Ramsey theory

Simple example:

Given any set of 6 people either there are 3 among them such that everyone knows everyone else, or there are 3 among them such that no one knows the other two.

In this statement 6 can be replaced with any integer larger than 6 but it cannot be replaced by 5.

Example of 5 people not satisfying the condition

In language of graphs

A 2-edge-colored graph: each edge is either Red or Blue (this coloring is not proper coloring).

Given a 2-edge-colored complete graph K_n and integers p & q what we are interested in is:

- either p vertices where every edge is Red
- or q vertices where every edge is Blue

Ramsey's Theorem

Given any two positive integers p & q there exists an integer $R(p, q)$ such that for $n \geq R(p, q)$ every 2-edge-colored K_n contains either a Red K_p or Blue K_q.

Definition. The smallest possible choice in this theorem for $R(p, q)$ is called Ramsey number of p and q.

Examples.

- $R(p,2) = p$

- $R(3,3) = 6$

Proof.

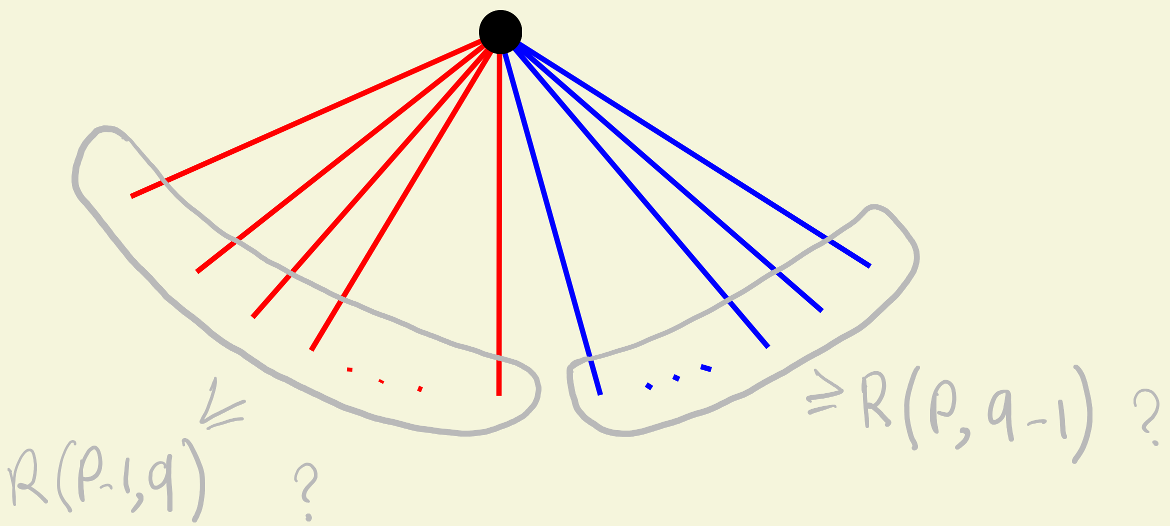
We have $R(p, 2) = R(2, p) = p$.

For the other values of p & q we apply induction on $p+q$,

taking $p=q=2$ as the base of induction $R(2, 2)=4$.

Thus we assume that $R(p,q)$ exists whenever $p+q \leq k$ and

consider a pair of p & q with $p+q=k+1$.



