Vizing theorem. \[ \chi'(G) \leq \Delta + 1 \quad (\chi'(G) > \Delta) \]

For multigraphs we have two inequalities:

1. \[ \chi'(G) \leq \Delta + \mu \] maximum multiplicity
2. \[ \chi'(G) \leq \frac{3}{2} \Delta \] external case:

Homework:

Determine \[ \chi'(K_n) \]
Proof.

Main technique: Kempe chain.

maximal RG path
We follow a proof by Ehrenfeucht, Fabor, Kierstead.

Stronger claim to prove:

Assum: 1) \( d(v) \leq k \)
2) for every \( u \sim v \), \( d(u) \leq k \)
3) for at most one \( u \sim v \), \( d(u) > k \)
4) \( G-v \) is \( k \)-edge-colorable.

Then \( G \) is \( k \)-edge-colorable.
We follow a proof by Ehrenfeucht, Fabor, Kierstead.

Stronger claim to prove:

Assum: 1) $d(v) \leq k$
2) for every $u \sim v$, $d(w) \leq k$
3) for at most one $u \sim v$, $d(w) \neq k$
4) $G - v$ is $k$-edge-colorable.

Then $G$ is $k$-edge-colorable.

$\Rightarrow \chi'(G) \leq \Delta(G) + 1$, moreover, if vertices of degree $\Delta(G)$
induce a forest (no cycle),

then $\chi(G) = \Delta(G)$
Proof of the stronger claim by induction on $k$.

Based on induction: $k=0 \checkmark \quad k=1 \checkmark$

Inductive assumption: the statement holds for $k-1$.

Our task: to prove it for $k$.

Let $G$ be a graph together with a vertex $v$ such that $G$ and $v$ satisfy the four conditions.
Step 1. Add pendant edges so that:
\[ d(u_1) = k, \quad d(u_2) = d(u_3) = \cdots = d(u_k) = k-1. \]
Step 2. Color edges of $G - v$ using $k$ colors.
Step 3. Define $X_i$:

vertices $u_i$ missing color $i$

$X_1 = \emptyset$
$X_2 = \{u_2\}$
$X_3 = \{u_2, u_4\}$
$X_4 = \{u_2, u_3, u_4, u_5\}$
$X_5 = \{u_3, u_5\}$
Step 3. Define $X_i$: vertices $U_i$ missing color $i$

- $X_1 = \emptyset$
- $X_2 = \{u_2\}$
- $X_3 = \{u_2, u_4\}$
- $X_4 = \{u_2, u_3, u_4, u_5\}$
- $X_5 = \{u_3, u_5\}$
We wish to make these sets of nearly equal size.

How to do that?
Observation:
\[ \sum_{i=1}^{k} |X_i| = 2k - 1 \]

Show that at least one $|X_i| = 1$. 
Construction of graphs of maximum degree 3 which are not 3-edge-colorable.

A gadget:

In every 3-edge-coloring of the gadget parallel edges of one side receive a same color and the three other pendant edges receive 3 different colors.
Corollary. The graph obtained from $K_{3,3}$ by subdividing one edge is not 3-edge-colorable.
Homework. The Petersen graph is not 3-edge-colorable.
Edge-coloring $\rightarrow$ vertex-coloring

$G$ $\rightarrow$ $L(G)$

$\Delta = k$ $\rightarrow$ $\Delta \leq 2^k - 2$
Beineke theorem: \((L(G)) < X(L(G)) < (L(G))^{1.1}\)

A graph \(H\) is the line graph of a graph \(G\) if and only if it does not have any of the following 9 graphs as a subgraph.
Strengthening Vizing theorem (Kierstead)

If $H$ is a graph with no induced $K_{1,3}$ or $K_5^-$, then $\chi(H) \in \{w(G), w(G)+1\}$. 
Improving Vizing theorem for bipartite graphs:

Theorem. If $G$ is a bipartite graph, then

$$X(G) = \Delta(G)$$

We will prove it using a min-max theorem.
Definition:

$m(G)$: size of maximum matching of $G$
- A set of edges with no common vertex

$c(G)$: order of a minimum cover of $G$
- A set vertices that cover all edges

Theorem. If $G$ is a bipartite graph, then

$m(G) = c(G)$
Example

$C = m = 4$
Lemma. If $G$ is a bipartite graph with at least one edge, then it has a vertex $u$ which belongs to every maximum matching.

Proof. In fact we prove that for every edge $uv$ one of $u$ or $v$ satisfies the condition of the lemma.

Toward a contradiction, suppose for an edge $uv$ there are matchings $M_u$ and $M_v$, each of maximum size where $M_u$ misses the vertex $u$ and $M_v$ misses the vertex $v$. 
Consider the subgraph induced by $m_u \cup m_v$.

Let $P$ be the connected component of this subgraph that contains the vertex $u$. 
Claim 1. $P$ is a path (not a cycle).

Claim 2. $P$ starts with a red edge and ends with a green one.

Claim 3. The vertex $v$ does not belong to $P$.

Because $G$ is bipartite, last vertex of $P$ is in the same part as $u$, but $v$ is in the other part.
In P switch red and green.
Theorem. If $G$ is a bipartite graph, then $m(G) = c(G)$

Proof. By induction on $m(G)$.

If $m(G)=0$ then $\checkmark$

Also if $m(G)=1$ then $\checkmark$

Assume that the claim is valid for $m(G_i) \leq k-1$ and let $G$ be a bipartite graph with $m(G) = k$. 
Theorem. If $G$ is a bipartite graph, then $m(G) = c(G)$

Let $u$ be a vertex which is in every maximum matching.

Consider $G-u$. 
Corollary. If $G$ is a $K$-regular bipartite graph, then $\chi'(G) = k$.
Corollary. For every bipartite graph $G$ we have

$$X'(G) = \Delta (G)$$
Corollary. Line graph of every bipartite graph $G$ satisfies:

$$X(L(G)) = W(L(G))$$

Valid for every induced subgraph of $L(G)$
Perfect graph: A graph $G$ where every induced subgraph $H$ satisfies

$$\chi(H) = \omega(H)$$
1. Show that for $k \geq 2$ the odd cycle $C_{2k+1}$ and its complement are not perfect.

2*. Show that there are graphs with $\omega(G) = 2$ and $\chi(G) = k$ (for every $k$).

3*. Show that for every $g$ and $k$ there exists a graph $G$ which has no cycle of length smaller than $g$ and has chromatic number $k$. 