ldots, u_n) \in \mathcal{P}(y) \). Then:

\[
\sum_{\sigma : [1, k] \rightrightarrows [1, n]} D^k f(x \mid \sigma(\pi))
\]

\[
\leq \sum_{i_1 \in [1, n]} \ldots \sum_{i_k \in [1, n]} D^k f(x \mid u_{i_1}, \ldots, u_{i_k})
\]

\[
= D^k f(x \mid y, \ldots, y)
\]

Since \( A = \sup_{\pi \in \mathcal{P}(y)} \sum_{\alpha : [1, k] \rightrightarrows [1, n]} D^k f(x \mid \alpha(\pi)) \), we see that \( A \leq D^k f(x \mid y, \ldots, y) \), which ends the proof.

With this characterization, [15] shows that the sequence of functions \( (x \in \mathcal{B} C \mapsto T_f^n(x \mid y)) \) is bounded by \( (x \in \mathcal{B} C \mapsto f(x + y)) \) in the cone of pre-stable functions from \( C \to D \).

**Lemma 3.21.** Let be \( y \in \mathcal{P} C \), and \( x \in \mathcal{B} C \). Then \( \forall N \in \mathbb{N} \), \( T_f^N(x \mid y) \leq f(x + y) \), and the function \( (x \in \mathcal{B} C \mapsto f(x + y) - T_f^N(x \mid y)) \) is pre-stable.

**Proof.** Let be \( y \in \mathcal{B} C \) and \( y \in \mathcal{B} X \). We are able to express \( f(x) \) by using \( f(x) \) and finite differences on any partition of \( y \); indeed, for every partition \( \pi \) of \( y \), it holds that:

\[
f(x + y) = f(x) + \sum_{1 \leq k \leq n} \frac{1}{k!} \sum_{\sigma : [1, k] \rightrightarrows [1, n]} D^k f(x, \sigma(\pi))
\]

(1)

**Proof.** It is an algebraical calculation, done in [15]. We give here the proof for \( n = 2 \). Let \( \pi = (y_1, y_2) \) a partition of \( y \). Then we see that:

\[
f(x) + \sum_{1 \leq k \leq n} \frac{1}{k!} \sum_{\sigma : [1, k] \rightrightarrows [1, n]} D^k f(x, \sigma(\pi))
\]

\[
= f(x) + \Delta_1 f(x, y_1) + \Delta_1 f(x, y_2)
\]

\[
+ \frac{1}{2} (\Delta_2 f(x, y_1, y_2) + \Delta_2 f(x, y_2, y_1))
\]

\[
= f(x) + D_1 f(x + y_1) + f(x + y_2) - f(x)
\]

\[
+ f(x + y_1 + y_2) + f(x + y_1) - f(x + y_2) + f(x)
\]

\[
= f(x + y_1 + y_2) = f(x + y).
\]

Moreover we are also able to express the derivatives of \( f \) at \( x \) towards the direction \( y \) also using the partitions of \( y \) (it is the sense
of Lemma 3.20). Accordingly:

\[
Tf^N(x | y) = f(x) + \sum_{k=1}^{N} \frac{1}{k!} \sup_{\pi \in P(y)} \sum_{[i] \subset \{1, \ldots, k\}} \Delta_k f(x | \sigma(\pi))
\]

Using Lemma 3.16, we see that it implies:

\[
Tf^N(x | y) \leq f(x) + \sum_{k=1}^{N} \frac{1}{k!} \sup_{\pi \in P(y)} \sum_{[i] \subset \{1, \ldots, k\}} \Delta_k f(x | \sigma(\pi))
\]

We can now use the Scott continuity of + and \(\cdot\), and we obtain:

\[
Tf^N(x | y) \leq \sup_{\pi \in P(y)} f(x) + \sum_{k=1}^{N} \frac{1}{k!} \sum_{[i] \subset \{1, \ldots, k\}} \Delta_k f(x | \sigma(\pi))
\]

We can now conclude using (1):

\[
Tf^N(x | y) \leq \sup_{\pi \in P(y)} f(x + y) \leq f(x + y)
\]

The proof of the pre-stability of the function can be found in [15]. It is based on the fact that each one of the above inequality can be seen as an inequality in the cone of pre-stable functions.

Since we have shown that the partial sum of the Taylor series of \(f\) was a bounded non-decreasing sequence in the complete cone of pre-stable functions from \(C^1_b\) to \(D\), we can define the Taylor series of \(f\) as its supremum.

**Definition 3.22.** We define \(Tf(x | \cdot) : \mathcal{B}^1_b \rightarrow D\) the Taylor series of \(f\) in \(x\), and \(Rf(x | \cdot) : \mathcal{B}^1_b \rightarrow D\) the Remainder of \(f\) in \(x\) as:

\[
Tf(x | y) = \sup_{N \in \mathbb{N}} Tf^N(x | y)
\]

\[
Rf(x | y) = f(x + y) - Tf(x | y).
\]

### 3.3.3 Extended Bernstein’s theorem

Our goal from here is to show that for any \(x \in \mathcal{B}^0 C\), \(Rf(0 | x) \neq 0\). We recall here the main steps of the proof of [15]. It is based on two technical lemmas, that analyze more precisely the behavior of the remainder of \(f\). The first one is actually a summary of several technical results shown separately in [15].

**Lemma 3.23.** Let be \(x \in \mathcal{B}^0 C\). Then it holds that both:

\[
Rf_x : y \in \mathcal{B}^1_x \mapsto Rf(x | y) \in D
\]

and \(Rf^y : x \in \mathcal{B}^1_y \mapsto Rf(x | y) \in D\) are pre-stable functions. Moreover \(Rf_x(0) = 0\), and for every \(x \in \mathcal{B}^1_y\), it holds that \(T(Rf^y)(0 | x) = 0\).

**Proof.** We give here only sketches of the proofs. The detailed proof can be found in [15].

- For \(Rf^y\), it is a consequence of the fact that both \(f^y : x \in \mathcal{B}^1_y \mapsto f(x + y) \) and \(Tf^x : x \in \mathcal{B}^1_y \mapsto Tf(x | y)\) are pre-stable functions, with \(Tf^y \leq f^y\) in the cone of pre-stable functions, and \(Rf^y \leq Tf^y\).
- The pre-stability of \(Rf_x\) is stated in Theorem 4.1. of [15]. It is based on a previous technical lemma shown in [15], which says it is sufficient for a function to be pre-stable, to have all its differences in \(\theta\) to be non-negative. Then the idea is to fix \(x\), and to consider for every \(N \in \mathbb{N}\), the function \(g_N : y \in \mathcal{B}^1_y \mapsto f(x + y) - Tf^N(x | y)\). It is then possible to show that for any \(n \in \mathbb{N}\), and \(u \in \mathcal{B}^0_{x^n}\), \(\Delta_n g_N(0 | u) = \Delta_n f(x | u) - \Delta_n(Tf^N)(0 | u)\), with \(Tf^N(x | y) : y \mapsto Tf^N(x | y)\). By a computation on the \(\Delta_n(Tf^N)(0 | u)\), we see that the \(\Delta_n g_N(0 | u)\) are non-negative. Then, we conclude using the fact that \(Rf_x(y) = \inf_{x \in \mathcal{B}^0} Tf^N(y)\).

- The fact that \(Rf_x(y) = 0\) is a direct consequence of the \(n\)-linearity of the map \(u \mapsto D^u f(x | u)\), and \(D^u f(x | u)\) allows us to show that if we take \(g(y) = D^u f(x | u)\), then \(D^u g(x | v) = D^{u+k} f(y_0 | u, v)\), and from there to compute the Taylor series of \(Rf^y\).

\[\Box\]

The second technical lemma gives us a way to decompose \(Rf(x | y)\) into smaller pieces. It is stated in Theorem 5.3 in [15].

**Lemma 3.24.** Let be \(x, y\) such that \(x+y \in \mathcal{B}^0 C\). Then \(Rf(0 | x+y) \leq Rf(y | x) + Rf(x | y)\), and furthermore \(Rf(0 | x+y) \geq Rf(x | y)\), and \(Rf(0 | x+y) \geq Rf(y | x)\), and moreover all the inequality are in the cone of pre-stable functions.

