Coloring & Signed Cycles

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Joint work with:
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and other colleagues
A classic observation:

\[ x(G) \leq 2k+1 \iff T_{2k-1}(G) \rightarrow C_{2k+1} \]
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Question. What about $2k$-colorability?

Note: $G \rightarrow C_{2k}$ iff $G$ is bipartite.
signed graph: A graph where each edge is assigned a sign
+ positive  - negative

Notation: \((G, \sigma)\)

Examples:

\((K_3, +)\)  \(\Delta_3\)  \((K_{4\,1}, -)\)  \((K_5, -)\)
Definition. Given signed graphs \((G, σ)\) & \((H, π)\), a mapping of \(V(G)\) to \(V(H)\) (and \(E(G)\) to \(E(H)\)) is said to be a homomorphism of \((G, σ)\) to \((H, π)\) if it preserves adjacencies, (incidences) and signs of closed walks.

It is said to be edge-sign preserving homomorphism if it furthermore, preserves signs of edges.
Comment. The edge mapping is implied unless $(H, \pi)$ contains a digon.

Theorem. A signed graph $(G, \sigma)$ admits a homomorphism to a signed graph $(H, \pi)$ if and only if for some switching $(G, \sigma')$ there exists an edge-sign preserving homomorphism of $(G, \sigma')$ to $(H, \pi)$. 
Important note:
Notions of "isomorphism" and "automorphism" depend on our view and choice of homomorphism: edge-sign preserving homomorphism or switch homomorphism. Associated definitions like "vertex transitive" and "edge-transitive" change accordingly.

under edge-sign preserving homomorphism

the only automorphism:

\[
\begin{align*}
  a &\leftrightarrow d \\
  b &\leftrightarrow c
\end{align*}
\]

under switch homomorphism

Dihedral group $D_4$ (8 elements)
Thus $C_4$ is:
vertex-transitive & edge-transitive.
Four types of closed walks:

00: positive even
01: positive odd
10: negative even
11: negative odd

$g_{ij}(G, \delta)$: the length of shortest walk of type $ij$.  

$ij$
A basic no-homomorphism lemma.

\((G, \phi) \rightarrow (H, \pi) \implies g_{ij}(G, \phi) \geq g_{ij}(H, \pi)\)
Today's focus:

$C_4$ : Negative 4-cycle
no edge-sign preserving hom
For signed bipartite graph this is the only obstacle.

Given \((G, \sigma)\) to decide if there exists a switching such that \((G, \sigma')\) does not contain this obstacle is NP-hard.
$0$-free $2k$-coloring of signed graphs

Color set: $\{\pm 1, \pm 2, \ldots, \pm k\}$

A notion of proper coloring:

$$C : \mathcal{V}(G) \rightarrow \{\pm 1, \pm 2, \ldots, \pm k\}$$

$$C(x) \neq C(y) \& \Phi(x, y)$$

Observation.

$O$-free $2k$-coloring of $(G, t)$ is the same as $2k$-coloring of $G$. 
Recent study


they also considered $(2k+1)$-coloring where 0 is the extra color and proved a number of results for the combined chromatic number.
Question (Máčajová, Raspaud and Škoviera.)

Is every simple planar signed graph \(\{\pm 1, \pm 2\}\)-colorable?
Counterexample built by Kardos & Narboni

$\pm 1, \pm 2^3$-coloring \hspace{1cm} \text{mapping to } \mathbb{C}_4$
$\pm 1, \pm 2^\frac{3}{2}$-coloring

$(G_1, 6)$

mapping to $C_4$

$T_2(G_1, 6)$
Thm. \((G, 6)\) is \(\pm 1, \pm 2\) 3-colorable iff \(T_2(G, 6) \to C_4\)
A restatement of the 4-color theorem:

For every planar graph $G$

$$\text{T}_2(G, +) \rightarrow C_{-4}$$
A restatement of the 4-color theorem:

For every planar graph $G$

$$T_2(G, +) \rightarrow C_{-4}$$

Note

$T_2(G, +)$ is bipartite and has negative girth at least 6.
Remark

Not every signed bipartite planar graph of negative girth 6 maps to $C_4$.