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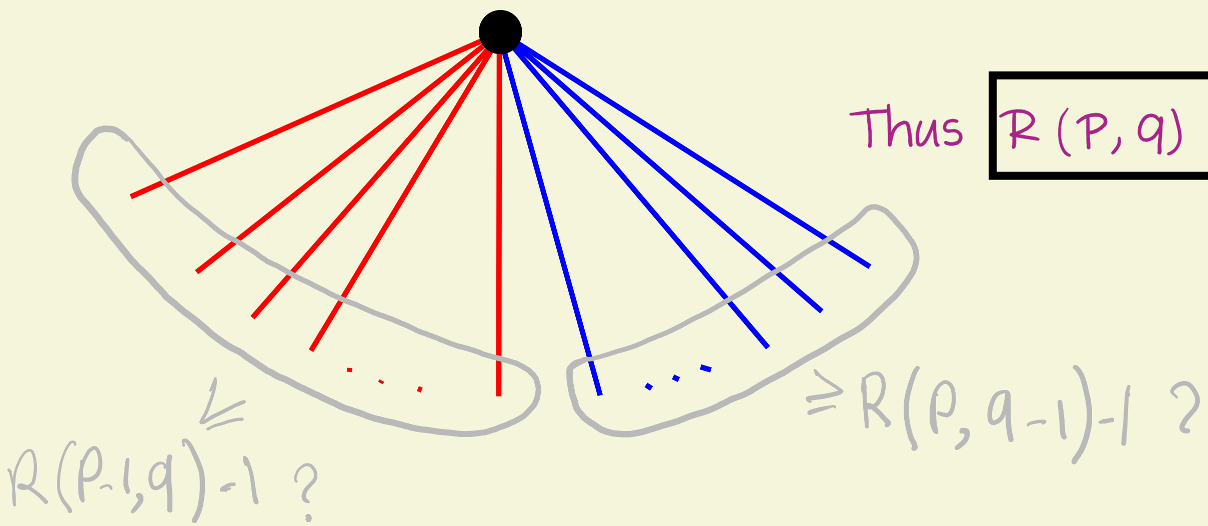


Thus $\boxed{R(p,q) \leq R(p,q-1) + R(p-1,q).}$

Generalizations:

H_k^n: k-uniform complete hypergraph on n vertices

Vertices: an n-set (e.g. $\{1, 2, \dots n\} = [n]$)

Hyper edge set: all k-subsets of $[n]$, $\binom{[n]}{k}$

L-edge-colored k-uniform complete hypergraph:

each hyperedge is assigned one of the L colors.

Ramsey's theorem: Given integers $k, r_1, r_2 \dots r_l$ $k \geq 2$, $r_1, r_2 \dots r_l \geq k$,

there exists an integer $f(K, r_1, r_2 \dots r_l)$ such that

for $n \geq f(K, r_1, r_2 \dots r_l)$ in any l-edge-colored k-uniform on n vertices

hypergraph there exists an index i for which we have:

an r_i-subset of vertices which induces a k-uniform

hypergraph all whose edges are colored with the i^{th} color.

Infinite Ramesy theory

Given.

- An infinite set A
- A positive integer k (k-subsets to be considered)
- A set of l colors (1, 2, --l)
- A coloring φ of the k-subset of A.

Conclusion.

An infinite subset A' of A where all k-subsets have a same color.

König's Lemma:

In every locally finite, connected, infinite tree there exists an infinite path.

Extremly difficult question:

Determine $R(p,q)$ or $R(r_1, r_2, \dots r_l)$ in general.

What is known:

$R(3, 3) = 6$

$R(3, 4) = 9$

$R(3, 5) = 14$

$R(3, 6) = 18$

$R(3, 10) \in \{40, 41, 42\}$

$\vdots$

$R(4, 4) = 18$

$R(4, 5) = 25$

$R(3, t)$ is of order $\dfrac{t^2}{\log t}$

$\hookrightarrow$ Every triangle-free graph on n vertices has an idependent set of order $\theta(\sqrt{n \log n})$

Most special cases that are open:

$$43 \leq R(5,5) \leq 48$$

$$102 \leq R(6,6) \leq 165$$

Best upper bound:

$$R(p,q) \leq R(p,q-1) + R(p-1,q)$$

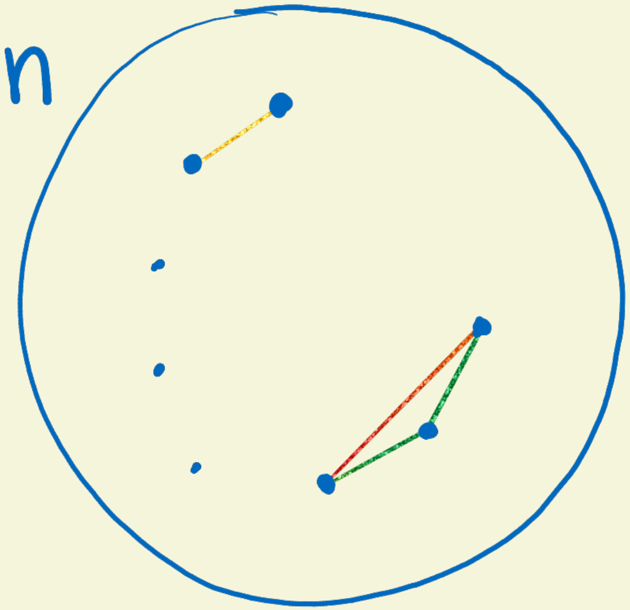
$$\binom{k+l}{k} = \binom{k+l-1}{k} + \binom{k-1+l}{l}$$