**Proof.** We give here a brief sketch of the proof of the first statement. More details can be found in [15]. We introduce the function \(Rf^{x+y} : y \in \mathcal{B}^1_y \mapsto Rf(0 | x+y)\). The proof is based on the fact that it is possible to establish (see [15]):

\[
T(Rf^{x+y})(0 | y) = T(Rf^x)(0 | y)
\]

and

\[
R(Rf^{x+y})(0 | y) = R(Rf^x)(0 | y)
\]

As a consequence, we can write:

\[
Rf(x | y) + Rf(y | x) = Rf_x(y) + Rf^y(x) \\
= T(Rf_x)(0 | y) + R(Rf_x)(0 | y) + T(Rf^y)(0 | y) + R(Rf^y)(0 | y)
\]

\[
= T(Rf_x)(0 | y) + R(Rf^{x+y})(0 | y) + T(Rf^{x+y})(0 | y) + R(Rf^{x+y})(0 | y)
\]

by (2) and (3).

\[
= Rf^{x+y}(y) + T(Rf_x)(0 | y) + R(Rf^y)(0 | y)
\]

\[
\geq Rf^{x+y}(y) = Rf(0 | x+y),
\]

and we see that we have also shown that the difference is pre-stable. The other two statement are shown in a similar way.

\[\Box\]

We use Lemma 3.24 to show the a more involved upper bound on \(Rf(0 | x)\).

**Lemma 3.25.** Let be \(x \in \mathcal{B}^0 C\), and \(\pi = [x_1, \ldots, x_n]\) a partition of \(x\), such that for every \(x_i \in \pi, x + x_i \in \mathcal{B}^0 C\). Then \(Rf(0 | x) \leq \sum_{1 \leq i \leq n} \inf_{x \in \pi_i} Rf(x | z)\).

**Proof.** For every \(x \in \mathcal{B}^0 C\), we denote \(g_x : y \in \mathcal{B}^1_y \mapsto f(x+y)\). From the definitions of the \(D^n\), we see that it holds that \(Rg_x(0 | y) = Rf(x | y)\).

Let be \(\pi_1, \ldots, \pi_n\) such that \(\pi_i\) is a partition of \(x_i\) over \(J_i\). Then \(\pi_1 + \ldots + \pi_n\) is a partition of \(x\). Lemma 3.24 applied several times, combined with the fact that \(Rg_x(0 | y) = Rf(x | y)\), tells us that:

\[
Rf(0 | x) \leq \sum_{z \in \pi_1 + \ldots + \pi_n} Rf(z | x)
\]

where \(z' = \sum_{i \in \pi_1 + \ldots + \pi_n} u_i\). Moreover, we know that \(Rf^z\) is pre-stable (by lemma 3.23). Since, for every \(z \in \pi_1 + \ldots + \pi_n\),

\[\Box\]
\[ z' \leq x \text{ (it is immediate, since } \pi_1 + \ldots + \pi_n \text{ is a partition of } x), \text{ it} \]
\[ \text{it holds that } Rf(z') \leq x \leq Rf(z) \text{.} \]
\[ \text{As a direct consequence, we see that } Rf(0 | x_0) \leq \sum_i \sum_{x \in A_i} Rf(x | z), \text{ which} \]
\[ \text{leads to the result.} \]

We are now ready to show the main result of this section.

**Proposition 3.26 (Extended Bernstein’s Theorem).** Let be \( C, D \)
\[ \text{directed-complete cones, and } f : \mathcal{P}C \rightarrow D \text{ a pre-stable function.} \]
\[ \text{Then for every } x \in \mathcal{P}C, \text{ it holds that } f(x) = \int f(0 | x). \]

**Proof.** Let be \( x \in \mathcal{P}C \). First, we consider the partition \( \pi = [\frac{x}{n}, \ldots, \frac{x}{N}] \)
\[ \text{of } x, \text{ with } N \text{ taken such as } x + \frac{x}{N} \in \mathcal{P}C. \text{ We know that such a } N \]
\[ \text{exists since } x \text{ is in the open unit ball } \mathcal{P}C. \text{ We use Lemma 3.25 on} \]
\[ Rf(0 | x), \text{ and the partition } \pi, \text{ and it tells us that:} \]
\[ Rf(0 | x) \leq \sum_{1 \leq j \leq n} \inf_{\pi=(u_1, \ldots, u_n)} Rf(x | u_j) \] \[ \sum_{1 \leq j \leq n} Rf(x | u_j). \] \[ (4) \]
\[ \text{Observe that the above expression is valid, since for every } u_j \text{ in} \]
\[ \text{a partition } \pi \text{ of } \frac{x}{n}, x + u_i \in \mathcal{P}C. \text{ We know, by Lemma 3.23 that} \]
\[ Rf(x | 0) = 0. \text{ Therefore, we can rewrite (4) as} \]
\[ Rf(0 | x) \leq \sum_{1 \leq j \leq n} \inf_{\pi=(u_1, \ldots, u_n)} Rf(x | u_j) \sup_{1 \leq j \leq n} Rf(x | u_j) = Rf(x | 0). \] \[ (5) \]
\[ \text{Moreover, we are able to express the right part of (5) by the finite differences of the} \]
\[ \text{pre-stable function } Rf_x; \text{ indeed for each } i, \]
\[ \Lambda_{Rf}^1(0, u_i) = Rf(x | u_i) - Rf(x | 0). \] \[ (6) \]
\[ \text{By the definition of derivatives (see Definition 3.14), we see that} \]
\[ D^1Rf_x(0 | x) = \inf_{\pi \in \mathcal{P}(\frac{x}{N})} \sum_{i \in \pi} \Lambda_{Rf}^1(0, v). \] \[ (7) \]
\[ \text{We see now that combining (5), (6) and (7) leads us to } Rf(0 | x) \]
\[ \leq \sum_{1 \leq j \leq n} D^1Rf_x(0 | \frac{x}{n}). \text{ Moreover, we know that for every} \]
\[ \gamma \in \mathcal{P}_N, D^1Rf_x(0 | \gamma) \leq T(Rf_x)(0 | \gamma). \text{ Hence by using again} \]
\[ \text{Lemma 3.23, which says that } T(Rf_x)(0 | \frac{x}{n}) = 0, \text{ it holds that} \]
\[ Rf(0 | x) = 0. \]
\[ \square \]

### 4 Cstab is a conservative extension of Pcoh:

Probabilistic coherence spaces (PCS) were introduced by Ehrhard and Danos in [4] as a model of higher-order probabilistic computation. It was successful in giving a fully abstract model both of PCF\_\_ and of a discrete probabilistic extension of Levy’s Call-by-Push-Value. In this section, we present briefly basic definitions from [4] and highlight an embedding from PCSs into cones.

#### 4.1 Probabilistic Coherence Spaces

The definition of the PCS model of Linear Logic follows the tradition
\[ \text{initiated by Girard with Coherence Spaces in [9], and followed for} \]
\[ \text{instance by Ehrhard in [6] when defining hypercoherence spaces.} \]
\[ \text{A coherent space interpreting a type can be seen as a symmetric} \]
\[ \text{graph, and the interpretation of a program of this type is a clique} \]
\[ \text{of this graph. Interestingly, such a graph A can be alternatively} \]
\[ \text{characterized by giving its set of vertices (that we will call web),} \]
\[ \text{and a family of subset of this web, meant to be the family of the} \]
\[ \text{cliques of A. Then we know that an arbitrary family of subsets of a} \]
\[ \text{given web arises indeed as a family of cliques for some graph when} \]
\[ \text{some duality criterion is verified.} \]

PCS are designed to express probabilistic behavior of programs.

As a consequence, a clique is not a subset of the web anymore, but a quantitative way to associate a non-negative real coefficient to every element in the web.

**Definition 4.1 (Pre-Probabilistic Coherent Spaces).** A Pre-PCS is a pair \( X = (|X|, PX) \), where \(|X|\) is a countable set called web of \( X \), \( PX \) is a subset of \( \subseteq \mathbb{R}_+^{|X|} \) whose elements are called cliques of \( X \).

