A refinement of choosability of graphs

Xuding Zhu

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The reason why a k-colorable graph is not L-colorable for a k-list assignment L is that lists assigned to vertices by L may be complicately entangled.

Definition 1. By a partition of a positive integer k we mean a finite multiset $\lambda = \{k_1, k_2, \ldots, k_q\}$ of positive integers with $k_1 + k_2 + \ldots + k_q = k$. Each integer $k_i \in \lambda$ is called a *part* of λ.

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Definition 2. Assume $\lambda = \{k_1, k_2, \ldots, k_q\}$ is a partition of k and G is a graph. A λ-*assignment* of G is a k-assignment L of G in which the colours in $\cup_{x \in V(G)} L(x)$ $= C$ can be partitioned into sets $C_1, C_2, \ldots, C_q$ so that for each vertex x and for each $1 \leq i \leq q$, $|L(x) \cap C_i| = k_i$. Each C_i is called a *colour group* of L. We say G is λ-*choosable* if G is L-colourable for any λ-assignment L of G.

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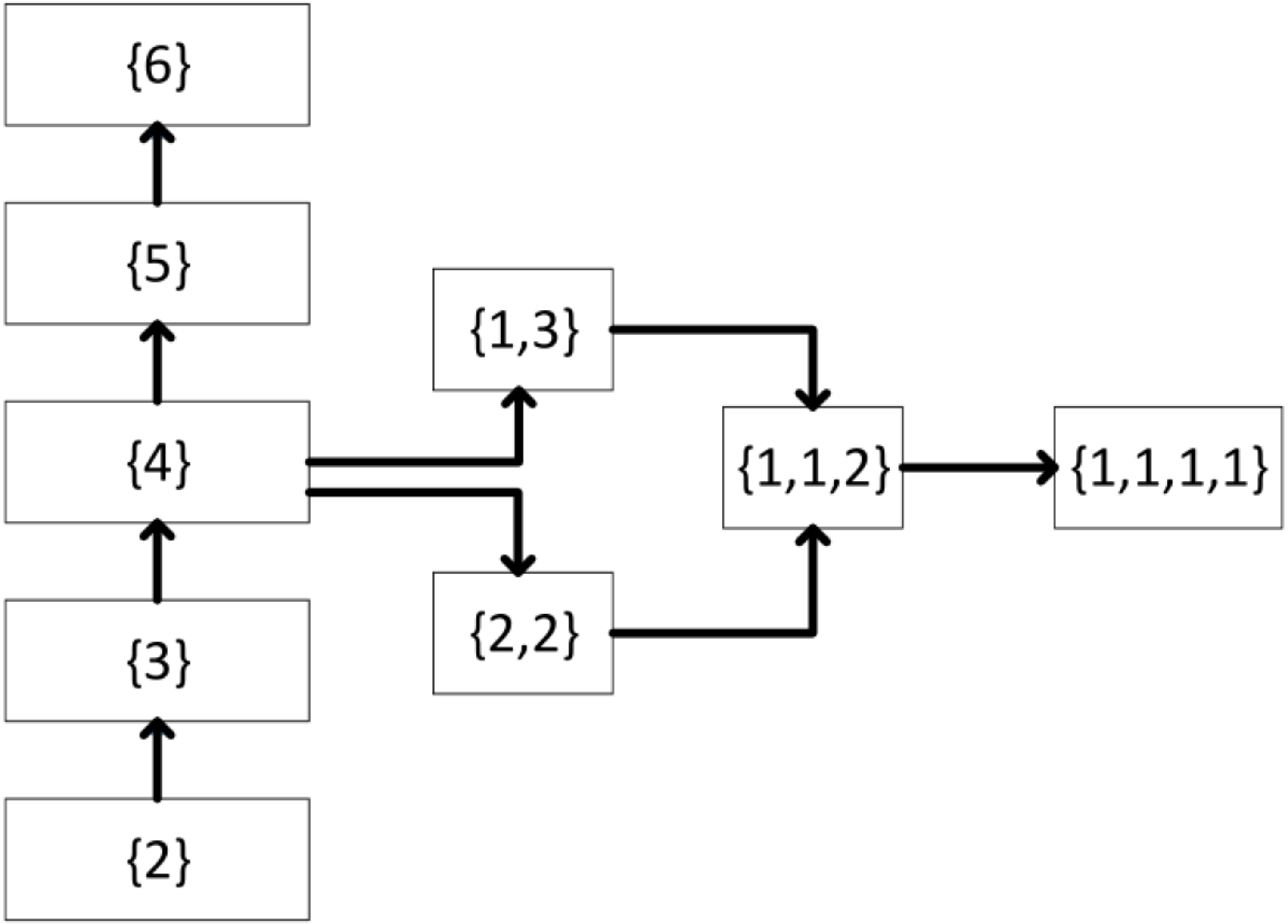
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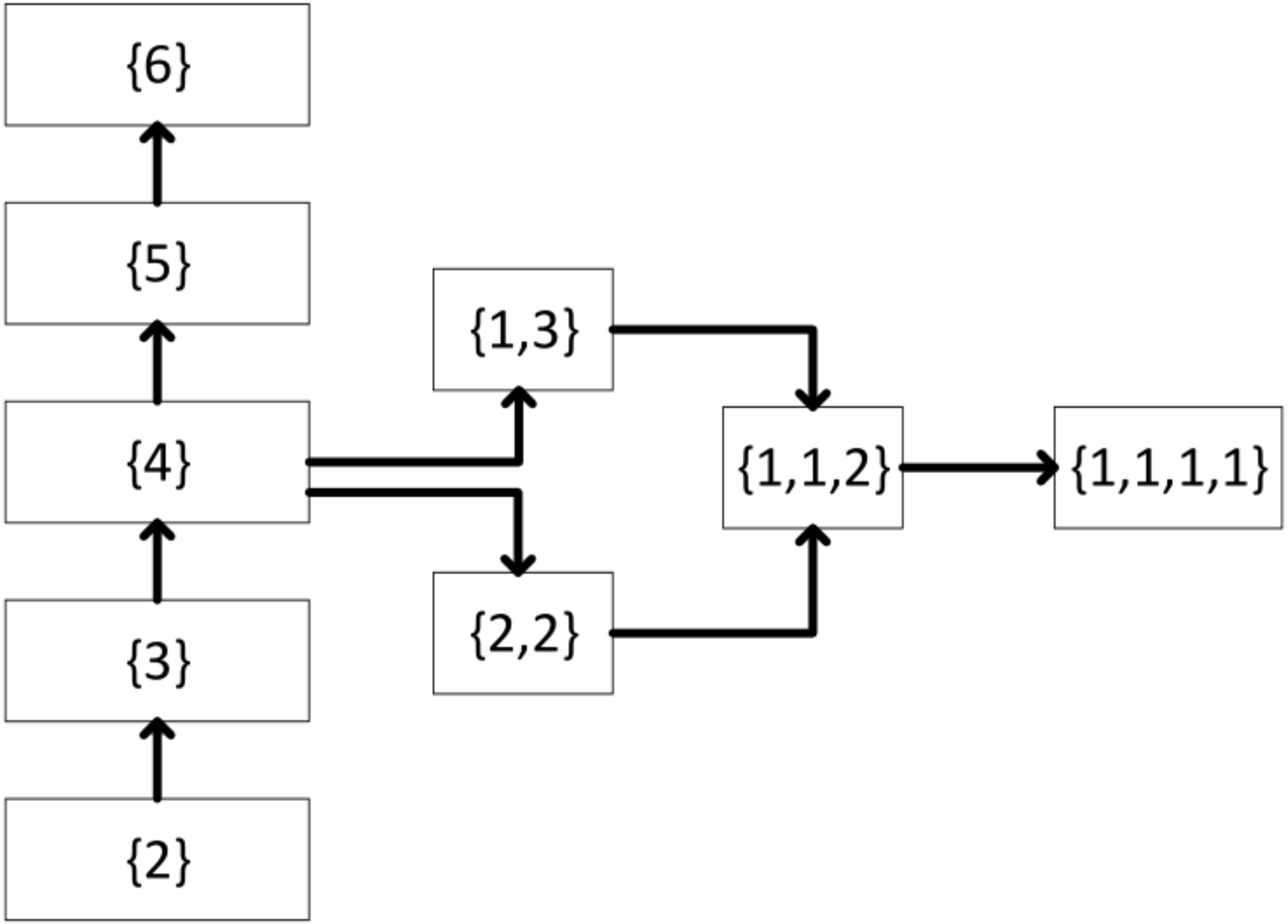

4CT is tight in the refined scale of λ-choosability.

