Vizing theorem. \( X'(G) \leq \Delta + 1 \) \( \quad ( \ X'(G) \geq \Delta \) simple

For multigraphs we have two inequalities:

1. \( X'(G) \leq \Delta + \mu \rightarrow \text{maximum multiplicity} \)

2. \( X'(G) \leq \frac{3}{2} \Delta \text{ external case:} \)

Homework:

Determine \( X'(K_n) \).
Proof.

Main technique: Kempe chain.
We follow a proof by Ehrenfeucht, Fabor, Kierstead.

Stronger claim to prove:

Assum: 1) $d(v) \leq k$

2) for every $u \sim v$, $d(u) \leq k$

3) for at most one $u \sim v$, $d(u) = k$

4) $G - v$ is $k$-edge-colorable.

Then $G$ is $k$-edge-colorable.
We follow a proof by Ehrenfeucht, Fabor, Kierstead.

Stronger claim to prove:

Assum: 1) \( d(v) \leq k \)

2) for every \( u \sim v \), \( d(u) \leq k \)

3) for at most one \( u \sim v \), \( d(u) \neq k \)

4) \( G-v \) is \( k \)-edge-colorable.

Then \( G \) is \( k \)-edge-colorable.

\[ \Rightarrow \chi'(G) \leq \Delta(G) + 1 \], moreover, if vertices of degree \( \Delta(G) \) induce a forest (no cycle),

then \( \chi(G) = \Delta(G) \)
Proof of the stronger claim by induction on $k$.

Based on induction: $k=0$ ✓ $k=1$ ✓

Inductive assumption: the statement holds for $k-1$.

Our task: to prove it for $k$.

Let $G$ be a graph together with a vertex $v$ such that $G$ and $v$ satisfy the four conditions.
Step 1. Add pendant edges so that:

\[ d(u_1) = k, \quad d(u_2) = d(u_3) = \ldots = d(u_k) = k - 1. \]
Step2. Color edges of $G-v$ using $k$ colors.
Step 3. Define $X_i$: vertices $U_i$ missing color $i$

- $X_1 = \emptyset$
- $X_2 = \{U_2\}$
- $X_3 = \{U_2, U_4\}$
- $X_4 = \{U_2, U_3, U_4, U_5\}$
- $X_5 = \{U_3, U_5\}$
Step 3. Define $X_i$: vertices $U_i$, missing color $i$

$X_1 = \emptyset$

$X_2 = \{u_2\}$

$X_3 = \{u_2, u_4\}$

$X_4 = \{u_1, u_3, u_4, u_5\}$

$X_5 = \{u_3, u_5\}$
We wish to make these sets of nearly equal size.

How to do that?

Observation:

$$\sum_{i=1}^{k} |x_i| = 2k - 1$$
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$$\sum_{i=1}^{k} |x_i| = 2k - 1$$

Show that at least one $|x_i| = 1$. 
Construction of graphs of maximum degree 3 which are not 3-edge-colorable.

A gadget:

In every 3-edge-coloring of the gadget parallel edges of one side receive a same color and the three other pendant edges receive 3 different colors.
Corollary. The graph obtained from $K_{3,3}$ by subdividing one edge is not 3-edge-colorable.

$\chi' = 3$
Homework. The Petersen graph is not 3-edge-colorable.
Edge-coloring $\rightarrow$ vertex-coloring

$G \rightarrow L(G)$

$\Delta = k \rightarrow \Delta \leq 2^{k-2}$

$\Delta=k$ $\rightarrow$ ?

Diagram:
- A 4-vertex graph with a 2-coloring (blue and red).
- A 6-vertex graph with a 3-coloring (blue, red, green).
Beineke theorem: \( w(L(G)) \leq \chi(L(G)) \leq w(L(G)) + 1 \).

A graph \( H \) is the line graph of a graph \( G \) if and only if it does not have any of the following 9 graphs as a subgraph.
Strengthening Vizing theorem (Kierstead)

If $H$ is a graph with no induced $K_{1,3}$ or $K_5^-$ then $\chi(H) \in \{\omega(G), \omega(G) + 1\}$. 

Improving Vizing theorem for bipartite graphs:

Theorem. If $G$ is a bipartite graph, then $\chi(G) = \Delta(G)$

We will prove it using a min-max theorem.
Definition:

\[ m(G) : \text{size of maximum matching of } G \]

\[ \downarrow \]

A set of edges with no common vertex

\[ c(G) : \text{order of a minimum cover of } G \]

\[ \downarrow \]

A set vertices that cover all edges

\[ c(G) \geq m(G) \]

Theorem. If \( G \) is a bipartite graph, then

\[ m(G) = c(G) \]
Example

$C = m = 4$
Lemma. If \( G \) is a bipartite graph with at least one edge, then it has a vertex \( u \) which belongs to every maximum matching.

Proof. In fact we prove that for every edge \( uv \) one of \( u \) or \( v \) satisfies the condition of the lemma.

Toward a contradiction, suppose for an edge \( uv \) there are matchings \( M_u \) and \( M_v \), each of maximum size where \( M_u \) misses the vertex \( u \) and \( M_v \) misses the vertex \( v \).
Consider the subgraph induced by $m_u \cup m_v$.

Let $P$ be the connected component of this subgraph that contains the vertex $u$. 
Claim 1. $P$ is a path (not a cycle).

Claim 2. $P$ starts with a red edge and ends with a green one.

Claim 3. The vertex $V$ does not belong to $P$.

Because $G$ is bipartite, last vertex of $P$ is in the same part as $u$, but $v$ is in the other part.
In P switch red and green.
Theorem. If $G$ is a bipartite graph, then $m(G) = c(G)$

Proof. By induction on $m(G)$.

If $m(G)=0$ then $\checkmark$

Also if $m(G)=1$ then $\checkmark$

Assume that the claim is valid for $m(G) \leq k-1$ and let $G$ be a bipartite graph with $m(G) = k$. 
Theorem. If $G$ is a bipartite graph, then $m(G) = c(G)$

Let $u$ be a vertex which is in every maximum matching. Consider $G - u$. 
Corollary. If $G$ is a $k$-regular bipartite graph, then $\chi'(G) = k$
Corollary. For every bipartite graph $G$ we have

$$X'(G) = \Delta(G)$$
Corollary. Line graph of every bipartite graph $G$ satisfies:

$$X(L(G)) = W(L(G))$$

Valid for every induced subgraph of $L(G)$
Perfect graph: A graph $G$ where every induced subgraph $H$ satisfies $\chi(H) = \omega(H)$.
Homework.

1. Show that for $k \geq 2$ the odd cycle $C_{2k+1}$
   and its complement are not perfect.

2*. Show that there are graphs with
   $\omega(G) = 2$ and $\chi(G) = k$ (for every $k$).

3*. Show that for every $g$ and $k$ there exists
   a graph $G$ which has no cycle of length smaller
   than $g$ and has chromatic number $k$. 