We need here to introduce some notations to deal with infinite dimensional \( \mathbb{R} \)-vector spaces. Given a countable web \( A \), and an element of \( A \), we denote \( e_a \) the vector in \( \mathbb{R}_+^{|A|} \) which is 1 in \( a \), and 0 elsewhere. We are also going to introduce a scalar product on vectors in \( \mathbb{R}_+^{|A|} \): if \( u, v \in \mathbb{R}_+^{|A|} \), we will denote \( \langle u, v \rangle = \sum_{a \in |X|} u_a v_a \in \mathbb{R} \cup \{\infty\} \). Moreover, if \( A \) and \( B \) are countable sets, \( x \in \mathbb{R}_+^{A \times B} \), and \( u \in \mathbb{R}_+^{|A|} \), we denote by \( x \cdot u \) the vector in \( (\mathbb{R}_+ \cup \{\infty\})^{|B|} \) given by \( (x \cdot u)_b = \sum_{a \in A} x_{a,b} u_a \) for every \( b \in B \).

We are going to give examples of pre-PCS modeling discrete data-types. First, we define a pre-PCS \( 1 \) to correspond to unit type.

Since unit-type programs have only one possible outcome (that they can reach or not), we can have only one vertex: \( \{1\} = \star \). We want the denotation of a unit-type program to express its probability of termination: we take the set of cliques \( P1 \) as the interval [0, 1].

Let us now look at what happens when we consider programs of type \( N \): a program can now have a countable numbers of possible outcomes, while the web will consist in \( N \), and cliques will be sub-distributions on these vertices.

**Example 4.2 (Pre-PCS of Natural Numbers).** We define the Pre-PCS \( \mathbb{N}_{\text{Pcoh}} \) by taking \( [\mathbb{N}]_{\text{Pcoh}} = \mathbb{N} \), and \( P\mathbb{N}_{\text{Pcoh}} = \{ u \in \mathbb{R}_+^{|\mathbb{N}|} | \sum_{n \in \mathbb{N}} u_n \leq 1 \} \). It correspond to the denotational semantics of the base type \( N \) of PCF\_\_ in \( \text{Pcoh} \).

We now need to give a quantitative bi-duality criterion, to specify which one of the \( PX \subseteq \mathbb{R}_+^{|X|} \) are indeed valid families of cliques.

To do that, we first define a duality operator: if \( X = (|X|, PX) \) is a pre-PCS, we define the pre-PCS \( (|X|)^+ = (|X|, \{ u \in \mathbb{R}_+^{|X|} | \forall v \in PX, \langle u, v \rangle \leq 1 \} \). We are now ready to give conditions on pre-PCSs to actually be PCSs.

**Definition 4.3 (Probabilistic Coherent Spaces).** A pre-PCS \( X \) is a PCS if \( (|X|)^+ = X \) and moreover the following two conditions hold:

- \( \forall a \in |X| \), there exists \( \lambda > 0 \) such that \( \lambda e_a \in PX \).
- \( \forall a \in |X| \), there exists \( M \geq 0 \), such that for every \( u \in PX \), \( u_a \leq M \).

We may see easily that both \( 1 \) and \( \mathbb{N}_{\text{Pcoh}} \) are indeed PCSs.

As highlighted in Example 4.4 from [8], we can associate in a generic way a cone to any PCS. The idea is that we consider the extension of the space of cliques by all uniform scaling by positive reals. We formalize this idea in Definition 4.4 below.

**Definition 4.4.** Let be \( X \) a PCS. We define a cone \( \mathcal{C}X \) as the \( \mathbb{R}_+ \) semi-module \( \left\{ \alpha \cdot x \ s.t. \alpha \geq 0, x \in PX \right\} \) where the \( + \) is the usual addition on vectors. We endow it with \( \| \cdot \|_{\mathcal{C}X} \) defined by:

\[ \|x\|_{\mathcal{C}X} = \sup_{y \in \mathcal{P}(X)^+} \langle x, y \rangle = \inf_{r} \left\{ \frac{1}{r} | x \in PX \right\}. \]

It is easily seen that it is indeed a cone (observe that the proof uses the so-called technical conditions from Definition 4.3). Moreover, we can see that \( \mathcal{P}C_X \) consists exactly in the set \( PX \) of cliques of
X. Looking at the cone order $\leq_{\text{C}_X}$, as defined in Definition 3.2, we see that it coincides on $PX$ with the pointwise order in $\mathbb{R}_+^{|X|}$. It is relevant since we know already from [4] that $PX$ is a bounded-complete and $\omega$-continuous cpo with respect to this pointwise order.

**Lemma 4.5.** For every PCS $X$, it holds that $C_X$ is a directed-complete cone.

**Proof.** To show that $C_X$ is directed complete, we use the fact that $PX$ is a complete partial order. \hfill \Box

### 4.2 The Category Pcoh.

Intuitively a morphism in $\text{Pcoh}(X, Y)$ is a linear map from $\mathbb{R}_+^{|X|}$ to $\mathbb{R}_+^{|Y|}$ preserving the cliques.

**Definition 4.6** (Morphisms of PCSs). Let be $X, Y$ two PCSs. A morphism of PCSs between $X$ and $Y$ is a matrix $x \in \mathbb{R}_+^{|X| \times |Y|}$ such that for every $u \in PX$, it holds that $x \cdot u \in PY$.

We now illustrate Definition 4.6 by looking at the morphisms from $\text{Bool}$ to itself; they are the $x \in \mathbb{R}_+^{|2| \times |2|}$ such that $x_{1,1} + x_{1,2} \leq 1$, and similarly $x_{2,1} + x_{2,2} \leq 1$. We see that they are exactly those matrices specifying the transitions for a probabilistic Markov chain with two states $t$ and $f$.

We call $\text{Pcoh}$ in the following the category of PCS and morphisms of PCS. In [4], it is endowed with the structure of a model of linear logic. We are only going to recall here partly the exponential structure, since our main focus will be on the Kleisli category associated to $\text{Pcoh}$.

In [4], the construction of the exponential was done by defining a functor $!$, as well as hereditarily and digging making $\text{Pcoh}$ a Seely category, and consequently a model of linear logic. Here, we are only going to recall explicitly the effect of $!$ on PCSs. We denote by $M_{\{1\}}(\{X\})$ the set of finite multisets over the web of $X$, and we take it as the web of the PCS $PX$. Moreover, we will use the following notation: for every $x \in \mathbb{R}_+^{|X|}$, and $\mu \in M_{\{1\}}(\{X\})$, we denote $x^{\mu} = \prod_{i \in |X|} x_{\mu(i)} \in \mathbb{R}_+$.

**Definition 4.7.** Let be $X$ a PCS. We define the promotion of $x \in PX$, as the element $x^! \in M_{\{1\}}(\{X\})$ given by $x^! \mu = x^{\mu}$. We define $|X| = (M_{\{1\}}(\{X\}), \{x^! \mid x \in X \})^{\perp\perp}$.

### 4.3 The Kleisli Category of Probabilistic Coherence Spaces

The idea, as usual, is that morphisms in the Kleisli category can use several times their argument, while morphism in the original category are linear. The Kleisli category for $\text{Pcoh}$, denoted $\text{Pcoh}_K$, has also PCSs for objects, while $\text{Pcoh}_K(X, Y) = \text{Pcoh}(X, Y)$. We give here a direct characterization of $\text{Pcoh}_K$ morphisms.

**Lemma 4.8** (from [4]). Let be $f \in \mathbb{R}_+^{|X| \times |Y|}$. Then $f$ is a morphism in $\text{Pcoh}_K(X, Y)$, if and only if for every $x \in PX$, $f \cdot x^! \in PY$.

What Lemma 4.8 tells us is that any $f \in \text{Pcoh}(X, Y)$ is entirely characterized by the map $\tilde{f} : x \in PX \to f \cdot x^! \in PY$. We denote by $\tilde{E}^{X,Y}$ the set of all maps $PX \to PY$ that are equal to a $f$ with $f \in \text{Pcoh}(X, Y)$. It has been shown in [4] that $\tilde{\cdot}$ is actually a bijection from $\text{Pcoh}(X, Y)$ to $\tilde{E}^{X,Y}$.