Example

$T_2(G, \phi)$ where $(G, \phi)$ is not $\pm 1, \pm 2^3$-colorable
\( C_4 \)-critical signed graphs

\[(G, \phi) \rightarrow C_4 \]

but for every proper subgraph \( G' \)

\[(G', \phi) \rightarrow C_4 \]
Examples

\[ \hat{W} \]
Examples
Theorem. Suppose \((G, \phi)\) is \(C_4\)-critical signed graph which is not isomorphic to \(\hat{W}\). Then

\[|E(G)| \geq \frac{4}{3} |V(G)|\]
Theorem. Suppose \((G, \delta)\) is \(C_4\)-critical signed graph which is not isomorphic to \(\hat{W}\). Then

\[
|E(G)| \geq \frac{4}{3} |V(G)|
\]

Corollary. Every signed bipartite planar graph of negative girth at least 8 maps to \(C_4\).
Similar results from literature:

with a long history of studying k-critical graphs

Theorem. If \( k \geq 4 \) and \( G \) is \( k \)-critical, then:

\[
|E(G)| \geq \frac{((k+1)(k-2)|V(G)|-k(k-3))/2(k-1)}{3}
\]


Special case

Theorem. If \( G \) is \( C_3 \)-critical, then

\[
|E(G)| \geq \frac{5|V(G)|-2}{3}
\]

Introduction of $H$-critical graphs


Theorem. If $G$ is $C_5$-critical, then

$$|E(G)| \geq \frac{5|V(G)| - 2}{4}$$


Theorem. If $G$ is $C_7$-critical, then

$$|E(G)| \geq \frac{17|V(G)| - 2}{15}$$

Jaeger-Zhang Conjecture

Every planar graph of odd-girth at least $4k+1$ maps to $C_{2k+1}$.

Case $k=1$ is the Grötzsch theorem.

For $k \geq 2$ the exact statement remains open.

For general value of $k$ better and better supporting results are provided. The best one so far:

Theorem. If $G$ is a planar graph of odd girth at least $6k+1$, then $G$ maps to $C_{2k+1}$.

Bipartite analogue of Jaeger-Zhang:
Every signed bipartite planar graph of negative girth at least ?? maps to $C_{-2k}$.


Supportive result in $(8k-2)$


Recent progress $(6k)$

In collaboration with J. Li, Z. Wang & X. Zhu.
Restating the theorem:

If \((G, \phi)\) is \(C_{-4}\)-critical then,

\[
P(G) = 4|V(G)| - 3|E(G)| \leq 0
\]
Restating the theorem:

If \((G, \phi)\) is \(C_{-4}\)-critical then,

\[
P(G) = 4|V(G)| - 3|E(G)| \leq 0
\]

Take a minimum counterexample.

- \((G, \phi) \not\rightarrow C_{-4}\).
- every proper subgraph maps to \(C_{-4}\)
- \(P(G) \geq 1\)
Basic observations
- $G$ is 2-connected
- There is no 3-thread

Observations on $P(G)$
- $G$: nontrivial, bipartite, at most 4 vertices $\implies P(G) \geq 4$
- $P(\hat{W}) = 1$
- $P(P_2(G)) = P(G) - 2$

$P_2(G)$:
Two subdivisions of $\hat{\Omega}$:

Lemma 1. $P_2(\Omega_1) \rightarrow C_{-4}$

Lemma 2. either $P_2(\Omega_2) \rightarrow C_{-4}$ or $\hat{\Omega} \subseteq P_2(\Omega_2)$
The main Lemma

- $P(H) \geq 1$ if $G = H$
- $P(H) \geq 3$ if $G = P_2(H)$
- $P(H) \geq 4$ for all other subgraphs

Proof by contradiction

Take a maximal $H$

$P(H) \leq 3$
Two possibilities to consider:

I. 

