Université de Paris

Graphs Algorithms 2-29-1
MPRI - Final exam
March 1st 2021
3 hours

Instructions

There are two sections, the first one concerns the first part of the course given by R. Naserasr, and the second concerns the part of the course given by P. Charbit.

In both sections the exercises are independent and are not sorted by increasing difficulty.

You have to send your solutions as a files (preferably pdf, but if needed you can scan or take photos) and send them via email to reza@irif.fr before 19 :15.

First part of the course

Exercise 1 : Triangle-free graphs of high chromatic number
Prove that for every positive integer k there exists a triangle-free graph of chromatic number k.

Exercise 2 : List chromatic number
Let G be a graph and for each vertex v of G suppose there is a list $L(v)$ of colors which is the set of available colors for the vertex v.
The graph G is said to be k-list colorable if for any list assignment with each list being of size k there is a proper coloring of G where color of v is in $L(v)$.

1. Show that even cycles are 2-list colorable.

2. Show that odd cycles are not 2-list-colorable.

3. Show that $K_{3,3}$ is not 2-list colorable, but it is 3-list colorable.

4. Show that for any k there is a bipartite graph which is not k-list colorable.

5. Extend Brooks theorem to list-coloring. That is to say : if G is a graph with $\Delta(G) \geq 3$ and it is not a complete graph then G is $\Delta(G)$-list-colorable.

Exercise 3 : Edge-coloring question

1. Present a formula for the edge-chromatic number of the complete graph of order n, prove your claim.

Let G be a graph obtained from $K_{3,3}$ by subdividing an edge once.

2. Show that G is 2-connected.

3. Show that G is not 3-edge-colorable.

4. What is the edge-chromatic number of this graph ?

Exercise 4 : Ramsey Theory

1. Find $R(3,4)$, prove your claim.

A tournament order nm denoted T_n, is an orientation of the complete graph K_n. A transitive tournament od order n, denoted TT_n is a tournament with no directed cycle.

2. Prove that a tournament T_n is a transitive tournament if and only if it has no directed triangle.

3. Prove that for each positive integer k, there exists a (smallest) integer $n = f(k)$ such every tournament on n vertices contain TT_k as a subtournament. Determine $f(3)$ and $f(4)$.

Exercise 5 : Eigenvalues
Definition : $\lambda_1(G)$ is the largest eigenvalues of G. $\bar{d}(G)$ is the average degree of G.

1. Prove that $\bar{d}(G) \leq \lambda_1(G) \leq \Delta(G)$

2. Find all the eigenvalues of the complete graph K_n, the cycle C_n, and the Petersen graph.

Second part of the course

Exercise 6 : Well Quasi Orders
In this exercise, the partial order and containment relation on graphs that is considered is the *induced subgraph relation* (recall that H is an induced subgraph of G if H can be obtained from G by a sequence of vertex deletion). P_k denotes the induced path on k vertices.

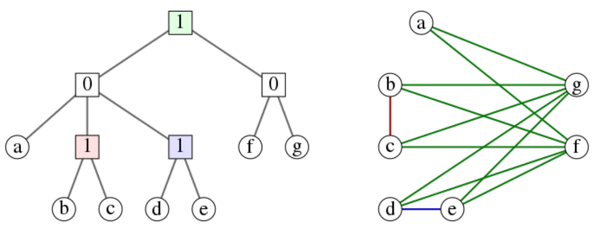
1. What is the class of P_2-free graphs ? Is it well quasi ordered (in short, wqo) ?

2. What is the class of P_3-free graphs ? Is it wqo ?

3. Prove that is G is P_4-free then either G is disconnected or its complement (the graph obtained by switching edges and non-edges) is disconnected.

4. A cotree is a tree whose internal nodes (i.e. non-leaves) are labeled with 0 or 1. Deduce from the previous question that for any P_4 free graph G, there exists a cotree T such that

- the leaves of T are in bijection with $V(G)$
- one can root T such that two leaves correspond to adjacent vertices of G if and only if their lowest common ancester in T is labeled 1.



5. Using a theorem seen in the course, prove that P_4-free graphs are wqo for the induced subgraph relation.

6. Prove that this is not true for P_5-free graphs.

Exercise 7 : Holes

A graph is *hole-free* if it does not contain as an induced subgraph any cycle C_n for $n \geq 4$ (note that it can contain some triangles).

1. Let G be a graph. Prove that the following statements are equivalent (hint : you can prove $i \Rightarrow ii \Rightarrow iii \Rightarrow iv \Rightarrow i$)

i) G is hole-free

ii) Every subgraph of G is a clique or admits a clique cutset (i.e. a subgraph C which induces a clique and such that $V(G) \setminus C$ induces a disconnected graph)

iii) G admits an optimal tree decomposition in which every bag induces a clique in G

iv) Every subgraph of G, admits a *simplicial vertex*, that is a vertex whose neighborhood is a clique.

2. Let G be a graph. Prove that

$$\mathrm{tw}(G) = \min\{\omega(G') - 1,\ G \text{ subgraph of } G' \text{ and } G' \text{ is hole-free}\}$$

Exercise 8 : FPT

Let t be a fixed integer and define an algorithmic problem :

Input : A graph G and an integer k

Output : YES if there exists $X \subset V(G)$ such that $|X| \leq k$ and $G \setminus X$ has treewidth at most t, and NO otherwise.

Prove that there exists an FPT algorithm when parameterized by k for this problem.

Exercise 9 : 3-Coloring

1. Using a result seen in the course, give a five line argument of the existence of an FPT algorithm parametrized by treewidth for the problem (3COL) of deciding whether a graph has chromatic number at most 3.

2. Using nice-tree decompositions as seen in the course, give an explicit such algorithm based on dynamical programming and give its complexity.