Observe that we can see the maps in $\tilde{E}^{X,Y}$ as entire series, in the sense that they can be written as the supremum of a sequence of polynomials. Indeed, for any morphism $f$, and $x \in PX$, we can write:

$$\tilde{f}(x) = \sup_{N \in \mathbb{N}} \sum_{b \in \mathbb{N}} \left( \sum_{\mu \text{ with } \mu(b) \leq N} f_{\mu, b} \cdot x^{\mu} \right) \cdot e_b$$

As the Kleisli category of the comonad ! in a Seely category, $\text{Pcoh}$ is a cartesian closed category. We give here explicitly the construction the product and arrow constructs: if $X$ and $Y$ are PCSs, $X \otimes Y$ is defined by $[X \otimes Y] = M_{\{1\}}(\{X\} \times \{Y\})$ and $P(X \otimes Y) = \text{Pcoh}(X, Y)$. If $(X_i)_{i \in I}$ is a family of PCSs, $\prod_{i \in I} X_i$ is defined by $\prod_{i \in I} X_i = \cap_{i \in I} [X_i] \times [X_i]$ and $PX = \{x \in \mathbb{R}_+^{|\prod_{i \in I} X_i|} \mid \forall i \in I, \pi_i(x) \in PX_i\}$, where $\pi_i(x)_{a} = x_{i,a}$.

### 4.4 A fully faithful functor $F : \text{Pcoh}_K \to \text{Cstab}_m$.

Recall that Definition 4.4 gave a way to see a PCS as a cone. Moreover, as stated in Proposition 4.9 below, a morphism in $\text{Pcoh}_K$ can also be seen as a stable function, in the sense that $\tilde{E}^{X,Y} \subseteq \text{Cstab}(\text{C}_X, \text{C}_Y)$.

**Proposition 4.9.** Let be $f \in \text{Pcoh}_K(X, Y)$. Then $\tilde{f}$ is a stable function from $\text{C}_X$ to $\text{C}_Y$.

**Proof.** We know from [4] that $\tilde{f} : PX \to PY$ is sequentially continuous with respect to the orders $\leq_{\text{C}_X} \leq_{\text{C}_Y}$. Moreover $f$ is pre-stable: it comes from the fact that $f$ can be written as a power series with all its coefficients non-negative. Finally, we have to show that $f(\tilde{\text{C}_X}) \subseteq \text{C}_Y$. Since $\tilde{\text{C}_X} = PX$, $\tilde{\text{C}_X} = PX$, and moreover $f$ is a morphism in $\text{Pcoh}_K(X, Y)$, we see that the result holds. \hfill \Box

Thus we can define a functor $F : \text{Pcoh}_K \to \text{Cstab}$, by taking $F X = \text{C}_X$, and $F f = \tilde{f}$. Our goal now is to show that $F$ is full and faithful, which will make $\text{Pcoh}_K$ a full subcategory of $\text{Cstab}$. As mentioned before, it was shown in [4] that $\tilde{\cdot}$ is a bijection from $\text{Pcoh}_K(X, Y)$ to $\tilde{E}^{X,Y}$. It tells us directly that $F$ is indeed faithful.

In the remainder of this section, we are going to show that $F$ is actually also full, hence makes $\text{Cstab}$ a conservative extension of $\text{Pcoh}_K$.

In the following, we fix $X$ and $Y$ two PCSs, and $g \in \text{Cstab}(\text{C}_X, \text{C}_Y)$. Our goal is to show that there exists $f \in \text{Pcoh}_K(X, Y)$ such that $f = g$. First, recall that we have shown in Lemma 4.5 that for every PCS $Z$, the cone $\text{C}_Z$ is directed complete. It means that all results in Section 3.3 can be used here: in particular, $g$ has higher-order derivatives $D^n g$, which makes Definition 4.10 below valid.

**Definition 4.10.** We define $f \in \mathbb{R}_+^{|M_{\{1\}}(\{X\}) \times |Y|}$ by taking:

$$f_{[a_1, \ldots, a_k], b} = \frac{\alpha_k}{(\alpha_1 a_1 + \cdots + \alpha_k a_k)} \left( D^k g(0 | e_{a_1} \cdots e_{a_k}) \right)_b \in \mathbb{R}_+^{|X|}$$

where $\alpha_k = \#((c_1, \ldots, c_k) \in |X|^k | \mu = (c_1, \ldots, c_k))$.

We have to show now that $f \in \text{Pcoh}_K(X, Y)$, and that $\tilde{f}$ coincides with $g$ on $PX$. The key observation here is that we have actually built $f$ in such a way that it is going to coincide with $T g(0 | \cdot)$—the Taylor series of $g$ defined in Definition 3.22. We first show it for the elements of $PX$ with finite support, by using finite additivity of the $D^k g(0 | \cdot)$.

**Lemma 4.11.** Let be $x \in PX$, such that $\text{Supp}(x) = \{a \in PX \mid x_a > 0\}$ is finite. Then it holds that $f \cdot x^!$ is finite, and moreover $f \cdot x^! = T g(0 | x)$.
Proof. Let $A = \{a_1, \ldots, a_m\} \subseteq |X|$ be the set $\text{Supp}(x)$. For any $b \in |Y|$, we can deduce from the definition of $f$ that:

$$(f \cdot x)^h = \sum_{k=0}^{\infty} \sum_{\mu \in [c_1, \ldots, c_k] \in M_k^X(A)} \frac{a_\mu}{k!} D^k g(0 | e_{c_1}, \ldots, e_{c_k}) b \cdot x^\mu.$$  

Looking at the definition of $a_\mu$, we see that this implies:

$$(f \cdot x)^h = \sum_{k=0}^{\infty} \frac{1}{k!} D^k g(0 | e_{c_1}, \ldots, e_{c_k}) b \cdot \prod_{i=1}^{k} x_{c_i} = T g(0 | x).$$

By Lemma 3.17, we know that $D^k g(0 | \cdot)$ is $k$-linear. As a consequence, and since $x = \sum_{i=m}^{\infty} x_{c_i} e_{c_i}$ and that moreover $A$ is finite, we see that (8) implies the result:

$$(f \cdot x)^h = T g(0 | x). \quad \square$$

We are now going to apply the generalized Bernstein’s theorem, as stated in Proposition 3.26, to the stable function $g$ from $\mathcal{F} X$ to $\mathcal{F} Y$. It tells us that:

$$\forall x \in \mathcal{B}^0 C_X, \quad g(x) = T g(0 | x). \quad (9)$$

Combining (9) with Lemma 4.11, we obtain that:

$$\forall x \in \mathcal{B}^0 C_X \text{ with } \text{Supp}(x) \text{ finite, } f \cdot x' = g(x). \quad (10)$$

We can now use (10) to show that $\tilde{f}$ and $g$ coincide on $PX$: the key point is that the subset of elements in $PX$ of norm smaller than 1 and finite support is dense, and that moreover $g$ is Scott-continuous.

Lemma 4.12. $\forall x \in PX$, $f \cdot x' = g(x)$, and moreover $f \in \text{Pcoh} (X, Y)$.

Proof. Let be $x \in PX$. It is easy to see that there exists a non-decreasing sequence $y_n$, with $x = \sup_{n \in \mathbb{N}} y_n$, and for every $n$, $\|y_n\|_{C_X} < 1$, and $y_n$ has finite support. For every $y_n$, we can use (10), and we see that $g(y_n) = f \cdot y_n^\prime$. Since $g$ is a morphism in $\text{Cstab}$, $g$ is sequentially Scott-continuous, hence:

$$g(x) = \sup_{n \in \mathbb{N}} g(y_n). \quad (11)$$

Moreover, we know from [4] that both $x \mapsto x'$ and $x \mapsto u \cdot x$ are Scott continuous. It means that:

$$f \cdot x' = \sup_{n \in \mathbb{N}} f \cdot y_n. \quad (12)$$

Combining (11) and (12), we obtain that $f \cdot x' = g(x)$.

