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Nowhere-zero flows of graphs and signed graphs

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IRIF-Graph Seminar, online, Feb 27, 2021

Overview

Four Coloring Theorem

Four Color Theorem (Appel-Haken 1976)

Every planar map is four colorable.

More compute-aided proofs found later. (RSST 1997 etc.)

Coloring Planar Graphs

For **vertex coloring**

4CT: Every planar graph is 4-colorable.

3CT: Every triangle-free planar graph is 3-colorable.

OB: A graph is 2-colorable iff it contains no odd cycle.

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How about coloring between 2 and 3 ?

2.9-coloring, 2.5-coloring, $(2+\frac{1}{p})$-coloring ?

Circular Coloring

A graph G is circular $\frac{k}{d}$-colorable if there exists a function $c : V(G) \mapsto \mathbb{Z}_k$ such that

$$||c(x) - c(y)||_k \geq d, \quad \text{for any edge } xy \in E(G)$$

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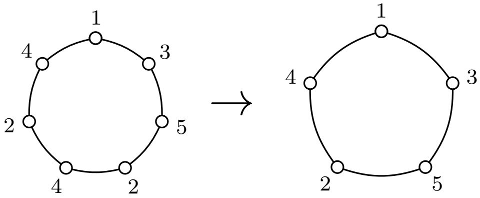
Definition of circular chromatic number $\chi_c(G)$

$\chi_c(G)$: the least rational number r such that G is circular r-colorable.

$\chi(G) = \lceil \chi_c(G) \rceil$, $\chi_c(C_{2p+1}) = 2 + \frac{1}{p}$

$\chi_c(G) \leq 2 + \frac{1}{p}$ iff G has a homomorphism to C_{2p+1}.

Homomorphism: $C_7 \to C_5$



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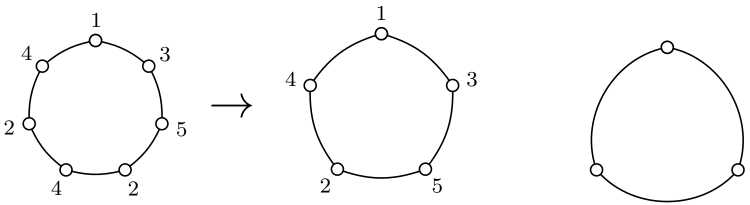

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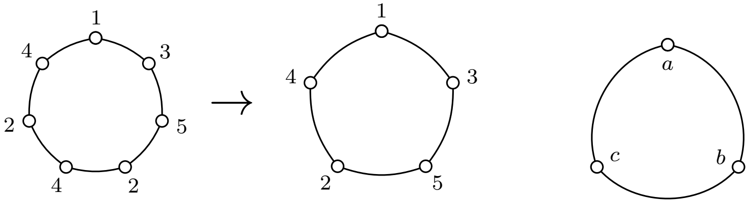
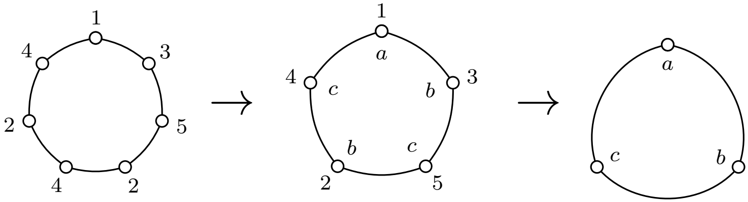
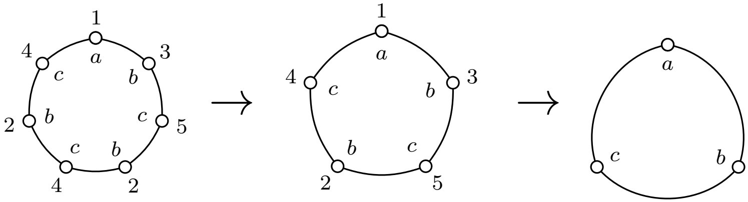


Homomorphism: $C_7 \to C_5$ and $C_5 \to C_3$



Homomorphism: $C_7 \to C_5$ and $C_5 \to C_3$, so $C_7 \to C_3$.



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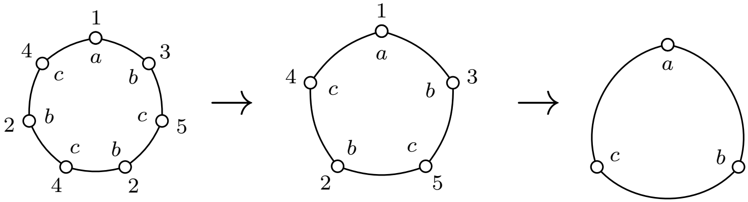

- Circular Coloring is monotonic.

Planar Circular Coloring Conjecture

The following is modified from Jaeger's conjecture.

Planar Circular Coloring Conjecture

$\chi_c(G) \leq 2 + \frac{2}{k}$ for any planar graph G of girth $\geq 2k$.

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- For $k = 4$, it is known that $\chi_c(G) \leq 2.5$ *for girth*$(G) \geq 10$ (by Dvořák&Postle 2017),
- For $k = 6$, it is known that $\chi_c(G) \leq 7/3$ *for girth*$(G) \geq 16$ (by Postle&Smith-Roberge 2019+, Cranston&Li 2020)

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Thm ((ii)LTWZ 2013 plus (i)(iii)LWZ 2020)

For a planar graph G,
(i) if $girth(G) \ge 6p - 2$, then $\chi_c(G) \le 2 + 2/(2p-1)$;
(ii) if $girth(G) \ge 6p$, then $\chi_c(G) \le 2 + 1/p$;
(iii) if $girth(G) \ge 6p + 2$, Then $\chi_c(G) < 2 + 1/p$.