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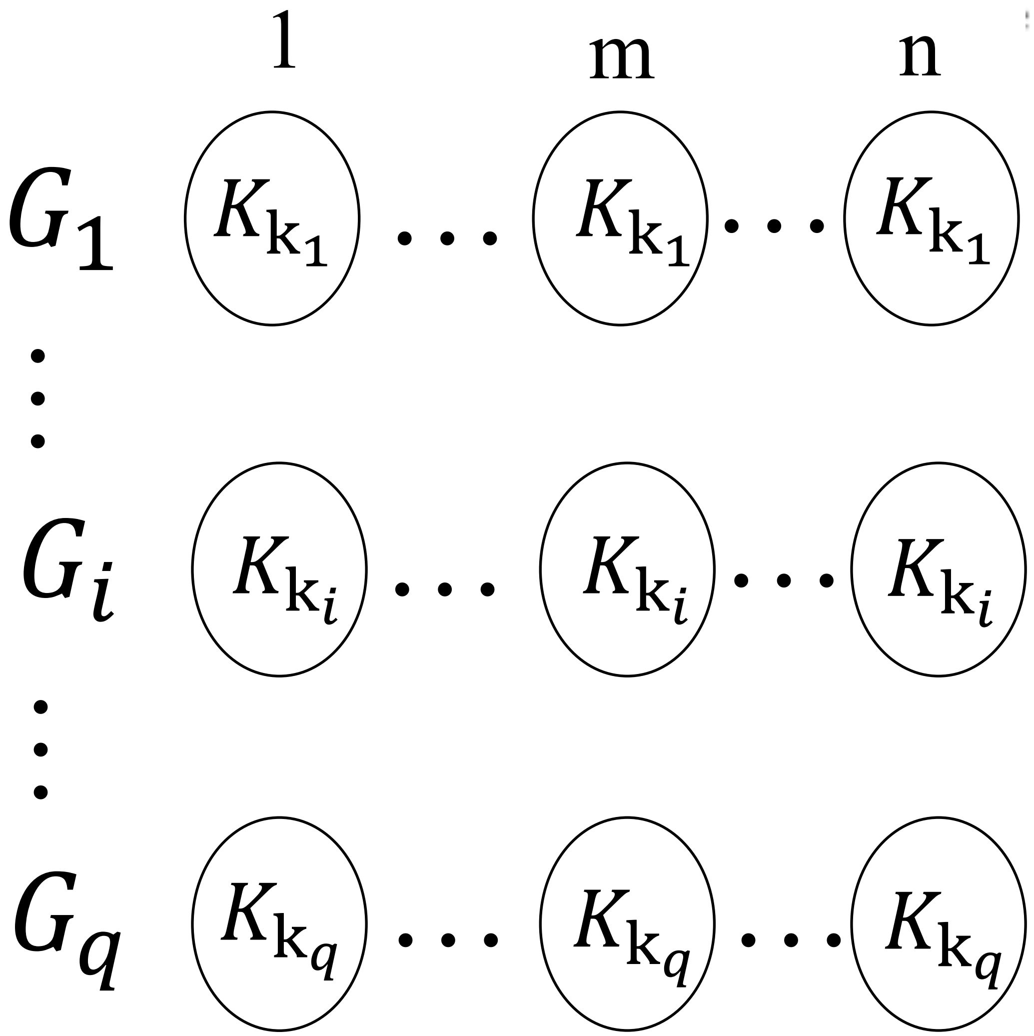

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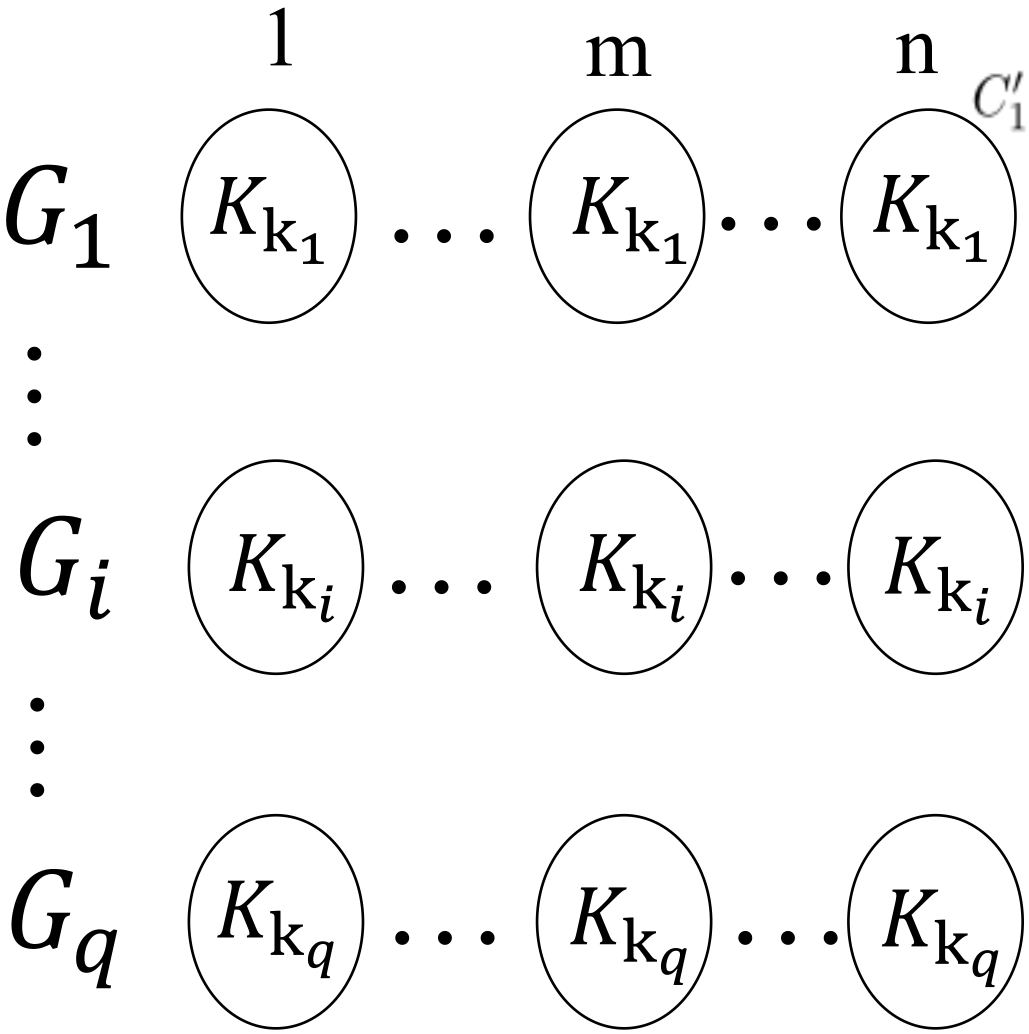

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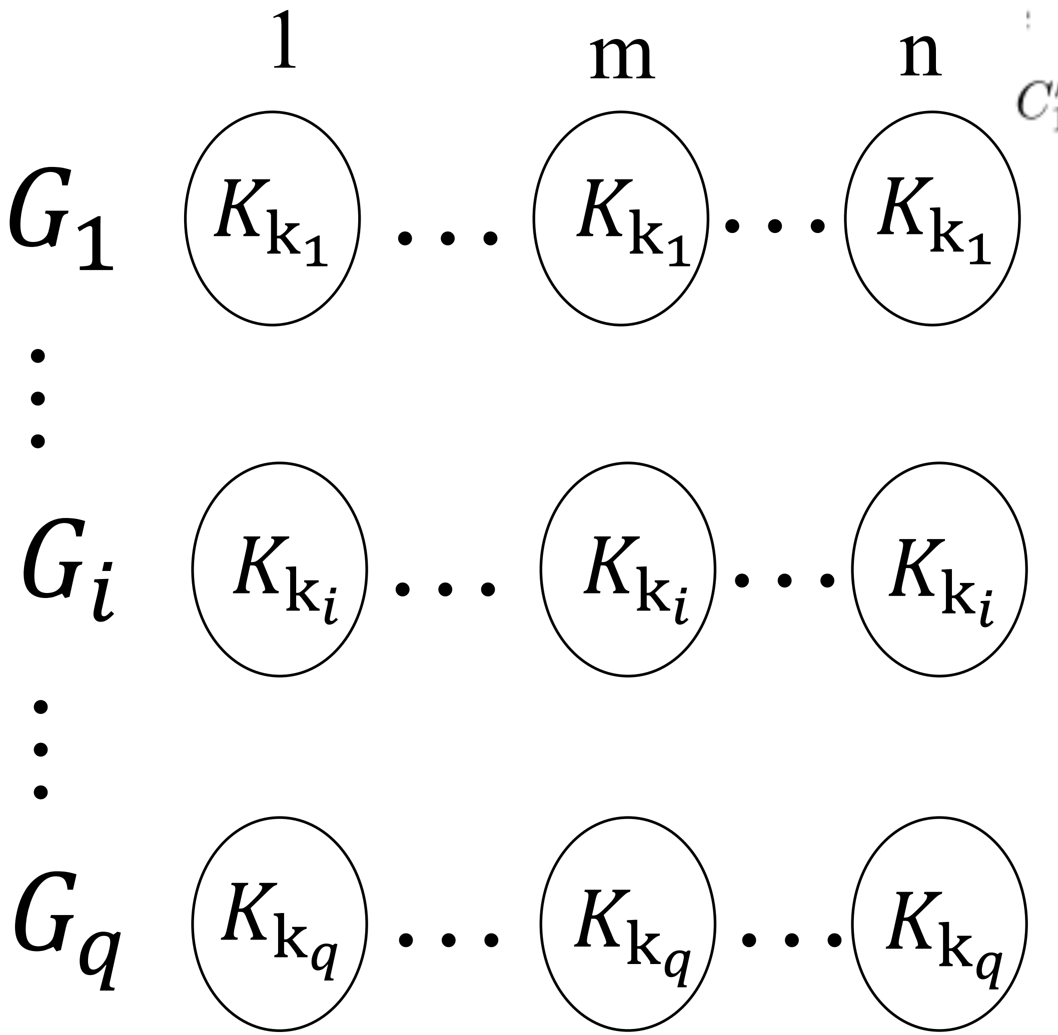

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$\binom{C'_j}{k'_j}$ is the family of all k'_j-subsets of C'_j.