Since $g(\mathcal{B} C_X) \subseteq \mathcal{B} C_Y$, it tells us also that $f(\mathcal{B} X) \subseteq PY$, and using Lemma 4.8 we see that $f \in \text{Pcoh} (X, Y)$. \quad \square

Since we have indeed been able to show in Lemma 4.12 that for any fixed stable function $g$ in $\text{Cstab}(\mathcal{F} X, \mathcal{F} Y)$, there exists a $f \in \text{Pcoh} (X, Y)$ such that $\mathcal{F} f = g$, we have indeed shown that $\mathcal{F}$ is full.

4.5 $\mathcal{F}$ preserves the cartesian structure.

We want now to give more guarantee on the functor $\mathcal{F}$: we want to show that it is a cartesian closed functor, meaning that it embeds the cartesian closed category $\text{Pcoh}$ into the cartesian closed category $\text{Cstab}$ in such a way that:

- $\mathcal{F}$ preserves the product: for every family $(X_i)_{i \in I}$ of PCSs, $\mathcal{F} \left( \prod_{i \in I} X_i \right)$ is isomorphic to $\text{Cstab}_i \mathcal{F} X_i$;
- $\mathcal{F}$ preserves function spaces: for every $X, Y$ PCSs, $\mathcal{F} (X \Rightarrow Y)$ is isomorphic to $\mathcal{F} X \Rightarrow \mathcal{F} Y$.

Lemma 4.13. $\mathcal{F}$ preserves cartesian products.

Proof. We fix a family $\mathcal{F} = (X_i)_{i \in I}$ of PCSs. In order to construct an isomorphism, we have a canonical candidate, given by:

$$\Psi^{\mathcal{F}} = (\pi_i | i \in I) \in \text{Cstab}(\mathcal{F} \left( \prod_{i \in I} X_i \right), \prod_{i \in I} Cstab_{\mathcal{F} X_i}).$$

We are now going to show that $\Psi^{\mathcal{F}}$ is an isomorphism. Looking at the definition of cartesian product in $\text{Pcoh}$ defined in Section 4.3, and the one of cartesian product in $\text{Cstab}$, defined in Section 3.2, we see that for every $x \in \mathcal{B} \mathcal{F} \left( \prod_{i \in I} \text{Pcoh}_{\mathcal{F} X_i} \right)$:

$$\Psi^{\mathcal{F}} (x) = (y_i)_{i \in I} \quad \text{where } \forall i \in I, y_i \in |X_i|, (y_i)_a = x_{(i,a)}.$$  

We want now to show that $\Psi^{\mathcal{F}}$ has an inverse. The only candidate is $\Theta^{\mathcal{F}} : y \in \mathcal{B}\mathcal{F} \left( \prod_{i \in I} \text{Cstab}_{\mathcal{F} X_i} \right) \mapsto \Theta(y) \in \left( \mathcal{F} \left( \prod_{i \in I} \text{Pcoh}_{\mathcal{F} X_i} \right) \right)_{\mathcal{F} X_i}$, defined by: $\forall i \in I, a \in |X_i|, \Theta(y)_{a,b} = (y_i)_a$. We see immediately that $\Theta^{\mathcal{F}}$ is linear, hence pre-stable, and that moreover it is Scott-continuous. Besides, it is also non-expansive: indeed for any $y \in \mathcal{B} \mathcal{F}$ it holds that:

$$||\Theta(y)||_{\mathcal{F} \left( \prod_{i \in I} \text{Pcoh}_{\mathcal{F} X_i} \right)} = ||y||_{\text{Cstab} \mathcal{F} X_i}.$$

We show now the non-expansiveness of $\Theta^{\mathcal{F}}$:

$$||\Theta(y)||_C = \inf \left\{ \frac{1}{r} r \cdot \Theta(y) \in \mathcal{P} \left( \prod_{i \in I} X_i \right) \right\}$$

$$= \inf \left\{ \frac{1}{r} \forall i \in I, r \cdot y_i \in PX_i \right\} = \sup \{||y_i||_{\mathcal{F} X_i} = ||y||_{\text{Cstab} \mathcal{F} X_i} \}.$$  

As a consequence, $\Theta^{\mathcal{F}}$ is a morphism in $\text{Cstab}(D, C)$, and the result folds. \quad \square

Lemma 4.14. $\mathcal{F}$ preserves the function space.

Proof. Let $X, Y$ two PCSs. As previously, there is a canonical candidate for the isomorphism: we define $\Theta^{X,Y}$ as the currying in $\text{Cstab}$ of the morphism:

$$\Theta^{X,Y} : \mathcal{F} (X \Rightarrow Y) \times \mathcal{F} X \xrightarrow{\Theta^{X,Y}} \mathcal{F} (X \Rightarrow Y \times X) \xrightarrow{\Theta^{X,Y}} \mathcal{F} Y,$$

where $\Theta^{X,Y}$ is as defined in the proof of Lemma 4.13 above.

Unfolding the definition, we see that actually: $\Theta^{X,Y} : f \in \mathcal{B}\mathcal{F} (X \Rightarrow Y) \mapsto f \in \left( \mathcal{F} X \Rightarrow \mathcal{F} Y \right)$. Since we have shown that $\mathcal{F}$ is full and faithful, we can consider $\Xi^{X,Y}$ the inverse function of $\Theta^{X,Y}$. Recall from the proof of the fullness of $\mathcal{F}$ in Section 4.4 that for every $\mu = \{a_1, \ldots, a_k\} \in M_{L(X)}$, and $b \in |Y|$:

$$\Xi^{X,Y} (f)_{\mu, b} = \frac{a_1 a_2 \ldots a_k}{k!} \left( D^k f(0 | e_{a_1}, \ldots, e_{a_k}) \right)_b.$$
Probabilistic Stable Functions on Discrete Cones are Power Series.

Recall from Lemma 3.17 that for any \( \tilde{u} \in \mathcal{C}(X,Y) \), the function \( f \in \text{Cstab}(\mathcal{F}X, \mathcal{F}Y) \mapsto D^X f(x | \tilde{u}) \in \mathcal{F}Y \) is linear and Scott-continuous. As a consequence, \( \Xi^X, Y \) too is linear and Scott-continuous.

To know that \( \Xi^X, Y \) is stable, we have still to show that it is bounded: we are actually going to show that it preserves the norm. Indeed, for every \( f \in \mathcal{F}(\mathcal{F}X \Rightarrow \mathcal{F}Y) \), we see using the definition of the norm on a cone obtained from a PCS (see Definition 4.4), that:

\[
\| \Xi^X, Y (f) \|_{\mathcal{F}(X \Rightarrow Y)} = \inf \left\{ \frac{1}{r} \mid r \cdot \Xi^X, Y (f) \in P(X \Rightarrow Y) \right\}
\]

(13)

It was shown in [4] that:

\[
r \cdot \Xi^X, Y (f) \in P(X \Rightarrow Y) \iff \forall x \in PX, (r \cdot \Xi^X, Y (f)) \cdot x' \in PY.
\]

(14)

We see that \( (r \cdot \Xi^X, Y (f)) \cdot x' = r \cdot f(x) \) since \( \Xi^X, Y \) has been defined as the inverse of \( YX \). It means that we can rewrite (14) as:

\[
r \cdot \Xi^X, Y (f) \in P(X \Rightarrow Y) \iff \forall x \in PX, r \cdot f(x) \in PY.
\]

(15)

Since for every PCS \( Z \), it holds that \( PZ = \mathcal{F}Z \), we can now use (15) to rewrite (13) as:

\[
\| \Xi^X, Y (f) \|_{\mathcal{F}(X \Rightarrow Y)} = \inf \left\{ \frac{1}{r} \mid \forall x \in \mathcal{F}X, r \cdot f(x) \in \mathcal{F}Y \right\}
\]

(16)

Looking now at the definition of the norm in the cone \( \mathcal{F}X \Rightarrow \mathcal{F}Y \), we can complete the proof using (16) and the homogeneity of the norm. Indeed:

\[
\| \Xi^X, Y (f) \|_{\mathcal{F}(X \Rightarrow Y)} = \inf \left\{ \frac{1}{r} \mid \| r \cdot f \|_{\mathcal{F}X \Rightarrow \mathcal{F}Y} \leq 1 \right\}
\]

\[
= \inf \left\{ \frac{1}{r} \mid r \cdot f \|_{\mathcal{F}X \Rightarrow \mathcal{F}Y} \leq 1 \right\}
\]

\[
= \| f \|_{\mathcal{F}X \Rightarrow \mathcal{F}Y}
\]

(17)

As a direct consequence of Lemma 4.13 and Lemma 4.14, we can state the following theorem:

**Theorem 4.15.** \( \mathcal{F} \) is full and faithful, and it respects the cartesian closed structures.

### 5 Adding Measurability Requirements

In [8], the authors developed a sound and adequate model of \( \text{PCF}_{\text{sample}} \) based on stable functions. However, as explained in more details in [8], they need to add to their morphisms some measurability requirements, both on cones and on functions between them, since the denotational semantics of the \( \text{let}(X, M, N) \) construct uses an integral, to modelize the fact that \( M \) is evaluated before being passed as argument to \( N \).