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- Those results are proved via **flows and orientations**.

Circular Coloring Table

Circular Coloring and Girth in Planar Graphs

Girth	Conjectured $\chi_c(G)$	Known $\chi_c(G)$
2	$\chi_c \leq 4$(4CT)	$\chi_c \leq 4$(4CT)
4	$\chi_c \leq 3$(3CT)	$\chi_c \leq 3$(Grotzsch)
6	$\chi_c \leq \frac{8}{3}$	?
8	$\chi_c \leq 2.5$	?
10	$\chi_c \leq \frac{12}{5}$	$\chi_c \leq 2.5$(DP17,CL20)
12	$\chi_c \leq \frac{7}{3}$	$\chi_c \leq \frac{5}{2}$(LTWZ13)
16	$\chi_c \leq 16/7$	$\chi_c \leq 7/3$(PS19+,CL20)
...	...	...
$4p$	$\chi_c \leq 2 + \frac{1}{p}$(Jaeger1981)	*
$6p-2$	*	$\chi_c \leq 2 + 2/(2p-1)$(LWZ20)
$6p$	*	$\chi_c \leq 2 + \frac{1}{p}$(LTWZ13)
$6p+2$	*	$\chi_c < 2 + \frac{1}{p}$(LWZ20)

The Flow Theory

Tutte initiated Integer Flows

Figure: W. T. Tutte(1917–2002)

Tutte's Flow Theory

Let $G = (V, E)$ be a graph (which may have parallel edges).

- $D = D(G)$: an orientation of a graph G
- $f : E(G) \mapsto A$ (where A is a subset of an Abelian group)

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- $D = D(G)$: an orientation of a graph G
- $f : E(G) \mapsto A$ (where A is a subset of an Abelian group)

(D, f) is called a **flow** if, under orientation D, for any vertex v,

balanced: sum of **in-flow** = sum of **out-flow** at v

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e), \forall v \in V(G).$$

Integer and Circular Flows

Flow: pair (D, f) with sum of **in-flow** = sum of **out-flow**, $\forall v$

(D, f): a nowhere-zero k-flow(k-NZF) if, in addition,

$$f : E \mapsto \{\pm 1, \pm 2, \ldots, \pm(k-1)\}$$

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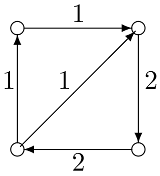
$$f : E \mapsto \{\pm 1, \pm 2, \ldots, \pm(k-1)\}$$

(D, f): a circular $\frac{k}{d}$-flow if, in addition,

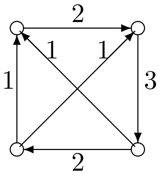
$$f : E \mapsto \{\pm d, \pm(d+1), \ldots, \pm(k-d)\}$$

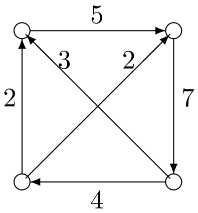
Examples of Flows

3-NZF



4-NZF





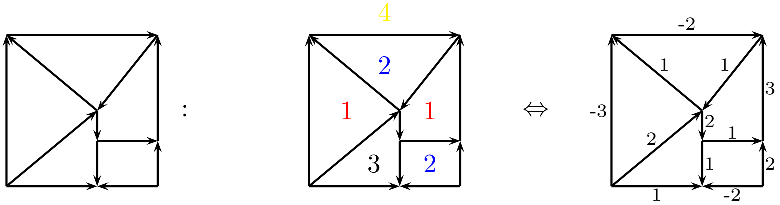
circular 9/2-flow (values in$\{\pm 2, \pm 3, \dots, \pm 7\}$)

Tutte's Coloring-Flow Duality

Flow Theory initiated by Tutte as generalization of map-coloring problems. ("map":= bridgeless plane graph "country":= face)

Coloring-Flow Duality Theorem (Tutte 1954)

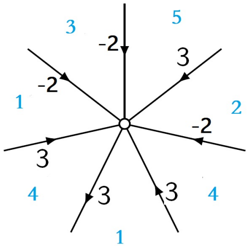
A bridgeless plane graph is k-face-colorable if and only if it admits a nowhere-zero k-flow.



Tutte's Coloring-Flow Duality

Circular Flow-Coloring Duality (Goddyn-Tarsi-Zhang, 1998)

For a plane graph G and its dual G^*,
G has a circular $\frac{k}{d}$-flow iff $\chi_c(G^*) \leq \frac{k}{d}$.



Circular Flow

Definition of circular flow index $\phi(G)$

$\phi(G)$: the smallest $r = \frac{k}{d}$ such that G admits a circular r-flow.

Circular Flow-Coloring Duality (Goddyn-Tarsi-Zhang, 1998)

For a plane graph G and its dual G^*, $\phi(G) \leq \frac{k}{d}$ iff $\chi_c(G^*) \leq \frac{k}{d}$.

Planar Circular Coloring Conjecture

$\chi_c(G) \leq 2 + \frac{2}{k}$ for any planar graph G of girth $\geq 2k$.

