MPRI - Graph Algorithms Part 1

Study questions

Exam questions would be similar to these questions.

Exercise 1 : Triangle-free graphs of high chromatic number
The goal of this exercise is to show that for every integer k there is a triangle-free graph of chromatic number k.

First construction :
Consider the following sequence of graphs :

— G_1 is K_1, that is the graph on one vertex with no edge.
— G_i is build from disjoint union of $G_1, G_2, \ldots, G_{i-1}$ by adding following vertices and edges : for each sequence $v = (x_1, x_2, \ldots, x_{i-1})$ where x_1 is a vertex of G_1, x_2 is a vertex of G_2 and generally x_j is a vertex of G_j ($j \leq i-1$), we add a new vertex v and connect v to each of $x_1, x_2, \ldots x_{i-1}$.

1. Build G_2 and G_3. Check that they are both triangle-free and of chromatic number 2 and 3 respectively.
2. Prove that G_i is a triangle-free graph for all values of i.
3. Prove that G_i has chromatic number exactly i.

Second construction :
Given a graph G define $M(G)$ as follows : We start with G, then for each vertex v of G, we add vertex v' that is adjacent to all neighbours of v in G. Then we add a new vertex u which is adjacent to all vertices v'.

4. Prove that $\chi(M(G)) = \chi(G) + 1$.
5. Use this to build a sequence of triangle-free graphs where the chromatic number goes to ∞.

Exercise 2 : List chromatic number
List coloring is a natural concept that appears when we want to extend coloring of some vertices of the graph to the uncolored vertices. The proper definition is as follows :
Let G be a graph and for each vertex v of G suppose there is a list $L(v)$ of colors which is the set of available colors for the vertex v.
The graph G is said to be k-list colorable if for any list assignment with each list being of size k there is a proper coloring of G where color of v is in $L(v)$.

1. Show that even cycles are 2-list colorable.

2. Show that odd cycles are not 2-list-colorable.
3. Show that $K_{3,3}$ is not 2-list colorable, but it is 3-list colorable.
4. Show that for any k there is a bipartite graph which is not k-list colorable.
5. Extend Brooks theorem to list-coloring. That is to say : if G is a graph with $\Delta(G) \geq 3$ and it is not a complete graph then G is $\Delta(G)$-list-colorable.

Exercise 3 : Edge-coloring question
1. Show that Petersen graph cannot be properly 3-edge-colored.
2. Present a formula for the edge-chromatic number of the complete graph of order n, prove your claim.
Let G be a graph obtained from $K_{3,3}$ by subdividing an edge once.
3. Show that G is 2-connected.
4. Show that G is not 3-edge-colorable.
5. What is the edge-chromatic number of this graph ?
6. Write the integer programming for the edge-chromatic number of this graph and for the Petersen graph.
7. [*] By solving the linear version of this program, or by any other techniques of your choice, find the fractional edge-chromatic number of these graphs (that us to solve the integer program in real values).

Exercise 4 : Application of matching algorithms
Given a weighted graph G (where each edge has a nonnegative weight), algorithm M can find, in polynomial time, a maximum weighted matching of G. That is a matching M of G whose sum of weights is maximum possible.

Let H be a graph (without weights) and let u and v be two vertices of H.

1. Using algorithm M find a polynomial time algorithm that determines if there is an even path connecting u and v and if so, finds a shortest such a path.
Hint : Consider two copies of H where in one copy u is a special vertex and in the other v. Build a new weighted graph where maximum matching of the new graph corresponds to such a path in H.

Exercise 5 : Interval graphs
Given a set of closed intervals $A = \{I_1, I_2, \ldots I_n\}$ of the real line the intersection graph of A is a graph where elements of A are the vertices with two vertices I_i and I_j being adjacent if $I_i \cap I_j \neq \emptyset$.

1. Find a best algorithm you can which for a given set A of intervals determines the clique number of the intersection graph of A.
2. Prove that the chromatic number of the intersection graph of A is equal to its clique number. Conclude an algorithm to determine the chromatic number of such intersection graphs.

3. [bonus] Give an algorithm which for a graph G determines if G is isomorphic to the intersection graph of a set A of intervals.

Exercise 6 : Ramsey Theory

1. Recall the definition of $R(s,t)$.
2. Determine $R(2,t)$.
3. Prove that $R(s,t) \leq R(s-1,t) + R(s,t-1)$.
4. Use this inequality to determine $R(3,3)$.
5. Improve the previous inequality as follows : Prove that if $R(s,t-1)$ and $R(s-1,t)$ are both even numbers, then $R(s,t) \leq R(s-1,t)+R(s,t-1)-1$. Use the improved inequality to determine $R(3,4)$ and $(4,3)$.
6. Define the Paley graph of order 17. Use this graph and the previous conclusions to determine $R(4,4)$.

Exercise 7 : Ramsey theory and signed graphs

Recall : A signed graph (G,σ) is balanced if we can switch at soem vertices to make all edges positive. It is antibalanced if we can switch at some vertices to make all edge negative.

1. Find the smallest value of n such that for any signature on K_n, there exists either a balanced K_4 or an antibalanced K_4.

Exercise 8 : Eigenvalues

Definition : $\lambda_1(G)$ is the largest eigenvalues of G. $\lambda_n(G)$ is the smallest eigenvalues of G. $\bar{d}(G)$ is the average degree of G.

1. Prove that $\bar{d}(G) \leq \lambda_1(G) \leq \Delta(G)$
2. Suppose G is connected. Prove that $\lambda_n(G) = -\lambda_1(G)$ if and only if G is bipartite.
3. Find all the eigenvalues of the complete graph K_n, the cycle C_n, and the Petersen graph.

4. Suppose G is a graph with exactly two distinct eigenvalues. Prove that G is a complete graph.
5. Prove, that if K_{10} is 3-edge-colored such that two of the colors induce isomorphic copies of the Petersen graph, then the 3rd color induces a bipartite graphs. Conclude that K_{10} cannot be edge decomposed to three disjoint copies of the Petersen graph.