Vizing theorem. \( \chi'(G) \leq \Delta + 1 \) \((\chi'(G) \geq \Delta)\)

For multigraphs we have two inequalities:

1. \( \chi'(G) \leq \Delta + \mu \rightarrow \) maximum multiplicity
2. \( \chi'(G) \leq \frac{3}{2} \Delta \) external case:

Homework:

**Determine** \( \chi'(K_n) \).
Proof.

Main technique: Kempe chain.
We follow a proof by Ehrenfeucht, Fabor, Kierstead.

Stronger claim to prove:

Assum: 1) $d(v) \leq k$

2) for every $u \sim v$, $d(u) \leq k$

3) for at most one $u \sim v$, $d(u) = k$

4) $G - v$ is $k$-edge-colorable.

Then $G$ is $k$-edge-colorable.
We follow a proof by Ehrenfeucht, Fábry, Kierstead.

Stronger claim to prove:

Assum: 1) \(d(v) \leq k\)

2) for every \(u-v\), \(d(w) \leq k\)

3) for at most one \(u-v\), \(d(w) = k\)

4) \(G-v\) is \(k\)-edge-colorable.

Then \(G\) is \(k\)-edge-colorable.

\[ \Rightarrow \chi'(G) \leq \Delta(G) + 1, \text{ moreover, if vertices of degree } \Delta(G) \text{ induce a forest (no cycle),} \]

then \(\chi'(G) = \Delta(G)\).
Proof of the stronger claim by induction on $K$.

Based on induction: \( k=0 \checkmark \quad k=1 \checkmark \)

Inductive assumption: the statement holds for $K-1$.

Our task: to prove it for $K$.

Let $G$ be a graph together with a vertex $v$ such that $G$ and $v$ satisfy the four conditions.
Step 1. Add pendant edges so that:
\[ d(u_1) = k, \quad d(u_2) = d(u_3) = \cdots = d(u_k) = k - 1. \]
Step 2. Color edges of $G - v$ using $k$ colors.
Step 3. Define $\mathcal{X}_i$:
vertices $U_i$, missing color i

$\mathcal{X}_1 = \emptyset$
$\mathcal{X}_2 = \{u_2\}$
$\mathcal{X}_3 = \{u_2, u_4\}$
$\mathcal{X}_4 = \{u_2, u_3, u_4, u_5\}$
$\mathcal{X}_5 = \{u_3, u_5\}$
Step 3. Define $X_i$:

vertices $U_i$ missing color $i$

\[ X_1 = \emptyset \]
\[ X_2 = \{ u_2 \} \]
\[ X_3 = \{ u_2, u_4 \} \]
\[ X_4 = \{ u_3, u_4, u_5 \} \]
\[ X_5 = \{ u_3, u_5 \} \]
Observation:
\[ \sum_{i=1}^{k} l_{x_i} l = 2k - 1 \]

We wish to make these sets of nearly equal size.

How to do that?
Observation:

\[ \sum_{i=1}^{k} |x_i| = 2k - 1 \]

Show that at least one \(|x_i| = 1\).
Construction of graphs of maximum degree 3 which are not 3-edge-colorable.

A gadget:

In every 3-edge-coloring of the gadget parallel edges of one side receive a same color and the three other pendant edges receive 3 different colors.
Corollary. The graph obtained from $K_{3,3}$ by subdividing one edge is not 3-edge-colorable.
Homework. The Petersen graph is not 3-edge-colorable.
Edge-coloring $\rightarrow$ vertex-coloring

$G$ $\rightarrow$ $L(G)$

$\Delta = k$ $\rightarrow$ $\Delta \leq 2^k - 2$
Beineke theorem: \( w(L(G)) \leq X(L(G)) \leq w(L(G)) + 1 \).

A graph \( H \) is the line graph of a graph \( G \) if and only if it does not have any of the following 9 graphs as a subgraph.
Strengthening Vizing theorem (Kierstead)

If $H$ is a graph with no induced $K_{1,3}$ or $K_5^-$ then $\chi(H) \in \{\omega(G), \omega(G)+1\}$.
Improving Vizing theorem for bipartite graphs:

**Theorem.** If $G$ is a bipartite graph, then

$$\chi(G) = \Delta(G)$$

We will prove it using a min-max theorem.
Definition:

\[ m(G) \]: size of maximum matching of \( G \)

\[ c(G) \]: order of a minimum cover of \( G \)

\[ c(G) \geq m(G) \] A set of vertices that cover all edges

Theorem. If \( G \) is a bipartite graph, then

\[ m(G) = c(G) \]
Example

$C = m = 4$
Lemma. If $G$ is a bipartite graph with at least one edge, then it has a vertex $u$ which belongs to every maximum matching.

Proof. In fact we prove that for every edge $uv$ one of $u$ or $v$ satisfies the condition of the lemma.

Toward a contradiction, suppose for an edge $uv$ there are matchings $M_u$ and $M_v$, each of maximum size where $M_u$ misses the vertex $u$ and $M_v$ misses the vertex $v$. 

Consider the subgraph induced by $m_u \cup m_v$.

Let $P$ be the connected component of this subgraph that contains the vertex $u$. 
Claim 1. P is a path (not a cycle).

Claim 2. P starts with a red edge and ends with a green one.

Claim 3. The vertex V does not belong to P.

Because G is bipartite, last vertex of P is in the same part as u, but v is in the other part.
In P switch red and green.
Theorem. If $G$ is a bipartite graph, then $m(G) = c(G)$.

Proof. By induction on $m(G)$.

If $m(G) = 0$ then \checkmark

Also if $m(G) = 1$ then \checkmark

Assume that the claim is valid for $m(G_i) \leq k-1$ and let $G$ be a bipartite graph with $m(G) = k$. 
Theorem. If $G$ is a bipartite graph, then $m(G) = c(G)$

Let $u$ be a vertex which is in every maximum matching. Consider $G-u$. 
Corollary. If $G$ is a $k$-regular bipartite graph, then $X'(G) = k$.
Corollary. For every bipartite graph $G$ we have

$$X'(G) = \Delta(G)$$
Corollary. Line graph of every bipartite graph $G$ satisfies:

$$X(L(G)) = W(L(G))$$

Valid for every induced subgraph of $L(G)$
Perfect graph: A graph $G$ where every induced subgraph $H$ satisfies $\chi(H) = \omega(H)$.
Homework.

1. Show that for $k \geq 2$ the odd cycle $C_{2k+1}$ and its complement are not perfect.

2*. Show that there are graphs with $\omega(G) = 2$ and $\chi(G) = k$ (for every $k$).

3*. Show that for every $g$ and $k$ there exists a graph $G$ which has no cycle of length smaller than $g$ and has chromatic number $k$. 