How about bounds of flow index ϕ ?

Flow Conjectures

k=1 4CT: $\phi \leq 4$ for every bridgeless planar graph.

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- How about general graphs?Peterson graph $\phi(P) = 5 > 4$!

Conj Tutte's 5-flow conjecture: $\phi \leq 5$ for every bridgeless graph.
Tutte's 4-flow conjecture: $\phi \leq 4$ for every bridgeless Peterson-minor-free graph.

Flow Conjectures

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Conj Tutte's 5-flow conjecture: $\phi \leq 5$ for every bridgeless graph.
Tutte's 4-flow conjecture: $\phi \leq 4$ for every bridgeless Peterson-minor-free graph.

k=2 3CT: $\phi \leq 3$ for every 4-edge-connected planar graph.

Conj Tutte's 3-flow conjecture: $\phi \leq 3$ for every 4-edge-connected graph.

- How about other values of k ?

Conj Jaeger's circular flow conjecture: for even k, $\phi \leq 2 + \frac{2}{k}$ for every $2k$-edge-connected graph.

Flow Theorems

Flow Theorems

- **Snark Thm (ERSST):** Every bridgeless cubic graph without Peterson-minor admits a nowhere-zero 4-flow.
- **4Flow Thm (Jaeger1979):** $\phi \leq 4$ for every 4-edge-conn graph.
- **6Flow Thm (Seymour1981):** $\phi \leq 6$ for every bridgeless graph.
- **3Flow Thm (LTWZ2013):** $\phi \leq 3$ for every 6-edge-conn graph.
- **Circular Flow Thm (LTWZ2013):** $\phi \leq 2 + \frac{1}{p}$ for every $6p$-edge-conn graph.

(coloring)near bipartite **VS** (flow) near Eulerian

Flow Index Table in **2020**

Flow Index and Edge Connectivity of Graphs

Edge-Conn.	Conjectured ϕ	Known ϕ
2	$\phi \le 5$(Tutte1954)	$\phi \le 6$(Seymour1981)
4	$\phi \le 3$(Tutte1972)	$\phi \le 4$(Jaeger1979)
6	$\phi < 3$(LTWZ18)	$\phi \le 3$(LTWZ13)
8	~~$\phi \le 2.5$~~(False)	$\phi < 3$(LTWZ18)
10	*	$\phi \le \frac{8}{3}$(LWZ20)
12	~~$\phi \le \frac{7}{3}$~~(False)(HLWZ18)	$\phi \le \frac{5}{2}$(LTWZ13)
14	*	$\phi < \frac{5}{2}$(LWZ20)
16	~~$\phi \le \frac{9}{4}$~~(False)	$\phi \le \frac{12}{5}$(LWZ20)
...	...	...
$6p-2$	*	$\phi \le 2 + \frac{2}{2p-1}$(LWZ20)
$6p$	*	$\phi \le 2 + \frac{1}{p}$(LTWZ13)
$6p+2$	*	$\phi < 2 + \frac{1}{p}$(LWZ20)

Flows from Modulo k-Orientation

An orientation D is called a modulo k-orientation of G if

$$indegree \equiv outdegree \pmod{k}, \forall v \in V(G)$$

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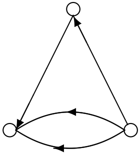
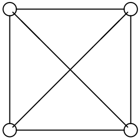
Proposition (Tutte, Steinberg-Younger)

A graph admits a nowhere-zero 3-flow if and only if it admits a modulo 3-orientation.

A mod 3-orientation



K_4 has no mod 3-orientation.



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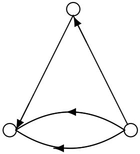

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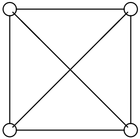

Theorem (Jaeger 1981)

$\phi(G) \leq (2 + \frac{1}{p})$ iff G admits a modulo $(2p+1)$-orientation.

Circular Flow Conjecture

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Every $4p$-edge-connected graph has a mod $(2p+1)$-orientation.

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Some history:

- $C\log n$-conn. (by Lai-Zhang 1992, Alon-Linial-Meshulam 1992, Lai-Shao-Wu-Zhou 2009)
- In 2012, Thomassen's breakthrough : $100p^2$-edge-conn. In 2013, Lovász-Thomassen-Wu-Zhang: $6p$-edge-conn.
- True for random graphs and random regular graphs by Sudokov 2001, by Alon-Pralat 2011
- ∃Counterexamples for $p \geq 3$ by Han-Li-Wu-Zhang 2018

The Circular Flow Conjecture of Jaeger is False

Theorem A (Han-Li-Wu-Zhang, 2018)

For every $p \geq 3$, there are infinite families of $4p$-edge-connected graphs admitting no modulo $(2p+1)$-orientations.

We will try to present a proof sketch in the rest of this lecture.

Some tools for modulo orientations

- **Orientation with boundary:** A function $\beta : V(G) \mapsto \mathbb{Z}_{2p+1}$ is called a $\mathbb{Z}_{2p+1}$-boundary if $\sum_{v\in V(G)} \beta(v) \equiv 0 \pmod{2p+1}$. An orientation D with $d_D^+(v) - d_D^-(v) \equiv \beta \pmod{2p+1}, \forall v$ is called a β-orientation.
- **Pre-orientation and extension**
- **Splitting**
- **graph contraction:** If H has some "strong" property, we try to work on G/H by induction, and then orient $E(H)$ to modify boundary, resulting a desired orientation of G.