$$\rightarrow R(p,q) \leq \binom{p+q-2}{r-1}$$

$$\Rightarrow \boxed{R(p,p) \leq \left(1+o(1)\right) \frac{4^{s-1}}{\sqrt{\pi s}}.}$$

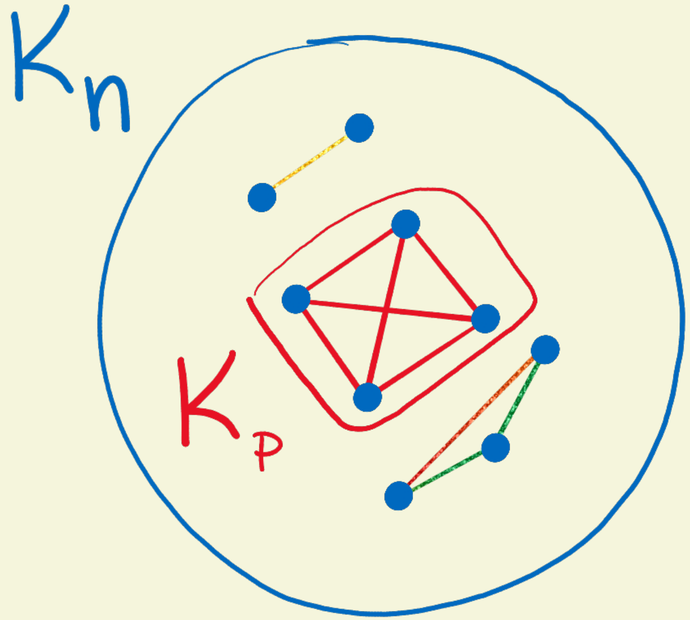
Best lower bound:

K_n



total number of 2-edge-colorings ?

Best lower bound: $\boxed{\text{probabilistic method}}$



total number of 2-edge-colorings?

total number of 2-edge-colorings where a given K_p is monochromatic?

$\Rightarrow$ if $\binom{n}{p} < 2^{\binom{p}{2}}$, then there exist an edg-coloring without a monochromatic K_p.

$$\Rightarrow \quad R(p,p) \geq (1+O(1)) \frac{\sqrt{2}}{e} \, 2^{p/2}$$

Lower bounds by (algebraic) constructions:

Γ: an additive) group

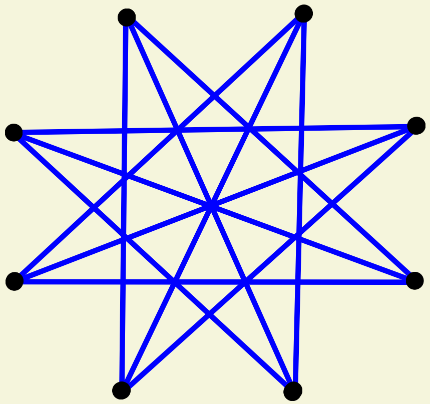
S: a subset of Γ, normally assumed to satisfy $x \in S \Rightarrow -x \in S$.