$\mathcal{S} = \binom{C'_1}{k'_1} \times \binom{C'_2}{k'_2} \times \ldots \times \binom{C'_p}{k'_p}$. $n = |\mathcal{S}|$.

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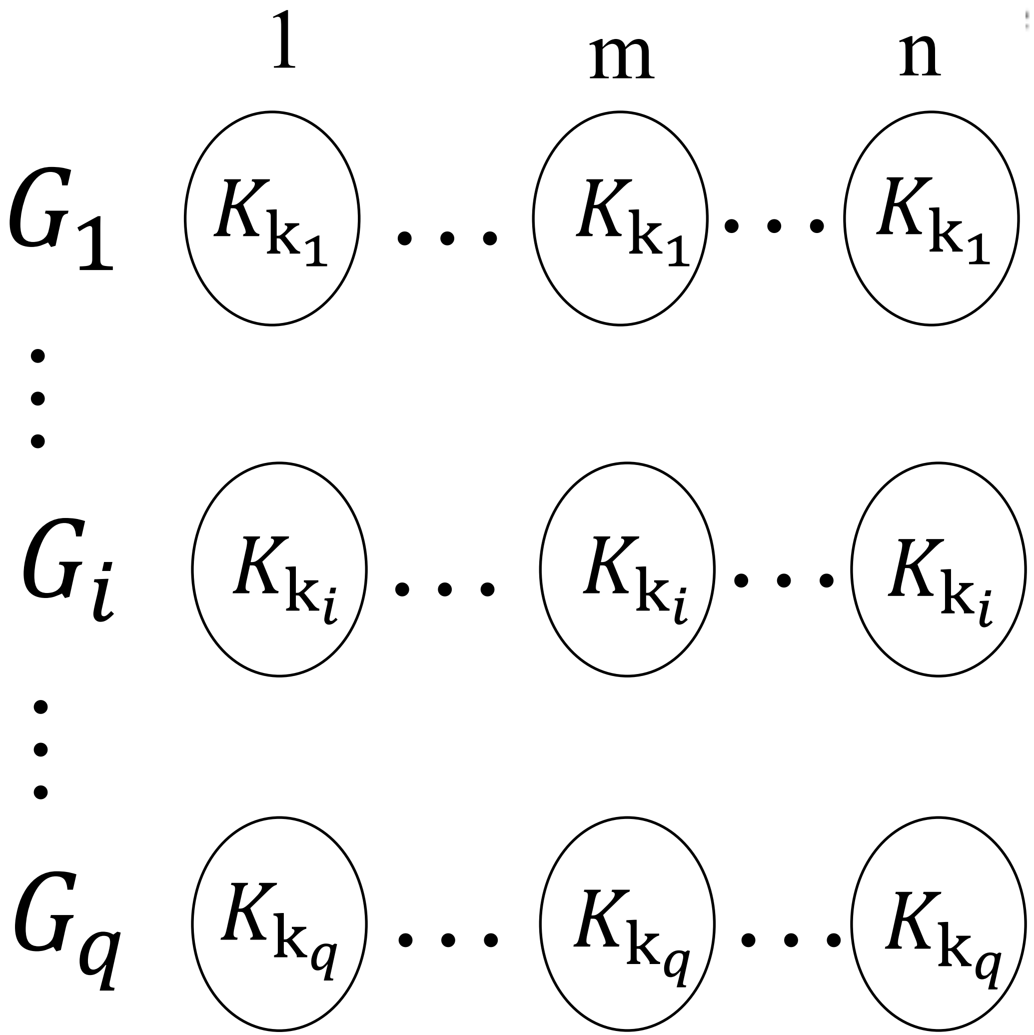

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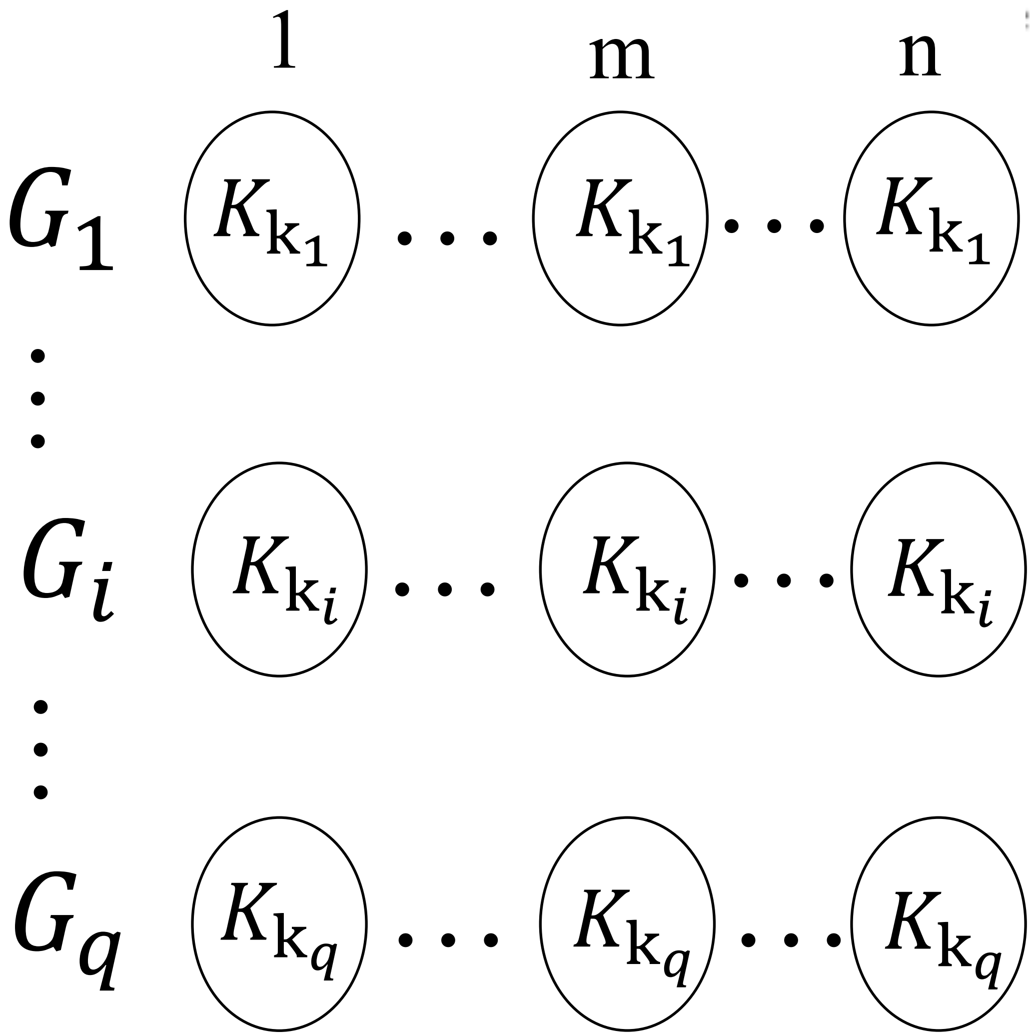