Orientations and β-orientations

- For a $\mathbb{Z}_{2p+1}$-boundary β, a β-orientation is an orientation D with $d_D^+(v) - d_D^-(v) \equiv \beta \pmod{2p+1}, \forall v$.

The above is taken $\pmod{2p+1}$. For orientation with prescribed value in $\mathbb{Z}$, there is a nice iff theorem of Hakimi.

Hakimi's Thm, 1960s

Let $b : V(G) \mapsto \mathbb{Z}$. Then G has an orientation D such that $d_D^+(v) - d_D^-(v) = b(v), \forall v$
iff the following holds:
$\sum_{v \in V(G)} b(v) = 0$, $b(v) - d_G(v)$ is even $\forall v$, and

$$|\sum_{v \in S} b(v)| \le d_G(S), \forall S \subset V(G)$$

Some observation on Orientations

For a $\mathbb{Z}_{2p+1}$-boundary β and a β-orientation D of G, if $d_G(v)$ is small (say $< 4p+2$), then (depends on the parity), $d_D^+(v) - d_D^-(v) = b(v)$ has two candidates:

- if $\beta(v)$ and $d_G(v)$ has the same parity, then $d_D^+(v) - d_D^-(v) = b(v) \in \{\beta(v), \beta(v) - 4p - 2\}$
- if $\beta(v)$ and $d_G(v)$ has the different parities, then $d_D^+(v) - d_D^-(v) = b(v) \in \{\beta(v) + 2p + 1, \beta(v) - 2p - 1\}$

Observation:

If G has max degree $< 4p+2$, in a modulo $(2p+1)$-orientation D of G (that is $\beta = 0, \forall v$), we have
(a) $d_D^+(v) - d_D^-(v) \in \{2p+1, -2p-1\}$ for each odd vertex v,
(b) $d_D^+(v) - d_D^-(v) = 0$ for each even vertex v.

Start from Complete Graph

- complete graph K_{4p} admits no mod $(2p+1)$-orientation.

Start from Complete Graph

- complete graph K_{4p} admits no mod $(2p+1)$-orientation.
- By contradiction. If $\exists D$,

$$d_D^+(v) - d_D^-(v) = (2p+1) \quad or \quad -(2p+1)$$

- $V^+ : v$ with $= (2p+1)$, $\quad V^- : v$ with $= -(2p+1)$. $|V^+| = |V^-| = 2p$. Then

$$\begin{aligned} 2p(2p+1) = \sum_{v \in V^+} (d_D^+(v) - d_D^-(v)) &= |\partial^+(V^+)| - |\partial^-(V^+)| \\ &\leq |\partial(V^+)| = 2p \cdot 2p \end{aligned}$$

a contradiction!

Construction 1

- complete graph plus $(p-1)$ triangles

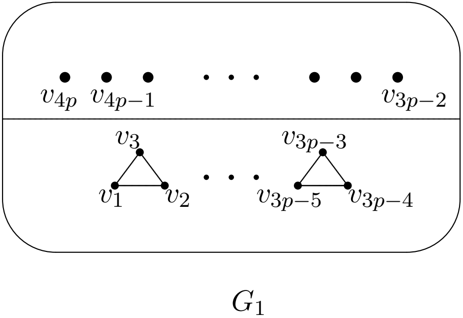

Figure: *The graphs G_1 and G_2.*

Construction 1

- add two new vertices and some edges

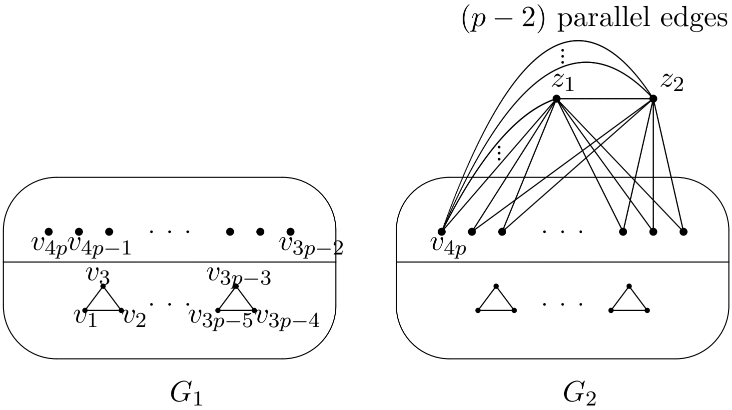

Figure: *The graphs G_1 and G_2.*

Construction 2

- $W = (2p-1)C_{4p+1} \cdot K_1$

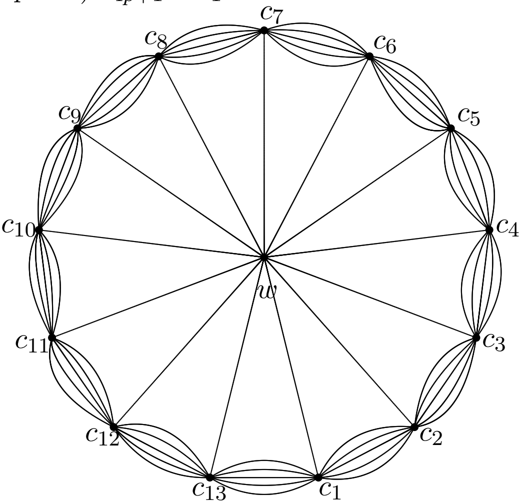

Figure: *The graph W for $p = 3$.*

2-sum (like dual of Hajos-join)

Lemma for 2-sum

Let $H = H_1 \oplus_2 H_2$. If neither H_1 nor H_2 admits modulo $(2p+1)$-orientation, then $H = H_1 \oplus_2 H_2$ admits no modulo $(2p+1)$-orientation.