Cayley graph (Γ, S)

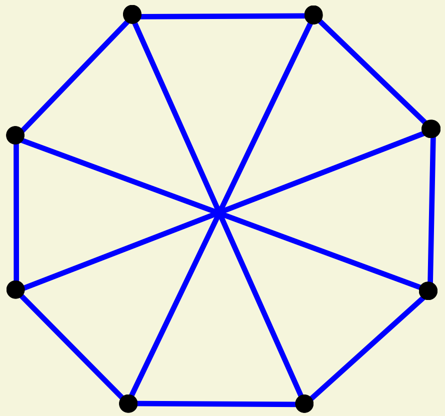
- vertex set: elements of
- edge set: $x - y \iff x - y \in S$

Examples

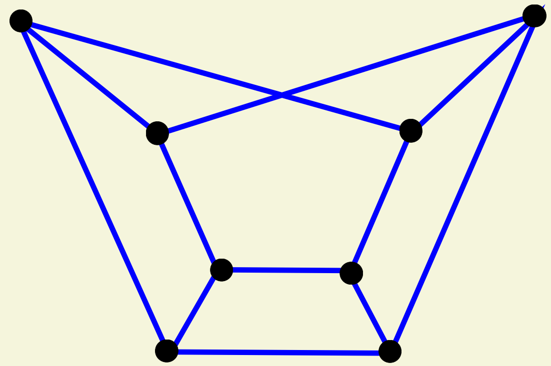
$G = (\mathbb{Z}_8, \{\pm 3, 4\})$.



$C(8,3)$



V_8

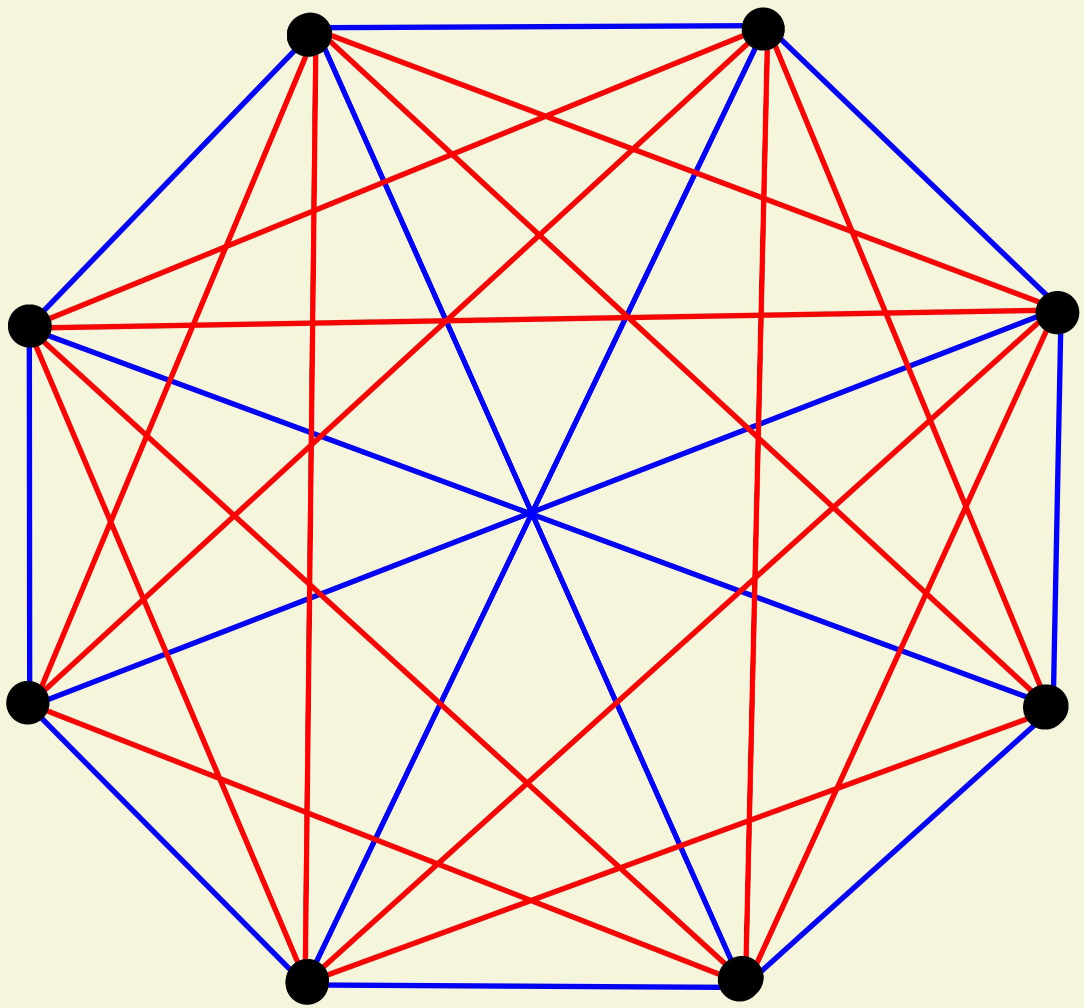


Möbius

Other names: Wagner graph

(in classification of K_5-minor-free graphs)

Examples



Field $(F,+,\cdot)$

$(F,+)$: an additive group with 0 as identity

$(F-0,\cdot)$: a multiplicative group with 1 as identity.