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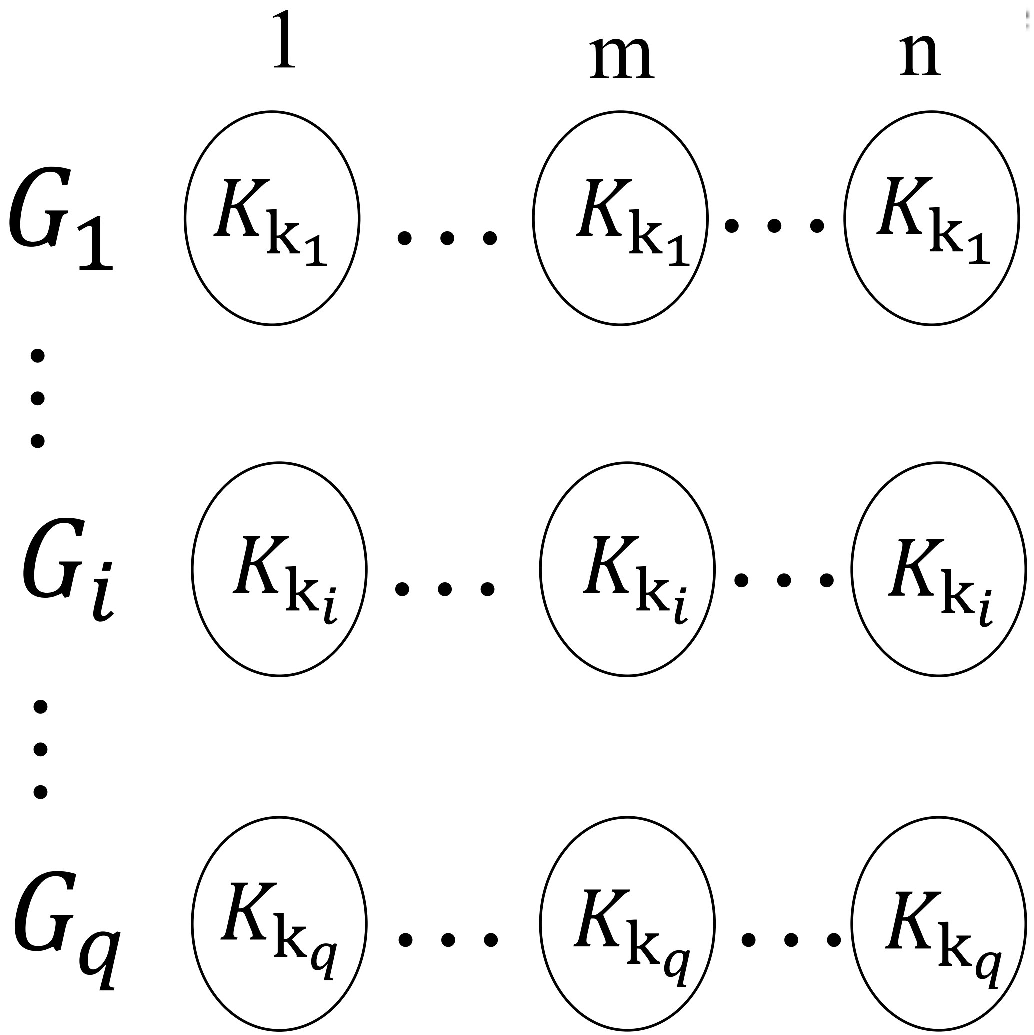

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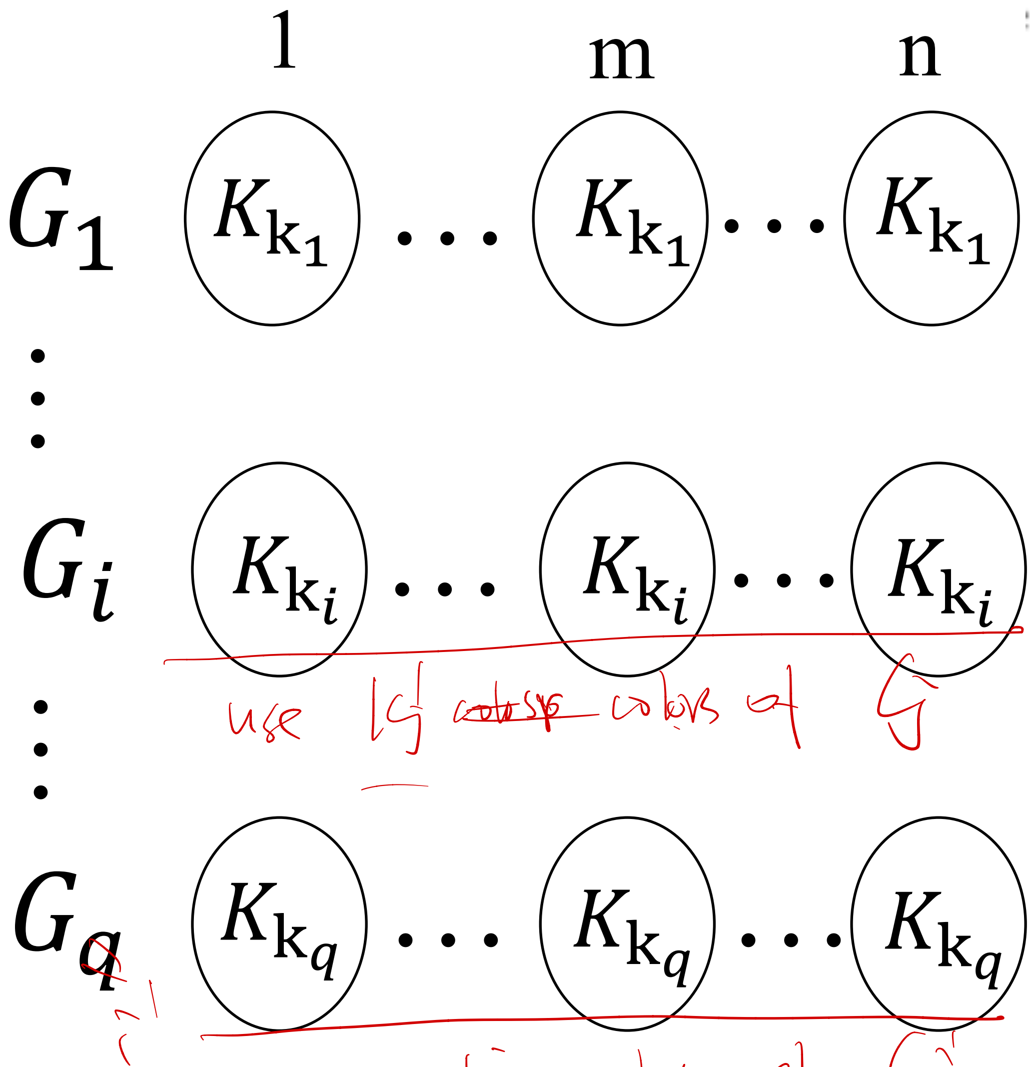

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$\lambda = (k_1, k_2, \ldots, k_q)$ and $\lambda' = (k_1', k_2', \ldots, k_p')$.

C_j' *is occupied by* G_i

if at least k_j' colours in C_j' are used by vertices in G_i.

$$J_i = \{j : C_j' \text{ is occupied by } G_i\}. \quad \neq \emptyset,$$

For $i, i' \in \{1, 2, \ldots, q\}$ *and* $i \neq i'$, *we have* $J_i \cap J_{i'} = \emptyset$.

$j \in J_i \cap J_{i'}$

$|C_j'| = 2k_j' - 1$

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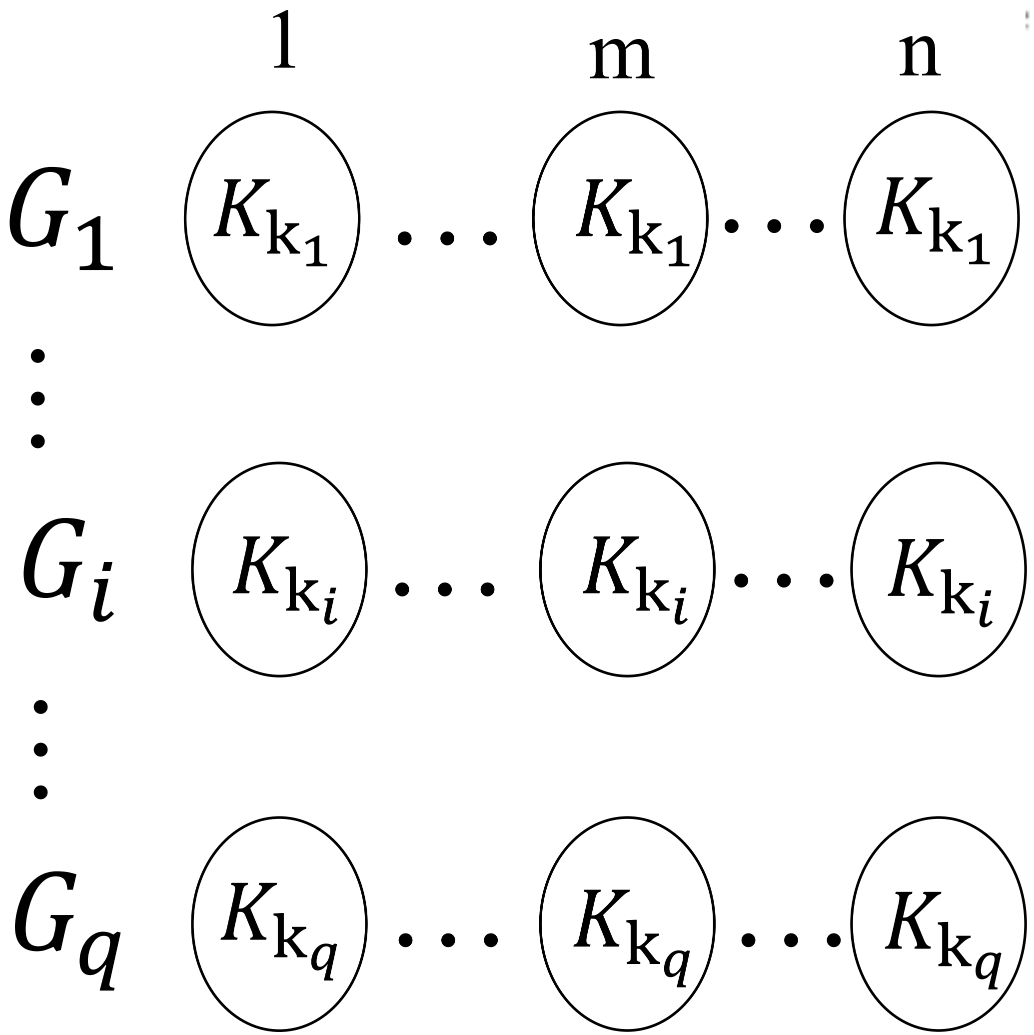