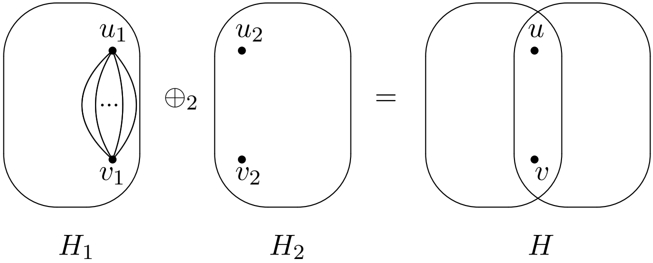

Figure: 2-*sum of* H_1 *and* H_2.

Final Construction via 2-sum

- M is $4p$-edge-connected without mod $(2p+1)$-orientation.

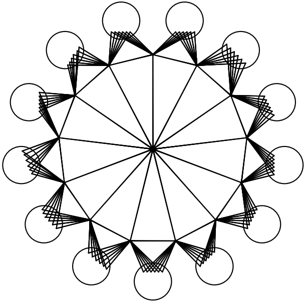

Figure: *The graph M for $p = 3$.*

Remarks

- Similar construction to obtain $(4p+1)$-edge-connected counterexamples.
- Extend to infinite many counterexamples via 2-sum.

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Several related conjectures are false.

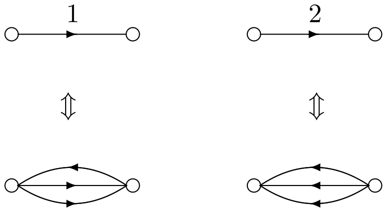
- Every odd-(4p+1)-edge-connected graph admits a circular (2+1/p)-flow. (Zhang, 2002 odd-connectivity version)
- Every (4p+1)-edge-connected graph admits all modulo $(2p+1)$ β-orientation. (Lai, 2007 all boundary version)
- Every (4p+1)-edge-connected graph admits a modulo $(2p+1)$-orientation. (Kochol 2001, and asked whether equivalent to $4p$-edge-connected)

$p = 2$ Case and Nowhere-zero 5-flow

Denote $3G$ to the graph obtained from G by replacing each edge with three parallel edges.

Proposition (Jaeger 1988)

$\phi(G) \leq 5$ if and only if $\phi(3G) \leq 2.5$.



$\mathbb{Z}_5$-NZF in G

Mod 5-orientation in $3G$

Connecting 5-Flow with Circular 5/2-flow

Proposition (Jaeger 1988)

$\phi(G) \leq 5$ if and only if $\phi(3G) \leq 2.5$.

5-flow conjecture equivalent form: $\phi(3G) \leq 2.5$ for any bridgeless graph G.

Connecting 5-Flow with Circular 5/2-flow

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Observation:

Equivalent Form of Seymour's 6-flow Thm:
$\phi(3G) \leq \frac{18}{7} \approx 2.572$ for any bridgeless graph G.

Jaeger's stronger Conjecture (1988)

$\phi(G) \leq 2.5$ for any 9-edge-connected graph G.

Connecting 5-Flow with Circular 5/2-flow

Jaeger's stronger Conjecture (1988)

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- True for 12-edge-connected graphs (L-T-W-Z 2013)
- Thomassen and CQ 2015 asked how to get a better estimation of ϕ for 9-edge-connected graphs. (or 8-conn)

Theorem (Li-Thomassen-Wu-Zhang, 2018)

$\phi(G) < 3$ for any 8-edge-connected graph G.

We develop a new tool: strongly connected modulo orientation

Strongly Connected Modulo Orientation

Theorem (Jaeger 1981)

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$\phi(G) < 2 + 1/p$ iff G admits a strongly connected modulo $(2p+1)$-orientation.

Theorem (Li-Thomassen-Wu-Zhang, 2018)

Every 8-edge-connected graph admits a strongly connected modulo 3-orientation.

We conjecture 6-edge-connectivity sufficient.

No circular 2.5-flow–but Not The End

In 2018, No counterexample was found when $p = 1, 2$.

- Tutte's 3-Flow Conjecture might be still true.
- The $p = 2$ case still leaves hope to approach Tutte's 5-Flow Conjecture. (Positive side: $\phi < 3$ for 8-conn)

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- The $p = 2$ case still leaves hope to approach Tutte's 5-Flow Conjecture. (Positive side: $\phi < 3$ for 8-conn)

But now(2020+), our counterexamples extends to $p = 2$.

Theorem (Li-Wu-Zhang, manuscript in preparing)

For every $p \geq 2$, there are infinite families of $(4p + 1)$-edge -connected graphs with flow index $\phi > 2 + \frac{1}{p}$.

Flow Index for given connectivity

New Question:

What is the correct (best) flow index for given connectivity?

(There is a related Additive Basis Conjecture of Jaeger-Linial-Payan-Tarsi 1992, if the strongest form of this conjecture is true, it would provide some good flow index bound.)