We call measurable functions \( \mathbb{R}^n \rightarrow \mathbb{R}^k \) the functions measurable when both \( \mathbb{R}^n \) and \( \mathbb{R}^k \) are endowed with the Borel \( \Sigma \)-algebra associated with the standard topology of \( \mathbb{R} \). The relevant properties of the class of measurable functions \( \mathbb{R}^n \rightarrow \mathbb{R}^k \) is that they are closed by arithmetic operations, composition, and pointwise limit, see for example Chapter 21 of [17].

5.1 The category \( \text{Cstab}_n \)

\( \text{Cstab}_n \) is built as a refinement of the category \( \text{Cstab} \). The objects of \( \text{Cstab}_n \) are going to be complete cones, endowed with a family of measurability tests.

If \( C \) is a complete cone, we denote by \( C' \) the set of linear and Scott-continuous functions \( C \rightarrow \mathbb{R}_+ \).

**Definition 5.1.** A measurable cone (MC) is a pair consisting of a cone \( C \), and a collection of measurability tests \( \mathcal{M}(C)_n \subseteq \mathcal{E}(C)_n \), where for every \( n, \mathcal{M}(C)_n \subseteq C^{\mathbb{R}^n} \), such that:

- for every \( n \in \mathbb{N}, 0 \in \mathcal{M}(C)_n \);
- for every \( n, p \in \mathbb{N} \), if \( I \in \mathcal{M}(C)_n \), and \( h : \mathbb{R}^p \rightarrow \mathbb{R}^n \) is a measurable function, then \( I \circ h \in \mathcal{M}(C)_n \);
- for any \( l \in \mathcal{M}(C)_n \), and \( x \in C \), the function \( u \in \mathbb{R}^n \mapsto l(u)(x) \in \mathbb{R} \) is measurable.

**Example 5.2.** (from [8]). Let \( X \) be a measurable space. We endow the cone of finite measures \( \text{Meas}(X) \) with the family \( \mathcal{M}(X) \) of measurable tests defined as:

\[
\mathcal{M}(X) = \{ \epsilon \in \mathbb{R} \mid \epsilon \in \Sigma_X \} \quad \text{where} \quad \epsilon_U(f) = \mu(U),
\]

where \( \Sigma_X \) is the set of all measurable subsets of \( X \). Observe that in this case, the measurable tests correspond to the measurable sets.

We define now measurable paths, which are meant to be the admissible ways to send \( \mathbb{R}^n \) into an MC \( C \).

**Definition 5.3.** (Measurable Paths). Let be \( C, (\mathcal{M}(C)_n)_{n \in \mathbb{N}} \) a measurable cone. A measurable path of arity \( n \) at \( C \) is a function \( \gamma : \mathbb{R}^n \rightarrow C \), such that \( \gamma(\mathbb{R}^n) \) is bounded in \( C \), and for every \( k \in \mathbb{N} \), for every \( l \in \mathcal{M}(C)_k \), the function \( (\tilde{f}, \tilde{s}) \in \mathbb{R}^{k+n} \mapsto l(\tilde{f}(\tilde{s})) \in \mathbb{R}_+ \) is a measurable function.

We denote \( \text{Paths}^n(C) \) the set of measurable paths of arity \( n \) for the MC \( C \). Using measurable paths, the authors of [8] add measurability requirements to their definition of stable functions.

**Definition 5.4.** Let be \( C, D \) two MCs. A stable function \( f : \mathcal{E} \rightarrow D \) is measurable if for all \( \gamma \in \text{Paths}^n(C) \) such that \( \gamma(\mathbb{R}^n) \subseteq \mathcal{E} \), it holds that \( f \circ \gamma \in \text{Paths}^n(D) \).

The category \( \text{Cstab}_n \) is therefore the category whose objects are MCs, and whose morphisms are measurable stable functions between MCs.

**Example 5.5.** Recall the function \( \lfloor \text{real} \rfloor_{\text{Cstab}_n} \) defined in Section 2:

\[
\lfloor \text{real} \rfloor_{\text{Cstab}_n} : \mu \in \text{Meas}(\mathbb{N}) \mapsto (U \in \Sigma_\mathbb{R} \mapsto \sum_{n \in \mathbb{N} \subseteq U} \mu(n)) \in \text{Meas}(\mathbb{R})
\]

We can see that \( \lfloor \text{real} \rfloor_{\text{Cstab}_n} \) is a measurable function from \( \text{Meas}(\mathbb{N}), \mathcal{M}(\mathbb{N}) \) into \( \text{Meas}(\mathbb{R}), \mathcal{M}(\mathbb{R}) \). Moreover it is linear and Scott-continuous, and norm-preserving, which makes it a morphism in \( \text{Cstab}_n \).

Observe that it would not be measurable, if we endowed \( \text{Meas}(\mathbb{N}) \) with for instance \( [0] \) as measurability tests instead of \( \mathcal{M}(\mathbb{N}) \): that is the reason why we take \( \text{Meas}(\mathbb{N}), \mathcal{M}(\mathbb{N}) \) as the denotational semantics of the type \( N \).

In [8], \( \text{Cstab}_n \) is endowed with a cartesian closed structure respecting the underlying \( \text{Cstab} \) structure. It means that we have only to specify which measurability tests we take on \( C \Rightarrow D \) and \( \prod_{i \in I} C_i \) (for \( I \) finite set). The construction is recalled in Figure 1.

5.2 Peoh: is a full subcategory of \( \text{Cstab}_n \)

We want now to convert the functor \( \mathcal{F} : \text{Peoh} \rightarrow \text{Cstab} \) into a functor \( \mathcal{F}^m : \text{Peoh} \rightarrow \text{Cstab}_n \). To build \( \mathcal{F}^m \), we are going to endow each \( \mathcal{F}X \) with measurability tests, in such a way that \( \mathcal{F}(f) \) will be a measurable stable function for any morphism \( f \in \text{Peoh} \).
We see that the \( C \) verified, and so is measurable from \( C_{\text{stab}} \). Closed by finite sum and pointwise limit, we see that it holds that for every \( M \) not the same measurable tests: cones of \( \text{Recall from the definition of section 5.3.} \)
also what happens at higher-order types, as we will explain in below, where we consider \( F \) measurable paths.
ability tests may be isomorphic in combinations), and moreoverScott-continuous (since we know we see that we have only to show that the \( C_{\text{stab}} \)-isomorphisms used in the proofs Lemmas 4.13 and 4.14 are also morphisms in \( C_{\text{stab}} \). Observe that it is immediate that the \( \Phi^X \) and \( \gamma^X \) are also morphisms in \( C_{\text{stab}} \); indeed we have defined them canonically by using the structural morphisms linked to the cartesian structure of \( C_{\text{stab}} \), which are the same in \( C_{\text{stab}} \) (see [8]).