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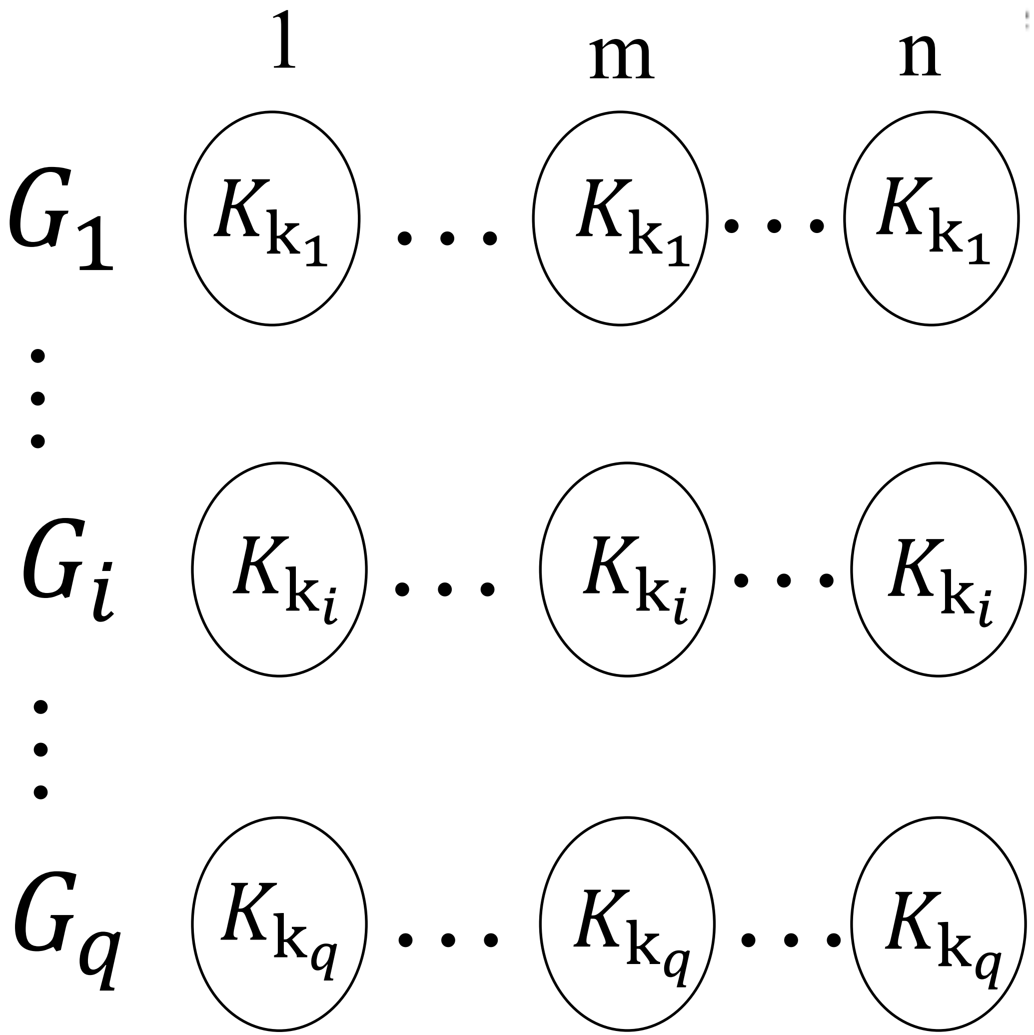

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For each index $i \in \{1, 2, \ldots, q\}$, $\sum_{j \in J_i} k'_j \geq k_i$.

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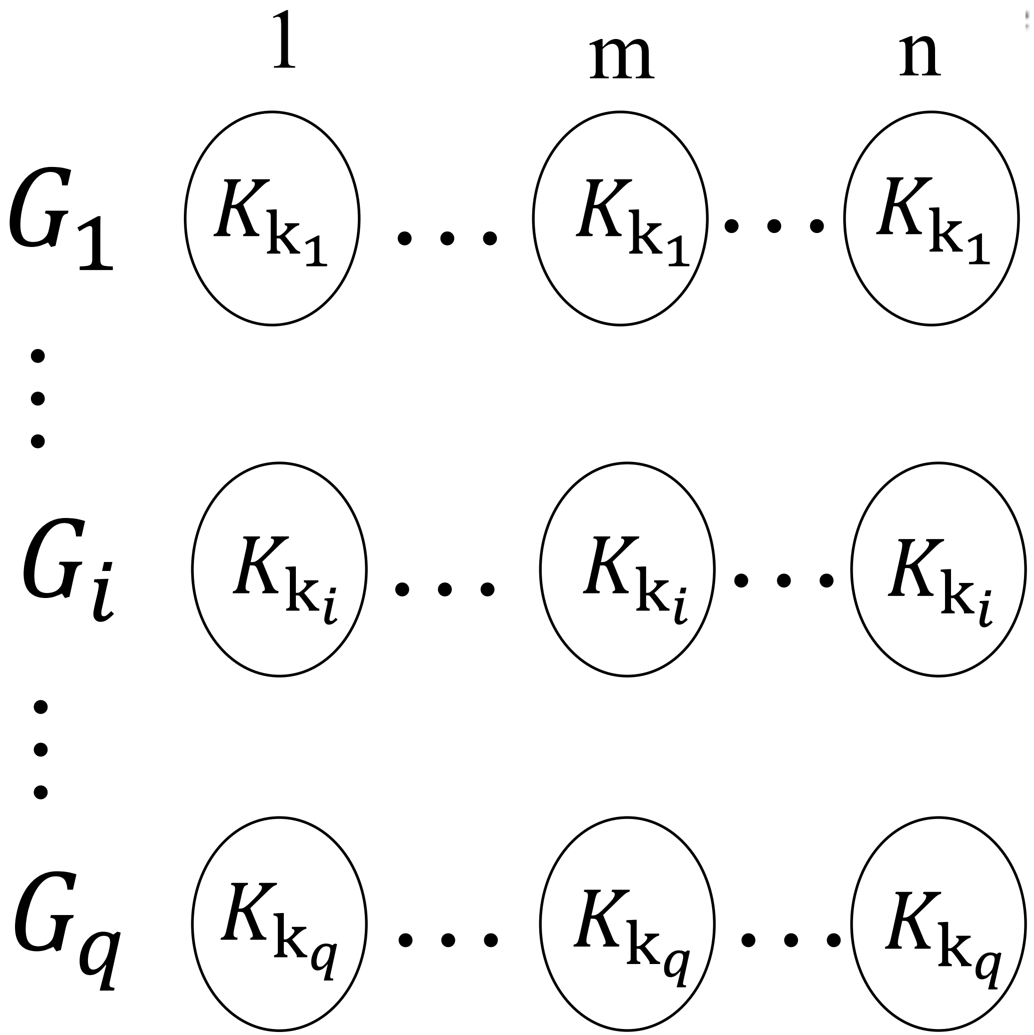

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Signed graph colouring and λ-choosability

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A k-colouring of (G, σ) is a mapping $f : V(G) \to N_k$ such that for each edge $e = xy$,

$$f(x) \neq \sigma(e) f(y).$$

$N_k = \{0, 1, -1, 2, -2, \ldots, q, -q\}$ if $k = 2q + 1$ is odd.

$N_k = \{1, -1, 2, -2, \ldots, q, -q\}$ if $k = 2q$ is even

a Z_k-colouring of (G, σ) is a mapping $f : V(G) \to Z_k$ such that for each edge $e = xy$, $f(x) \neq \sigma(e) f(y)$.

A graph G is signed k-colourable (respectively, signed Z_k-colourable) if for any signature σ of G, the signed graph (G, σ) is k-colourable (respectively, Z_k-colourable).