New values for flow index

Flow Index and Edge Connectivity of Graphs

Edge-Conn.	Conjectured ϕ	Known ϕ
2	$\phi \le 5$(Tutte1954)	$\phi \le 6$(Seymour1981)
4	$\phi \le 3$(Tutte1972)	$\phi \le 4$(Jaeger1979)
6	$\phi < 3$(LTWZ2018)	$\phi \le 3$(LTWZ2013)
8	~~$\phi \le 2.5$~~(False)	$\phi < 3$(LTWZ2018)
10	$*$	$\phi \le \frac{8}{3}$(LWZ2020)
12	~~$\phi \le \frac{7}{3}$~~(False)	$\phi \le \frac{5}{2}$(LTWZ2013)
14	$*$	$\phi < \frac{5}{2}$(LWZ2020)
16	~~$\phi \le \frac{9}{4}$~~(False)	$\phi \le \frac{12}{5}$(LWZ2020)
...	...	...
$6p-2$	$*$	$\phi \le 2 + \frac{2}{2p-1}$(LWZ2020)
$6p$	$*$	$\phi \le 2 + \frac{1}{p}$(LTWZ2013)
$6p+2$	$*$	$\phi < 2 + \frac{1}{p}$(LWZ2020)

New Tools: Extended Tutte Orientation

Theorem (Li-Wu-Zhang, 2020)

A graph admits a circular $\frac{k}{p}$-flow if and only if it has a (k,p)-extended-Tutte-orientation (ETO) D, which is an orientation D of $(k-2p)G$ with

$$d_D^+(v) - d_D^-(v) \equiv k d_G(v) \pmod{2k}, \quad \forall v \in V(G).$$

This extends Tutte's fact and Jaeger's modulo orientation to all rational numbers.

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This extends Tutte's fact and Jaeger's modulo orientation to all rational numbers.

Theorem (For $(2+\frac{2}{2p-1})$-flow)

For any $(6p-2)$-edge-connected graph G, $2G$ has a $(4p, 2p-1)$-ETO, and hence $\phi(G) \leq 2 + \frac{2}{2p-1}$.

New Tools: Extended Tutte Orientation

Theorem (For $(< 2 + \frac{1}{p})$-flow)

For any $(6p+2)$-edge-connected graph G, $\phi(G) < 2 + \frac{1}{p}$.

The key idea is to *extend a $(2 + \frac{1}{p} - \epsilon_1)$-flow of G/H to a $(2 + \frac{1}{p} - \epsilon_2)$-flow of G.*

(strongly connected mod orientation seems not applicable to prove it!)

New Tools: Extended Tutte Orientation

Theorem (For $(< 2+\frac{1}{p})$-flow)

For any $(6p+2)$-edge-connected graph G, $\phi(G) < 2+\frac{1}{p}$.

The key idea is to *extend a $(2+\frac{1}{p}-\epsilon_1)$-flow of G/H to a $(2+\frac{1}{p}-\epsilon_2)$-flow of G.*
(strongly connected mod orientation seems not applicable to prove it!)

Summery Table

Edge-Conn.	Known ϕ
2	$\phi \leq 6$(Seymour1981)
$6p-2$	$\phi \leq 2+\frac{2}{2p-1}$(LWZ20)
$6p$	$\phi \leq 2+\frac{1}{p}$(LTWZ2013)
$6p+2$	$\phi < 2+\frac{1}{p}$(LWZ20)

More on β-orientation

Definition

(a) A function $\beta : V(G) \mapsto \{0, \pm 1, \ldots, \pm k\}$ is called a $(2k, \beta)$-*boundary* if $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod{2k}$ and $\beta(v) \equiv d(v) \pmod 2$ for every $v \in V(G)$. For a vertex subset $A \subset V(G)$, define its boundary $\beta(A) \in \{0, \pm 1, \ldots, \pm k\}$ such that $\beta(A) \equiv \sum_{v \in A} \beta(v) \pmod{2k}$.

(b) Given a $(2k, \beta)$-boundary, an orientation D of G is called a *$(2k, \beta)$-orientation* if, for every vertex $v \in V(G)$, $d^+_D(v) - d^-_D(v) \equiv \beta(v) \pmod{2k}$.

Fact for β-orientation

Fact:

(a) A graph G admits a circular $\frac{2t+1}{t}$-flow if and only if it has a $(2t+1, t)$-ETO, which is a $(4t+2, \beta)$-orientation of G with $\beta(v) \equiv (2t+1)d_G(v) \pmod{4t+2}, \forall v \in V(G)$.

(b) A graph G admits a circular $\frac{4p}{2p-1}$-flow if and only if it has a $(4p, 2p-1)$-ETO, which is an $(8p, \beta)$-orientation of $2G$ with $\beta(v) \equiv 4pd_G(v) \pmod{8p}, \forall v \in V(G)$.

This is for the convenience of considering parity.

The inductive Thm

Theorem B (LTWZ2013, Wu thesis2012)

Let G be a graph with a $(2k, \beta)$-boundary. Let z_0 be a vertex of $V(G)$, and let D_{z_0} be a pre-orientation of $E(z_0)$ which achieves boundary $\beta(z_0)$ at z_0. Let $V_0 = \{v \in V(G) - z_0 : \beta(v) = 0\}$. If $V_0 \neq \emptyset$, we let v_0 be a vertex of V_0 with smallest degree.