II. Contains a $C_4$-critical graph $\hat{G}_2$
I.

\[ \hat{G} \]

\[ \hat{G} \]

\[ \hat{G} + x \]

\[ P(\hat{H} + x) = P(\hat{H}) + 4 - 2 \times 3 \leq 1 \]

\[ \implies \text{either } \hat{G} = \hat{H} + x \]

\[ \text{or } \hat{G} = \rho_2(\hat{H} + x) \]

\[ d(x) = 2 \quad \& \quad G = \rho_2(\hat{H}) \]

\[ P(\hat{G}) = P(\rho_2(\hat{H} + x)) = P(\hat{H} + x) + 4 - 2 \times 3 \leq 1 \]
Contains a $C_4$-critical graph $\hat{G}_2$
\( P(H) \leq 3 \)

1. \( |V(H)| + |E(H)| \) maximized

2. \( H \rightarrow C_4 \)

3. \( \hat{G}_1 \leftrightarrow C_4 \)

\( \hat{G}_2 \): A \( C_4 \)-critical subgraph

\( \chi = \hat{G}_2 \cap \chi_1 \)

4. \( \hat{G}_3 \): unpacking \( \hat{G}_2 \cup \chi_1 \)

Each group of edges is presented by one edge

\[
|V(G_3)| = |V(G_2)| - |V(\chi)| + |V(H)|
\]

\[
|E(G_3)| = |E(G_2)| - |E(\chi)| + |E(H)|
\]

\[
|P(G_3)| = |P(G_2)| - |P(\chi)| + |P(H)|
\]

\( \leq 1 \quad \gg 4 \quad \leq 3 \)
Corollary. Two forbidden subgraphs:

\[ P(\text{C&C}) = 2 \quad \text{and} \quad P(\text{ }) = 3 \]
Lemma.
Corollary. Two forbidden configurations:
In conclusion

- \( s(G) \geq 2 \)

- No vertex of degree 2 is adjacent to a vertex of degree 2.

- Each vertex of degree 3 has at most one vertex of degree 2.

\[ \Rightarrow \quad \overline{d}(G) \geq \frac{8}{3} \]
Refinements of $\pm 1, \pm 2, \ldots, \pm k$-coloring:

Circular $r$-coloring of $(G, G)$

$$\varphi: V(G) \rightarrow \mathbb{C}^r$$

subject to: \[
\begin{cases}
    d(\varphi(x), \varphi(y)) > 1 & \text{if } xy \text{ is positive} \\
    d(\varphi(x), \overline{\varphi(y)}) > 1 & \text{if } x \text{ y is negative}
\end{cases}
\]
Two possible definitions of antipodal:

\[ X \quad \overline{X} \quad X \]

Y. Kang and E. Steffen.
Circular coloring of signed graphs.
J. Graph Theory, 87(2) (2018), 135–148.

Naserasr, R., Wang, Z., and Zhu, X.
Circular chromatic number of signed graphs.

\[ X_c(G, \omega) \]
Lemma.

\[-X_c(G, +) \leq 2 + \frac{1}{k} \iff G \rightarrow \mathcal{C}_{2k+1}\]

\[-X_c(G, \circ) \leq \frac{4k}{2k-1} \iff G \rightarrow \mathcal{C}_{-2k}\]
Theorem.

Given a signed bipartite planar graph on $n$ vertices

$$X_c(G, \phi) \leq 4 - \frac{4}{\left\lfloor \frac{n+2}{2} \right\rfloor}$$

Moreover, equality is realized for every $n \geq 2$.

Circular $(4-\epsilon)$-coloring of some classes of signed graphs.
Open questions:
- The best bound for the circular chromatic number of signed planar simple graphs?
  
  It is known to be in \([4 + \frac{2}{3}, 6]\)

- The best bound for the circular chromatic number of signed planar simple graphs of negative girth \(\geq 6\)?

  It is known to be in \([\frac{14}{5}, 3]\)
Open questions:

- Minimum number of the edges of a $\{\pm 1, \pm 2, \ldots, \pm k\}$-critical signed graph on $n$ vertices?

- Minimum number of the edges of a $C_k$-critical signed graph on $n$ vertices?

sufficiently large
Ďakujem