A set I of integers is called symmetric if for any integer i, $i \in I$ implies that $-i \in I$.

An assignment L of a graph G is symmetric if for each vertex v of G, $L(v)$ is symmetric.

We say G is weakly k-choosable if G is L -colorable for any symmetric assignment L.

Theorem 13. *Every signed 4-colourable graph is weakly 4-choosable.*

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$$\sigma(e) = \begin{cases} -1, & \text{if } \min L^+(u) = \max L^+(v) \text{ or } \min L^+(v) = \max L^+(u), \\ 1, & \text{otherwise.} \end{cases}$$

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$$\phi(v) = \begin{cases} \max L^+(v), & \text{if } f(v) = 2, \\ -\max L^+(v), & \text{if } f(v) = 1, \\ \min L^+(v), & \text{if } f(v) = -2, \\ -\min L^+(v), & \text{if } f(v) = -1. \end{cases}$$

Theorem 13. *Every signed 4-colourable graph is weakly 4-choosable.*

There is a graph which is weakly 4-choosable but not signed 4-colorable.

Conjecture 1 : [*Máčajová*, Raspaud and *Škoviera*, 2016]
Every planar graph is signed 4-colorable.

Conjecture 2 : [K*ündgen* and *Ramamurthi*, 2002]
Every planar graph is weakly 4-choosable.

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$Kardoš$ and $Narboni$ constructed a planar graph which is not signed 4-colorable.

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$$\phi(v) = \begin{cases} \max L'(v), & \text{if } f(v) = 3, \\ \min L'(v), & \text{if } f(v) = 1, \\ 1, & \text{if } f(v) = 0, \\ 2, & \text{if } f(v) = 2. \end{cases}$$

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Conjecture: [Kang and Steffen , 2017]
Every planar graph is signed z_4-colorable.

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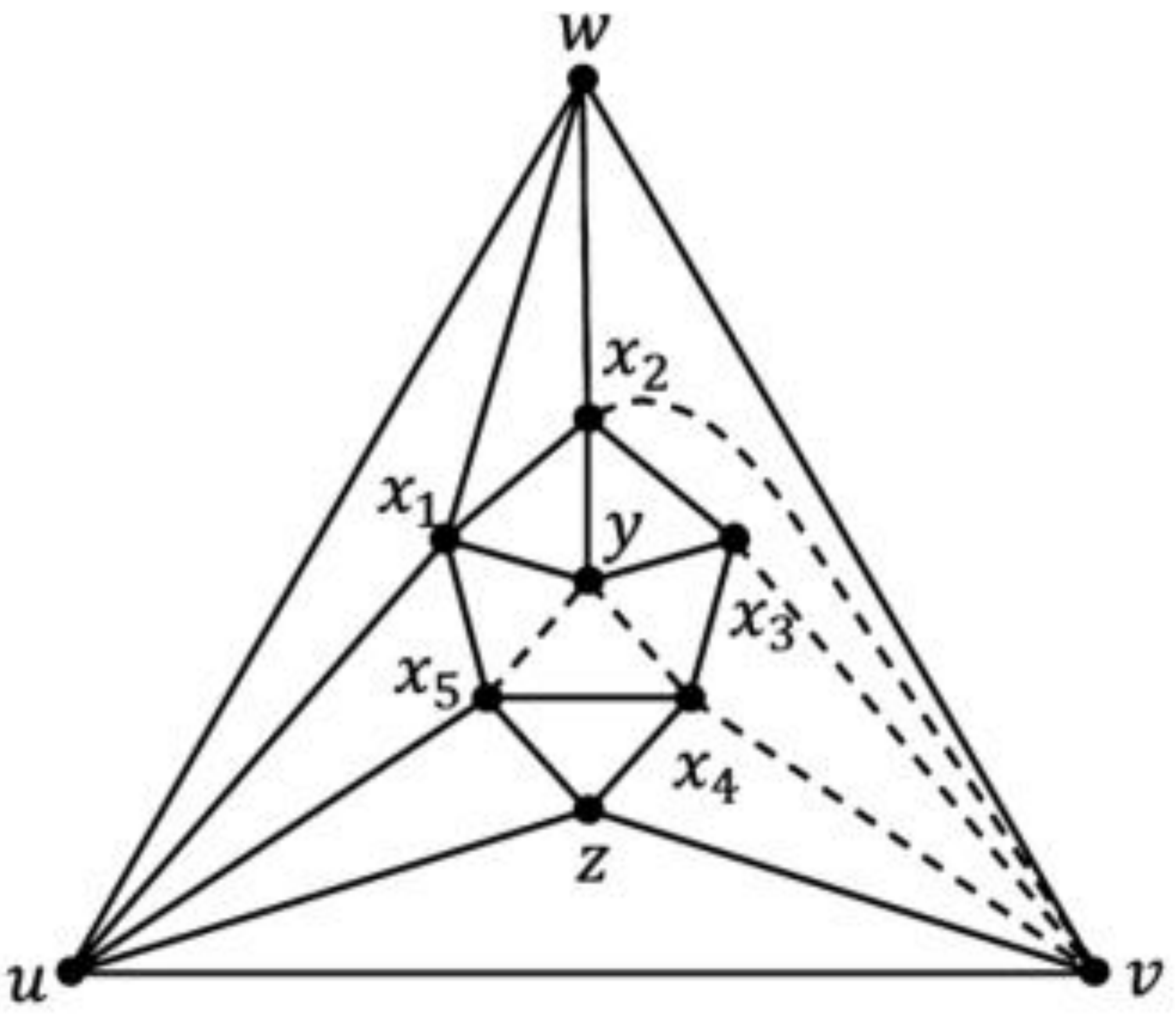
For any Z_4-colouring f of (H, σ), $\{f(u), f(v)\} \cap \{0, 2\} \neq \emptyset$.



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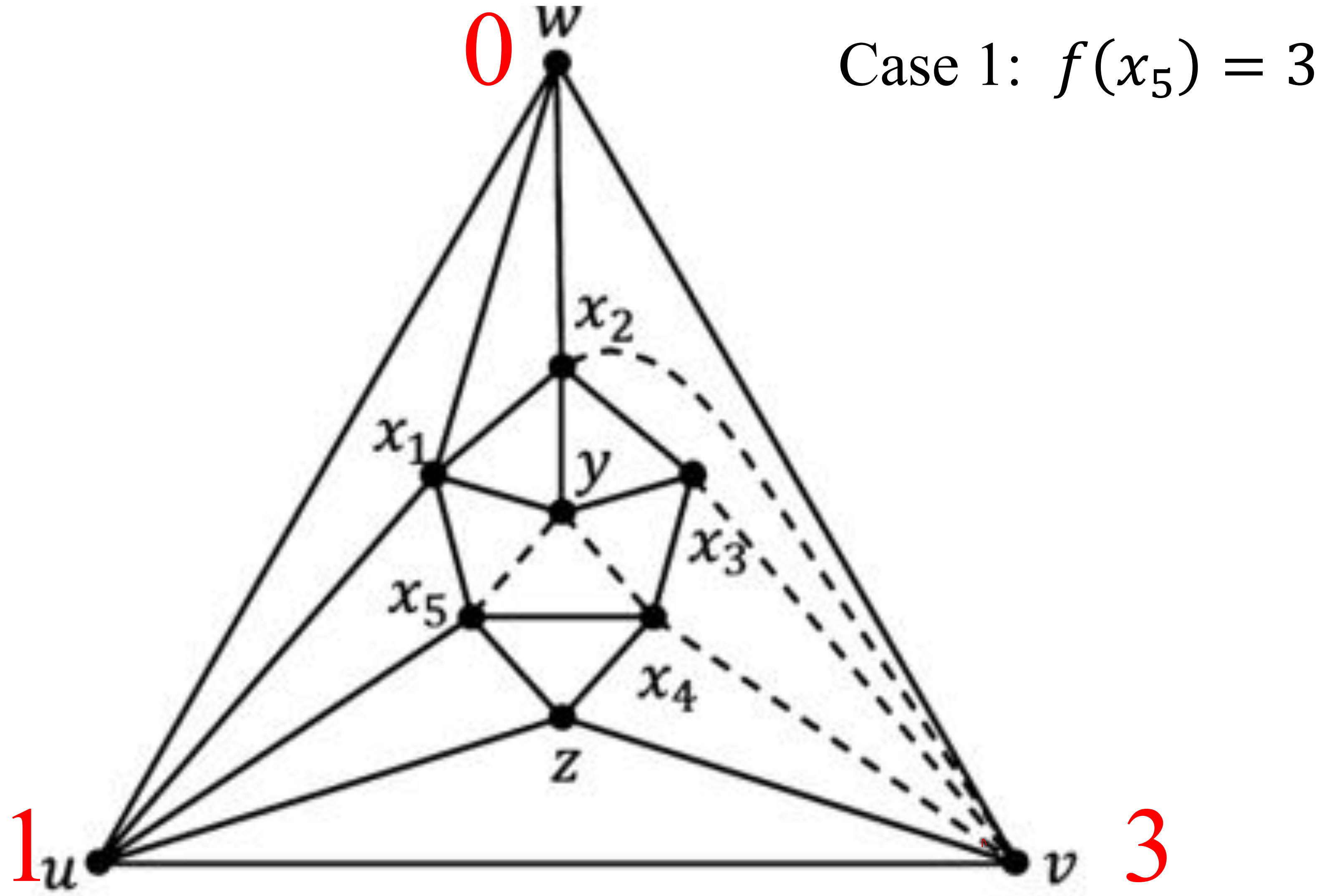