Assume that

(i) $|V(G)| \geq 3$;

(ii) $d(z_0) \leq 2k - 2 + |\beta(z_0)|$;

(iii) $d(A) \geq 2k - 2 + |\beta(A)|$ for any $A \subset V(G) \setminus \{z_0\}$ with $A \neq \{v_0\}$ and $|V(G) \setminus A| > 1$.

Then pre-orientation D_{z_0} at z_0 can be extended to a $(2k, \beta)$-orientation of the entire graph G.

Proof Techniques:

- **Contraction**
- **Splitting**
- **Deletion**

Each graph is (somehow) reducible by one of the above operations!

Proof Sktech:

- **Claim A.** for any nontrivial A, $d(A) \geq 2k + |\beta(A)|$
(nontrivial edge-cuts are large by Contraction)

- **Claim B.** for any nontrivial $A = \{v\}$,
$d(v) = 2k - 2 + |\beta(v)|$
(each vertex is "regular" by Splitting)

- **Claim C.** for each vertex v, $\beta(v)$ has the same sign (say > 0)
(each vertex has positive boundary by Deletion)

- **Final.** Modify to get a vertex negative boundary, resulting a contra!

Thm and applications

Theorem B' (LTWZ2013, Wu thesis2012)

Let G be a $(3k-3)$-edge-connected graph, where $k \geq 2$ is an integer (OK for both even and odd). Then for any $(2k, \beta)$-boundary β, G admits a $(2k, \beta)$-orientation.

Two Applications:

- For any $(3k-3)$-edge-connected simple graph G with $|E(G)|$ being a multiple of k, $E(G)$ can be decomposed into $K_{1,k}$'s.
- **Thomassen 2020:** Every 7-odd-edge-connected 9-regular-graph can be edge-decomposed into three 3-regular subgraphs.

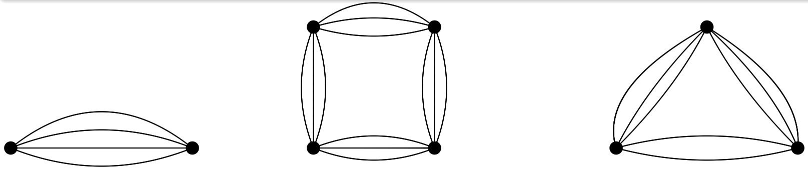
Similar technique for planar mod orientations

Theorem (Cranston-Li 2020)

Every 11-odd-edge-connected graph admits mod 5-orientation.

Corollary (Cranston-Li 2020)

(i) Every planar graph with girth ≥ 10 admits a hom to C_5.
(ii) Every directed planar graph of girth$\geq$ 11 admits a homomorphism to any 5-vertex tournament.



H_1 H_2 H_3

Figure: Some easy configurations, but not enough

Spanning trees and Modulo Orientation

Necessary Condition for modulo orientations

Proposition

If G admits modulo $(2p+1)$-orientation with all possible boundaries, then G contains $2p$ edge-disjoint spanning trees.

Nash-Williams and Tutte Theorem

G contains k edge-disjoint spanning trees if and only if for any partition $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$, $\sum_{i=1}^{t} d(P_i) - 2kt + 2k \geq 0$.

Motivated by those facts, we are trying to use spanning tree packing number to provide Sufficient Condition

Partitions for mod 5-orientation

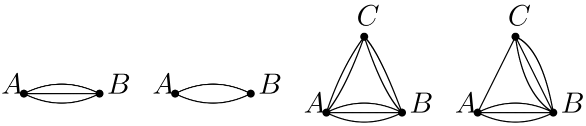
We define a weight function below to have some flexibility.

Definition

Let $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ be a partition of $V(G)$. Define $w_G(\mathcal{P}) = \sum_{i=1}^{t} d(P_i) - 11t + 19$ and

$$w(G) = \min\{w_G(\mathcal{P}) : \mathcal{P} \textit{ is a partition of } V(G)\}.$$

For example, $w(2K_2) = 1$, $w(J_1) = w(J_2) = 0$, $w(3K_2) = 3$.

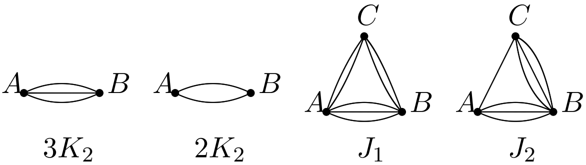


$3K_2$ $2K_2$ J_1 J_2

The Theorem for induction

Theorem

Let G be a planar graph and β be a Z_5 boundary of G. If $w(G) \geq 0$, then G admits a mod 5 β-orientation, unless G is one of the following problematic cases that there is a partition $\mathcal{P}$ such that $G/\mathcal{P}$ is isomorphic to one of the graphs $2K_2, 3K_2, J_1, J_2$.

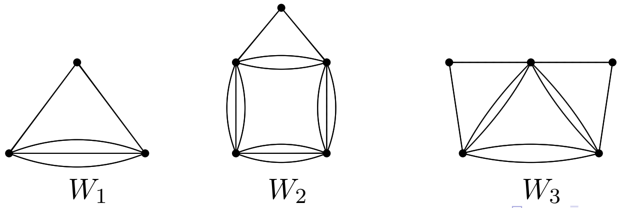


Idea in the proof

In a minimal counterexample G,

- Claim A: $w_G(\mathcal{P}) \geq 8$ for any "nontrivial" partition $\mathcal{P}$. This allows to use splitting and contraction.
- Claim B: W_1, W_2, W_3 are all forbidden in G by splitting and contraction.