**Lemma 5.11.** For all \( \mathcal{X} = \{ X_i \}_{i \in I} \) a finite family of PCSs,
\[
\Theta^\mathcal{X} \in C_{\text{stab}}( \bigcap_{i \in I} F^m X_i, \bigcap_{i \in I} X_i ).
\]

**Proof.** Since we already know that \( \Theta^\mathcal{X} \) is a morphism in \( C_{\text{stab}} \), we have only to show that it is measurable, i.e. that it preserves measurable paths. Let \( \gamma \) be in \( \text{Paths}^n(\bigcap_{i \in I} C_{\text{stab}} F^m X_i) \). We have to show that \( \Theta^\mathcal{X} \circ \gamma \) is a measurable path for \( \bigcap_{i \in I} F^m (\bigcap_{i \in I} X_i) \). Recall that Lemma 5.7 gives us a characterization of such paths. Since both \( \Theta^\mathcal{X} \) and \( \gamma \) are bounded, it is immediate that \( \Theta^\mathcal{X} \circ \gamma \) is bounded too. Now we show that for all \( (i, a) \in [\bigcap_{i \in I} X_i] \), \( (\Theta^\mathcal{X} \circ \gamma)(i, a) \) is measurable.

By looking at the definition of \( \Theta^\mathcal{X} \), we see that \( (\Theta^\mathcal{X} \circ \gamma)(i, a) \) is measurable.
We see that we can construct a measurable test \( m \in M^m(\bigcap_{i \in I} C_{\text{stab}} F^m X_i) \) such that \( (\gamma(i))(a) = m(\gamma(i)) \); it is enough to take \( m = \oplus_{j \in I} l_j \), with \( l_j = 0 \) if \( j \neq i \), and \( l_i = e_{a} \). Since \( y \) is a measurable tests, it means that \( \gamma \in R^n \Rightarrow m(\gamma(i))(a) \in R_+ \) is measurable, and the result folds.

**Lemma 5.12.** For all \( X, Y \) PCSs, it holds that
\[
\Xi^X \gamma \in C_{\text{stab}}(F^m X \Rightarrow F^m Y, F(X \Rightarrow Y)).
\]

**Proof.** We have to show that \( \Xi^X \gamma \) preserves measurable paths. Let \( \gamma \) be in \( \text{Paths}^n(F^m X \Rightarrow F^m Y) \). We fix \( \mu \in M^m(\mathcal{X}) \), and \( b \in \mathcal{Y} \). Our goal is to show that \( (\Xi^X \gamma)(\mu, b) : R^n \Rightarrow R_+ \) is measurable.

\[
M^n(\bigcap_{i \in I} C_i) = \{ \bigcup_{l_i \in I} l_i \mid \forall m, l_i \in M^n(C_i) \}
\]
\[
M^n(\bigcap_{i \in I} C_i) = \{ \gamma \mid m \gamma \in \text{Paths}^n(\bigcap_{i \in I} C_i, m \in M^n(\bigcap_{i \in I} C_i)) \}
\]
with \( \{ \bigcup_{l_i \in I} l_i(\gamma) \mid (x_i, \gamma) \in \mathcal{X} \} = \sum_{l_i \in I} l_i(\gamma)(x_i) \in R_+ \).

\[
\text{and } \gamma \Rightarrow m(\gamma)(f) = m(\gamma)(f(\gamma(f))).
\]

**Proof.** We have to show that \( \mathcal{F} \gamma \) preserves measurable paths. Let \( \gamma \) be in \( \text{Paths}^n(\bigcap_{i \in I} C_i, m \in M^n(\bigcap_{i \in I} C_i)) \). We know that \( \mathcal{F} \gamma \) is a measurable path for \( \bigcap_{i \in I} F^m (\bigcap_{i \in I} X_i) \). We see now that we can construct a measurable test \( m \in M^m(\bigcap_{i \in I} C_{\text{stab}} F^m X_i) \) such that \( m(\gamma(i))(a) = m(\gamma(i))(a) \); it is enough to take \( m = \oplus_{j \in I} l_j \), with \( l_j = 0 \) if \( j \neq i \), and \( l_i = e_{a} \). Since \( y \) is a measurable tests, it means that \( \gamma \in R^n \Rightarrow m(\gamma(i))(a) \in R_+ \) is measurable, and the result folds.

\[
\text{and } \gamma \Rightarrow m(\gamma)(f) = m(\gamma)(f(\gamma(f))).
\]

**Example 5.8.** Recall from the definition of \( \mathcal{F} \) that the underlying cones of \( \mathcal{F}^m \) are the same. We have not the same measurable tests:
\[
M^n(\mathcal{F}^m P_{\text{Proh}}) = \{ \epsilon_n \mid n \in \mathbb{N} \}; \quad M^n(\text{Meas}(\mathcal{N})) = \{ \epsilon_{U} \mid U \subseteq \mathbb{N} \}.
\]

We consider the identity function \( \text{id} : x \in P^{\text{Proh}}_{\text{Proh}} \mapsto (A \mapsto \sum_{A \in A} u) \in \text{Meas}(\mathcal{N}) \). Let \( \gamma \) be in \( \text{Paths}^n(\mathcal{F}^m P_{\text{Proh}}) \). We want to show that for every \( U \subseteq \mathbb{N} \):
\[
\text{id}(\gamma) = \sum_{n \in U} e_n(\gamma) \in R_+.\]

We see that \( \epsilon_U(\text{id}(\gamma)) = \sum_{n \in U} e_n(\gamma) \). Since \( \gamma \) is in \( \text{Paths}^n(\mathcal{F}^m P_{\text{Proh}}) \) it holds that for every \( n \in \mathbb{N} \), the function \( u = R^n \rightarrow \epsilon_U(\gamma) \in R_+ \) is measurable. Since the class of measurable functions are closed by finite sum and pointwise limit, we see that \( (u \in R^n \rightarrow \epsilon_U(\text{id}(\gamma)) \in R_+) \) is measurable too, and therefore \( \text{id} \) is a morphism in \( C_{\text{stab}} \). We show in the same way that its inverse is measurable too, and we have the result.

**Lemma 5.9.** Let \( X, Y \) be two PCSs, and \( f \in P_{\text{Proh}}(X, Y) \). Then \( \mathcal{F} f \) is measurable from \( C_{\text{X}} \) into \( C_{\text{Y}} \).
We recall here a classical result of real analysis on power series, \( \ell \) of functions from a bounded interval \( \mathbb{R}^m \times X \rightarrow \mathbb{R}^m \), and therefore:
\[
(f, \bar{u}) \in \mathbb{R}^{p+m} \mapsto (e_b \triangleright (f')((\bar{u})))(\bar{u}) \in \mathbb{R}_+ \text{ is measurable}. \] (17)
We are going to apply (17) to a particular measurable function \( f \). Let \( p_b \) be the cardinality of \( \text{Supp}(\mu) \), and \( \{a_1, \ldots, a_p\} = \text{Supp}(\mu) \). We define \( l^p : \mathbb{R}^p \rightarrow \mathbb{R}^m \times X \) as:
\[
l^p : r \in \mathbb{R}^p \mapsto \begin{cases} \sum_{1 \leq i \leq m} r_i \cdot e_{a_i} & \text{if } r_i \geq 0 \forall i \text{ and } \sum_{1 \leq i \leq m} r_i \leq 1; \\
0 & \text{otherwise}. \end{cases}
\]
We see that \( l^p(\mathbb{R}^p) \) is bounded in \( \mathbb{R}^m \times X \), and moreover for every \( a \in [X] \), the function \( r \in \mathbb{R}^p \mapsto l^p(r)_a \in \mathbb{R}^p \) is measurable. Using the characterization of \( \text{Paths}^\ell(\mathbb{R}^m \times X) \) in Lemma 5.7, we see that \( l^p \) is in \( \text{Paths}^\ell(\mathbb{R}^m \times X) \). Thus we can apply (17) with \( l = l^p \). Observe that
\[
(\ell_b \triangleright l^p(r)(\bar{u})) = (\gamma(\bar{u})(l^p(r)))^b.
\]
Therefore (17) tells us that \( \phi^{\mu, b} : \mathbb{R}^{p+m} \rightarrow \mathbb{R}_+ \), is measurable, with \( \phi^{\mu, b} \) defined as \( \phi^{\mu, b} : (f, \bar{u}) \in \mathbb{R}^{p+m} \mapsto (\gamma(\bar{u})(l^p(f)))^b \in \mathbb{R}_+ \). We define \( J \subseteq \mathbb{R}^p \) as \( [0, p\mathbb{R}] \). We are going to look at the restriction of the function \( \phi^{\mu, b} \) to \( J \times \mathbb{R}^m \), indeed we are going to show that \( \phi^{\mu, b} \) has partial derivatives on that interval. We define \( \psi^{\mu, b} : J \times \mathbb{R}^m \rightarrow \mathbb{R}_+ \), as the restriction of \( \phi^{\mu, b} \) to \( J \times \mathbb{R}^m \). Since \( \phi^{\mu, b} \) is a measurable function, and \( J \times \mathbb{R}^m \) a measurable subset of \( \mathbb{R}^{p+m} \), \( \psi^{\mu, b} \) also is measurable.