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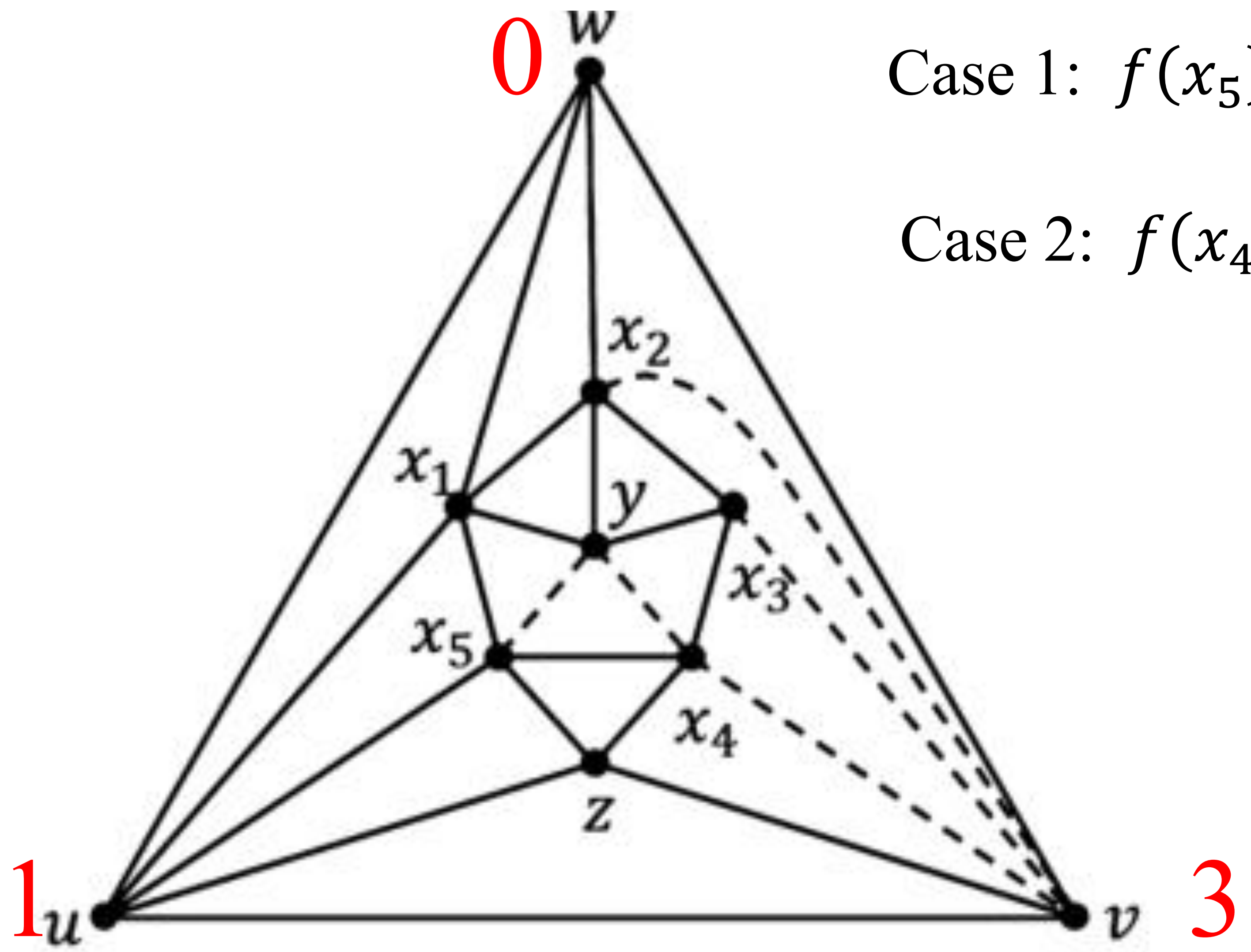

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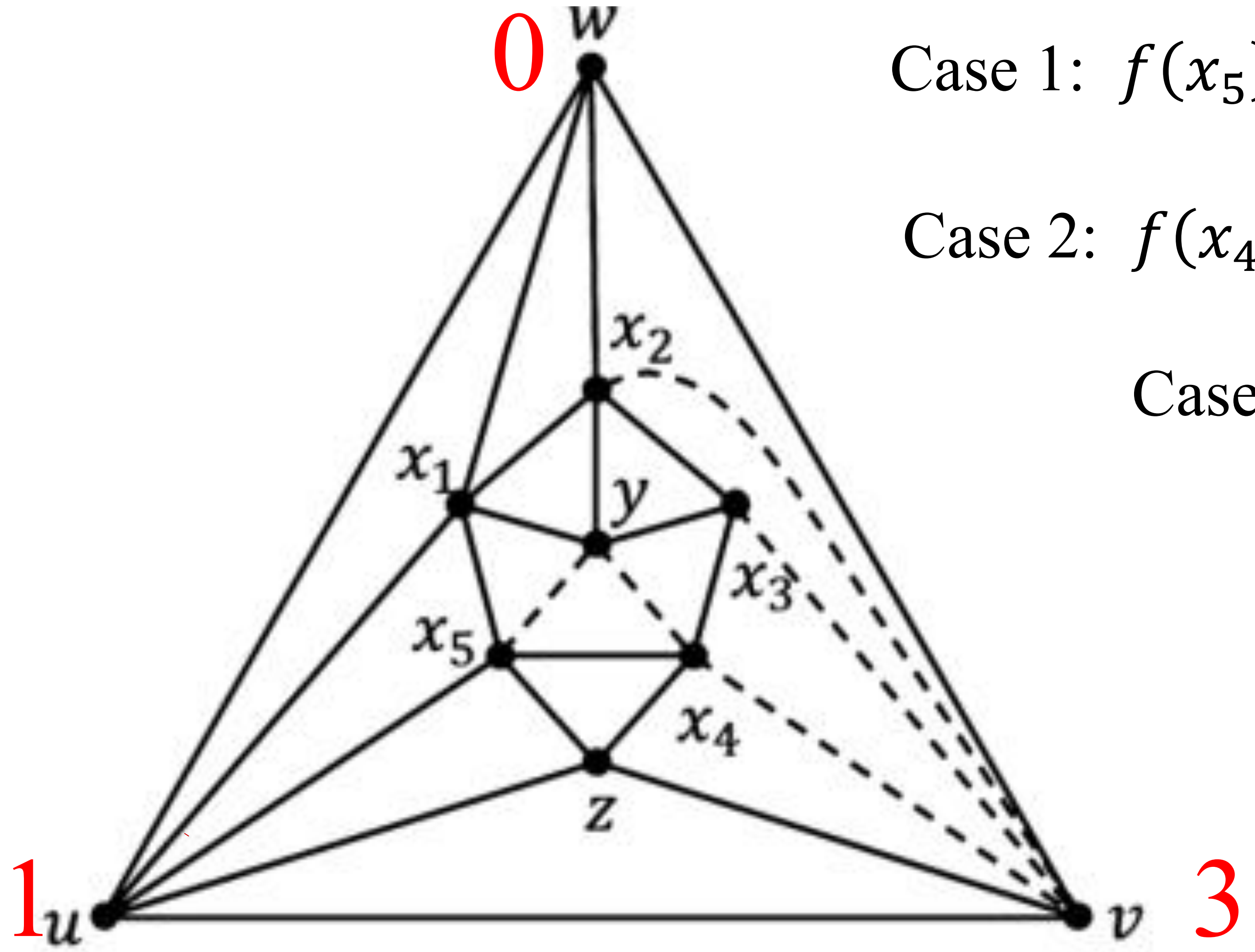