Then a simple discharging would finish the proof. It would be nice to extend to general mod $(2p+1)$-orientation.



Flows for bidirected signed graphs

André Bouchet in 1983 initiated Integer Flows of bidirected signed graphs from the dual of face-coloring in projective plane

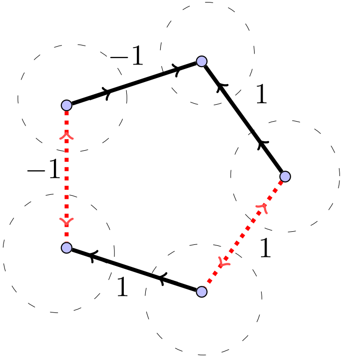
Flows in signed graphs

Let G be a **signed graph** and τ be an orientation of G.

- A pair (τ, f) is called a **flow** of G if

  inflow=outflow at every vertex

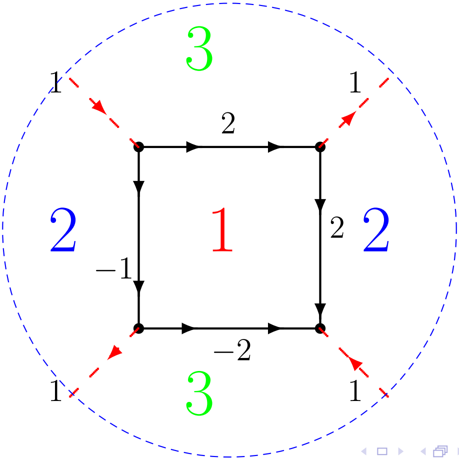
  (looking at those 'half-edges' incident with a vertex)



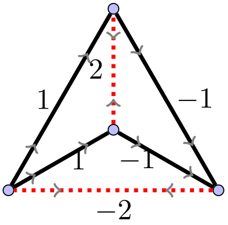
Coloring-Flow Duality of Signed Graphs

- For a nonorientable surface,
  'k-face-coloring' $\Rightarrow$ Signed Graph k-NZF

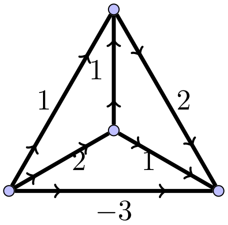
Coloring-Flow of K_4 in **projective plane**:



NZF of Ordinary K_4 and Signed K_4

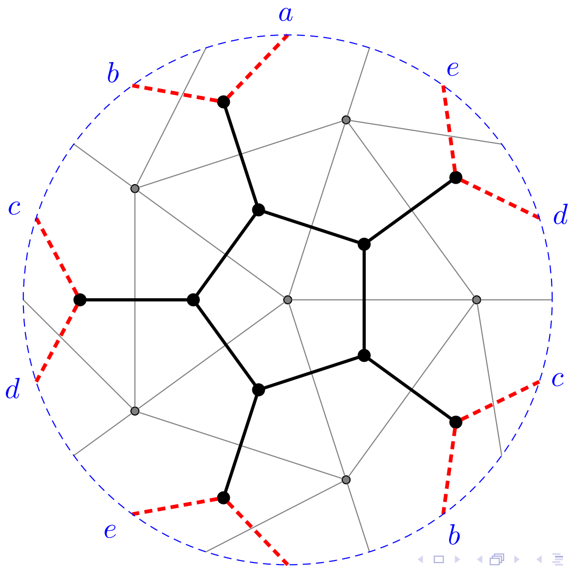


3-NZF of a signed K_4

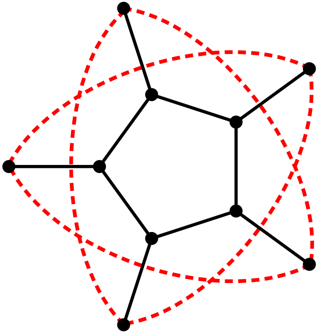


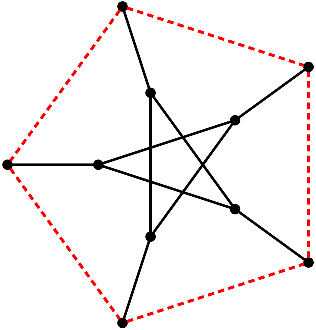
4-NZF of K_4

K_6 dual to Signed Petersen Graph in Projective Plane



K_6 dual to Petersen (Re-drawing)





It has no 5-NZF by Bouchet's dual Thm

Flow Index Table of Signed Graphs

Flow Index and Edge Connectivity of Signed Graphs

Edge-Conn.	Conjectured ϕ	Known ϕ
0,1	$\phi \leq 6$(Bouchet83)	$\phi \leq 11$(DLLLZZ20)
2,3	$\phi \leq 6$(Bouchet83)	?
4	*	$\phi \leq 4$(D03?)(RZ11)
5	$\phi \leq 3$(WYZZ15)	*
6	*	$\phi < 4$(RZ11)
8	*	$\phi \leq 3$(WYZZ15)
...	...	...
12	*	?$\phi < 3$(maybe??)
20	*	?$\phi \leq 2.5$(maybe??)
$12p-1$	*	$\phi \leq 2+\frac{1}{p}$(Zhu15)

Thank you for your attention !

Thank you very much!