Lemma 5.13. For every multiset \( \nu \in M_f([1, \ldots, p]) \), there exists an interval \( K \) of the form \([1, c] \) such that the partial derivative
\[
\frac{\partial^\nu \psi^{\mu, b}}{\partial r(\nu)} \bigg|_{r=0} : K \times \mathbb{R}^m \rightarrow \mathbb{R}_+ \text{ exists}, and moreover:
\[
\frac{\partial^\nu \psi^{\mu, b}}{\partial r(\nu)}(\bar{u}, \bar{u}) = \mathbb{E}^{X, Y}(\gamma(\bar{u}))_{\nu, b} \cdot \prod_{1 \leq i \leq \nu} \nu(i)!
\]
Proof. Since \( l^p(\mathbb{R}) = \sum_{1 \leq i \leq p} r_i \cdot e_i \) for \( \bar{r} \in J \), we see that:
\[
\phi^{\mu, b}(\bar{r}, \bar{u}) = \sum_{\nu \in M_f([1, \ldots, p])} \mathbb{E}^{X, Y}(\gamma(\bar{u}))_{\nu, b} \cdot \prod_{1 \leq i \leq \nu} \nu(i)!
\]
For a fixed \( \bar{x} \), we can see it as a generalization of entire series in real analysis. There are well-known results about the differentiation of such series: for instance, a uniformly convergent entire series is differentiable on its (open) domain of convergence. Here, we are going to show the counterpart of some properties on entire series, on what we call \textit{multisets series of p real variables}: those are the series of the form
\[
S(\bar{r}) = \sum_{\nu \in M_f([1, \ldots, p])} a_\nu \cdot \bar{r}^\nu \text{ where } \bar{r} \in \mathbb{R}^p.
\]
First, we observe that for each \( \bar{r} \), we can look at \( S(\bar{r}) \) as an infinite sum over natural numbers:
\[
S(\bar{r}) = \sum_{n \in \mathbb{N}} \sum_{\nu \in M_f([1, \ldots, m]) | \mu(\nu) = n} a_\nu \cdot \bar{r}^\nu.
\]
We recall here a classical result of real analysis on power series, that we will use in the following.

Lemma 5.14. [Derivation of a series] Let \( f_n : I \rightarrow \mathbb{R} \) be a sequence of functions from a bounded interval \( I \). We suppose that \( f(x) = \sum_{n \in \mathbb{N}} f_n(x) \) is convergent for every \( x \in I \), and moreover for each \( n \in \mathbb{N} \), \( f_n \) is derivable and \( \sum_{n \in \mathbb{N}} \int f_n(x) \) is uniformly convergent on \( I \). Then \( f \) is derivable, and moreover \( f' = \sum_{n \in \mathbb{N}} f'_n \).
Lemma 5.16. Let \( S(\tilde{r}) = \sum_{v \in M_f(\tilde{r})} a_v \cdot (\tilde{r})' \) with \( a_v \geq 0 \).
We suppose \( S \) convergent on \( I = [-b, b] \), with \( b > 0 \). Then for every
multiset \( v \in M_f((1, \ldots, p)) \), there exists 0 < \( b \leq \alpha \), such that, when
we define \( g : \tilde{r} \in [-b, b] \rightarrow S(\tilde{r}) \), the partial higher-order derivative
\( \frac{\partial g(v)}{\partial \tilde{r}^{(i)} \ldots \partial \tilde{r}^{(p)}} \) exists, and moreover:
\[
\frac{\partial g(v)}{\partial \tilde{r}^{(i)} \ldots \partial \tilde{r}^{(p)}} \left( \eta \right) = \sum_{\eta \in M_f((1, \ldots, p))} a_{\eta+v} \cdot \eta^p \cdot \prod_{1 \leq i \leq p} \left( \frac{(\eta + v)(i)!}{\eta(i)!} \right).
\]

Proof. The proof is by induction on \( \text{card}(v) \), and uses Lemma 5.15.

It is clear that the result holds for \( v = \emptyset \). Now, suppose that it
holds for every \( v \) of cardinality \( N \). Let \( k \) be a multiset of cardinality
\( N + 1 \), and we take \( v \) and \( i \) such that \( k = v + [i] \). By the induction
hypothesis, there exists \( c > 0 \) such that, when we define \( g : \tilde{r} \in [-c,c] \rightarrow S(\tilde{r}) \), the partial derivative \( \frac{\partial g(v)}{\partial \tilde{r}^{(i)} \ldots \partial \tilde{r}^{(p)}} \) exists, and
is equal to:
\[
T(\tilde{r}) = \sum_{\eta \in M_f((1, \ldots, p))} a_{\eta+v} \cdot \eta^p \cdot \prod_{1 \leq i \leq p} \left( \frac{(\eta + v)(i)!}{\eta(i)!} \right).
\]

We see we can apply Lemma 5.15 with \( T \) as multiset series, and
\( I = [-c, c] \). It means that there exist 0 < \( \alpha \leq c \), such that \( \frac{\partial T}{\partial \tilde{r}} \)
exists, and
\[
\frac{\partial T}{\partial \tilde{r}^{(i)} \ldots \partial \tilde{r}^{(p)}} = \sum_{\eta \in M_f((1, \ldots, p))} a_{\eta+v} \cdot \eta^p \cdot \prod_{1 \leq i \leq p} \left( \frac{(\eta + v)(i)!}{\eta(i)!} \right).
\]

We end the proof of Lemma 5.13 by using Lemma 5.16 for each
\( \tilde{u} \in \mathbb{R}^n \) on the multiset series given by
\( S_\rho(\tilde{r}) = \sum_{v \in M_f((1, \ldots, p))} \left( \exists X, Y \in \mathcal{Y}(\tilde{u}) \right) \rho(\tilde{r}, \tilde{u}) \cdot \tilde{r}^p \).

We see that it is indeed absolutely convergent on \( I = [-\frac{1}{2}, \frac{1}{2}] \), using
the fact that for \( \tilde{r} \in [0, \frac{1}{2}] \), \( S_\rho(\tilde{r}) = \psi^{k,b}(\tilde{r}, \tilde{u}) \) for \( \tilde{r} \in \mathbb{R}^p \). □

We can now end the proof of Lemma 5.12. Indeed, since the class
of real-valued measurable functions is closed on addition, multiplication
by a scalar and pointwise limit, and that \( \psi^{k,b} : K \times \mathbb{R}^p \rightarrow \mathbb{R} \)
is measurable, it holds that the partial derivatives (which they exist)
are measurable too. Indeed, observe that \( \frac{\partial \psi^{k,b}}{\partial r}((r_1, \tilde{u}), \tilde{u}) = \lim_{n \rightarrow \infty} \frac{\partial f_n((r_1, \tilde{r}), \tilde{u})}{\partial r} \), with
\( f_n((r_1, \tilde{r}), \tilde{u}) = n \cdot f((r_1, \tilde{r}), \tilde{u}) - f((r_1, \tilde{r}), \tilde{u}) \).

As a consequence, applying Lemma 5.13 with \( v = \emptyset \) leads us to:
\( \tilde{u} \in \mathbb{R}^n \mapsto \frac{\partial \psi^{k,b}(\tilde{r})}{\partial \tilde{r}^{(i)} \ldots \partial \tilde{r}^{(p)}}(\tilde{u}, \tilde{u}) \) is measurable.

Therefore (again by Lemma 5.13), \( \tilde{u} \in \mathbb{R}^n \mapsto \exists X, Y \in \mathcal{Y}(\tilde{u}) \rho(\tilde{r}, \tilde{u}) \cdot \tilde{r}^p \)
is measurable, and the result follows. □