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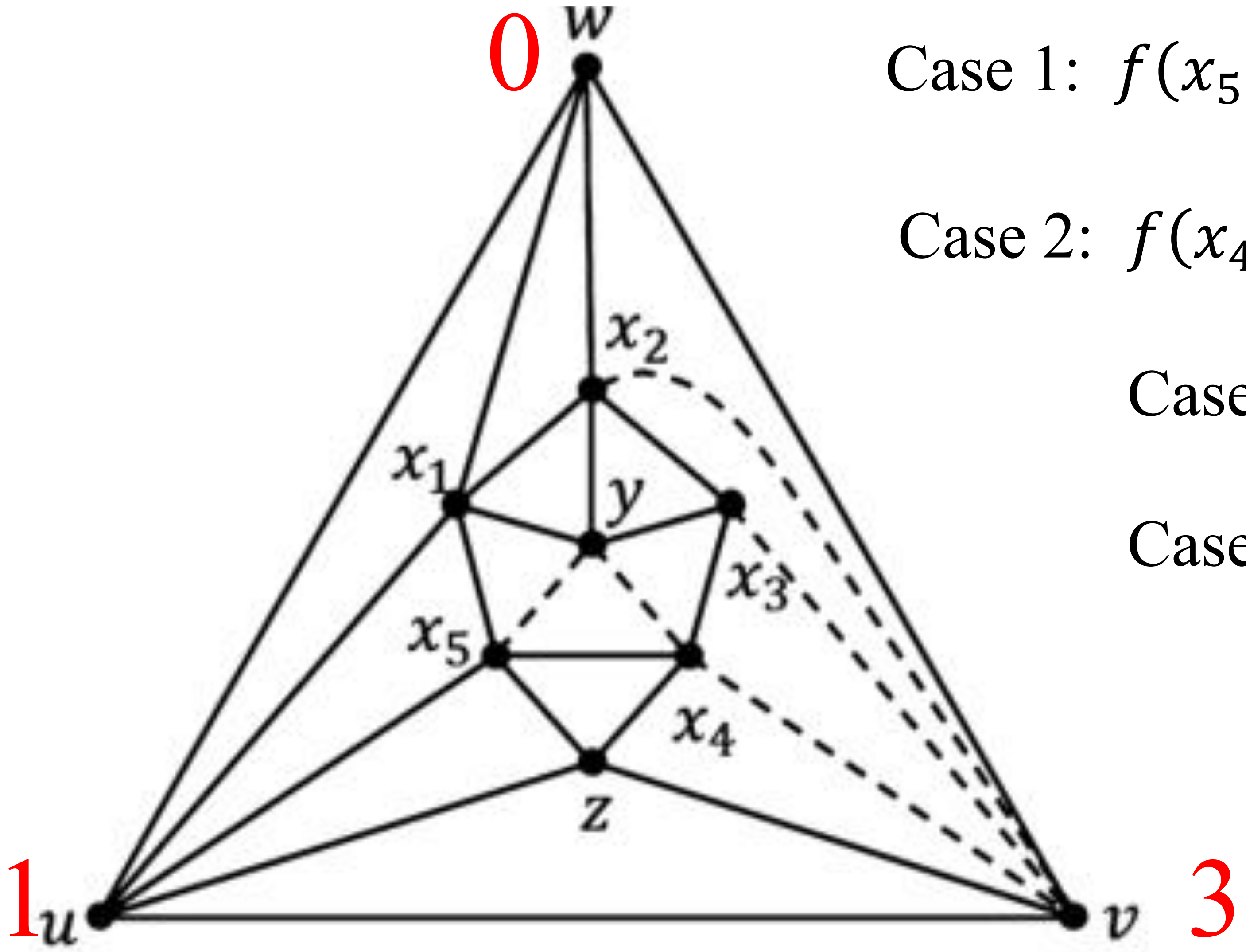

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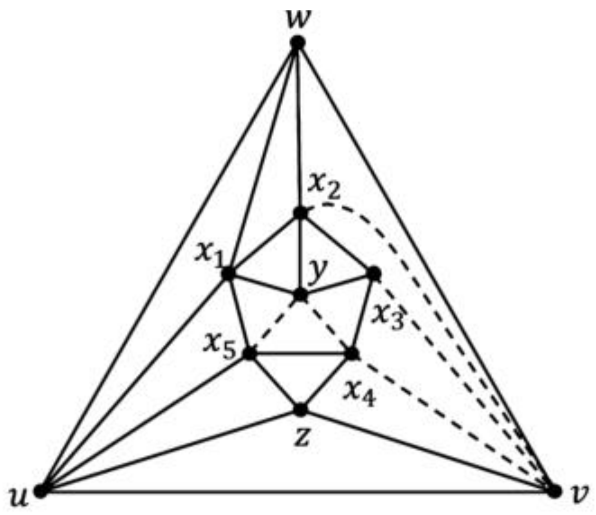

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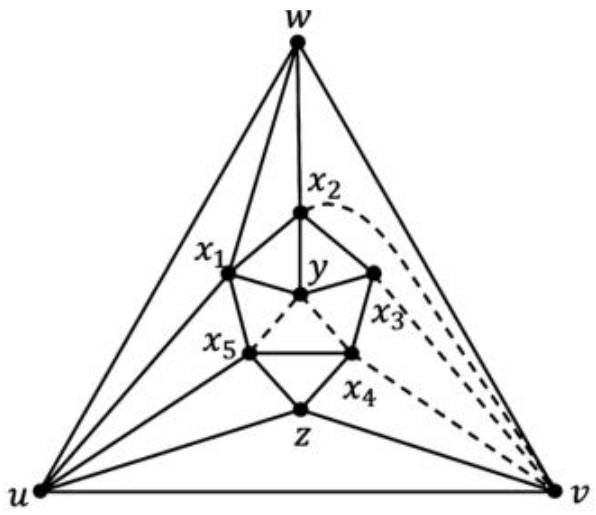

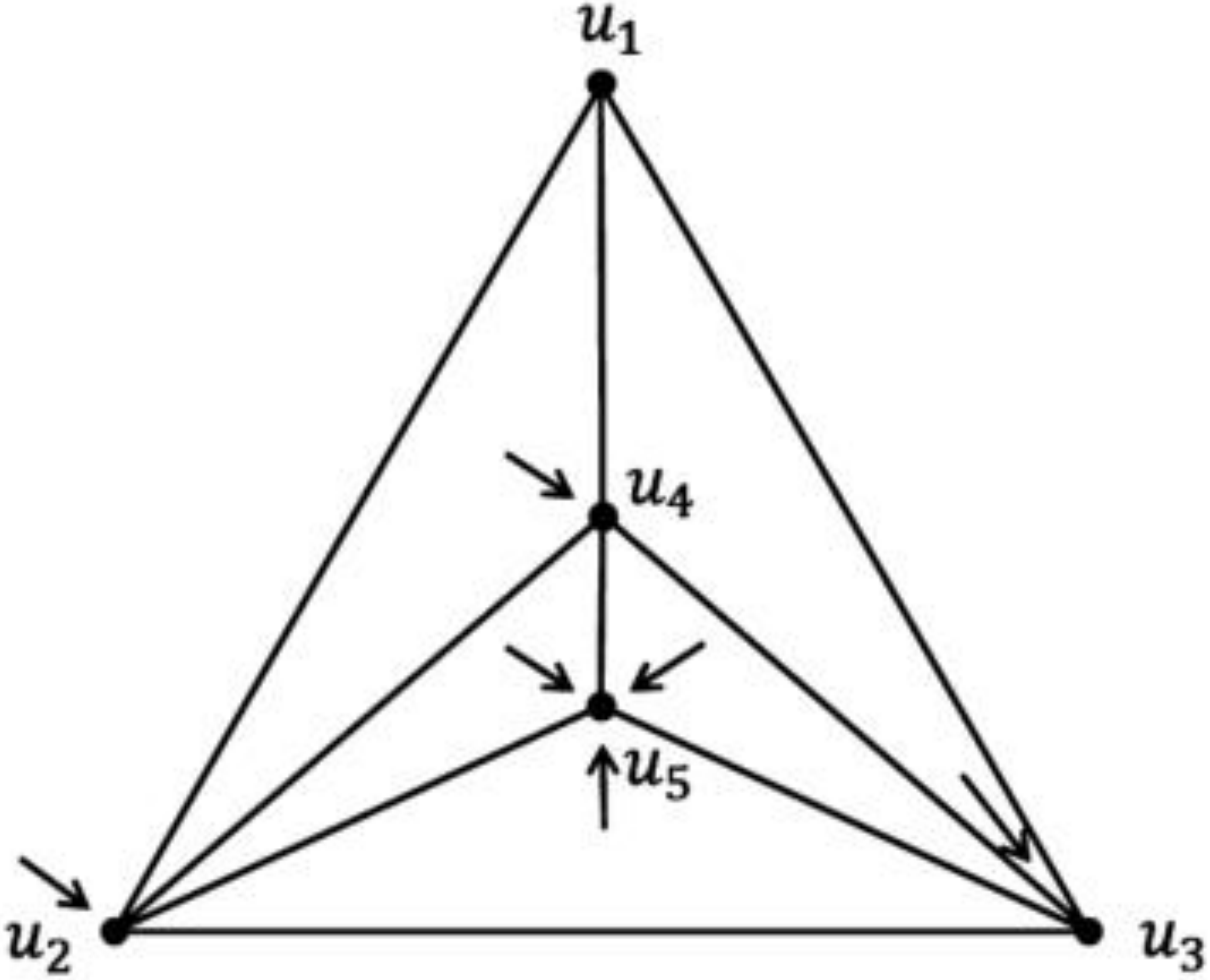

Thank you!