Both are commutative and, moreover, $a(b+c)=ab+ac$

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Finite Field: a Field where F is a finite set.

Examples: $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_8, \mathbb{Z}_9$?

$\mathbb{Z}_2 \times \mathbb{Z}_2$?

GF(4): Field on 4 elements

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

x	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

Golios theory. A finite field of order q exists if and only if $q = p^n$ for a prime number P.

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Question. How to build $GF(9)$?

Note: Any two finite fields of a same order are isomorphic.

For $n=1$, i.e. $q=p$, $(\mathbb{Z}_p,+,\times)$ is the finite field of order p.

For $n \geq 2$ we consider a polynomial $f(x)$ of degree n whose coefficients are from $\mathbb{Z}_p$, with the property that it is irreducible on $\mathbb{Z}_p[X]$.

$\hookrightarrow f(x) \neq g(x)h(x)$

Homework. There exists such a polynomial for every $n \geq 2$.

Theorem. $\mathbb{Z}_p[x]/f(x)$ is the field of order p^n.

Examples

In $GF(2)$ the polynomial $f(x) = x^n+x+1$ is irreducible.

To build $GF(8)$ we take x^3+x+1

(that means each time you see an x^3 you may replace it with $x+1$)

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$\cdot$	0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
0	0	0	0	0	0	0	0	0
1	0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
x	0	x	x^2	x^2+x	$x+1$	1	x^2+x+1	x^2+1
$x+1$	0	$x+1$	x^2+x	x^2+1	x^2+x+1	x^2	1	x
x^2	0	x^2	$x+1$	x^2+x+1	x^2+x	x	x^2+1	1
x^2+1	0	x^2+1	1	x^2	x	x^2+x+1	$x+1$	x^2+x
x^2+x	0	x^2+x	x^2+x+1	1	x^2+1	$x+1$	x	x^2
x^2+x+1	0	x^2+x+1	x^2+1	x	1	x^2+x	x^2	$x+1$

	0	1	X	$X+1$	X^2	X^2+1	X^2+X	X^2+X+1
0	0	0	0	0	0	0	0	0
1	0	1	X	$X+1$	X^2	X^2+1	X^2+X	X^2+X+1
X	0	X	X^2	X^2+X	$X+1$	1	X^2+X+1	X^2+1
$X+1$	0	$X+1$	X^2+X	X^2+1	X^2+X+1	X^2	1	X
X^2	0	X^2	$X+1$	X^2+X+1	X^2+X	X	X^2+1	1
X^2+1	0	X^2+1	1	X^2	X	X^2+X+1	$X+1$	X^2+X
X^2+X	0	X^2+X	X^2+X+1	1	X^2+1	$X+1$	X	X^2
X^2+X+1	0	X^2+X+1	X^2+1	X	1	X^2+X	X^2	$X+1$

Quderatic Residues:

Solutions of $x = a^2$ in $GF(q)$

Examples: $QR(\mathbb{Z}_5) = \{\pm 1\}$.

$QR(\mathbb{Z}_7) = \{1\ 2, -3\}$.

Homework: If $q \equiv 1 \pmod 4$, then $-1 \in QR(GF(q))$.

If $q \equiv 3 \pmod 4$, then $-1 \notin QR(GF(q))$.

Paley graph of order $q \equiv 1 \pmod 4$,

$$\left(GF(q),\ QR(GF(q))\right)$$

Paley graph of order 5

Paley graph of order 17

Analogue of Ramsey theory for oriented graphs:

T_n : tournament of order n

TT_n : transitive tournament of order n

Theorem. For every k there exists an $f(k)$ such that for $n \geqslant f(k)$ every T_n contains a copy of TT_k.