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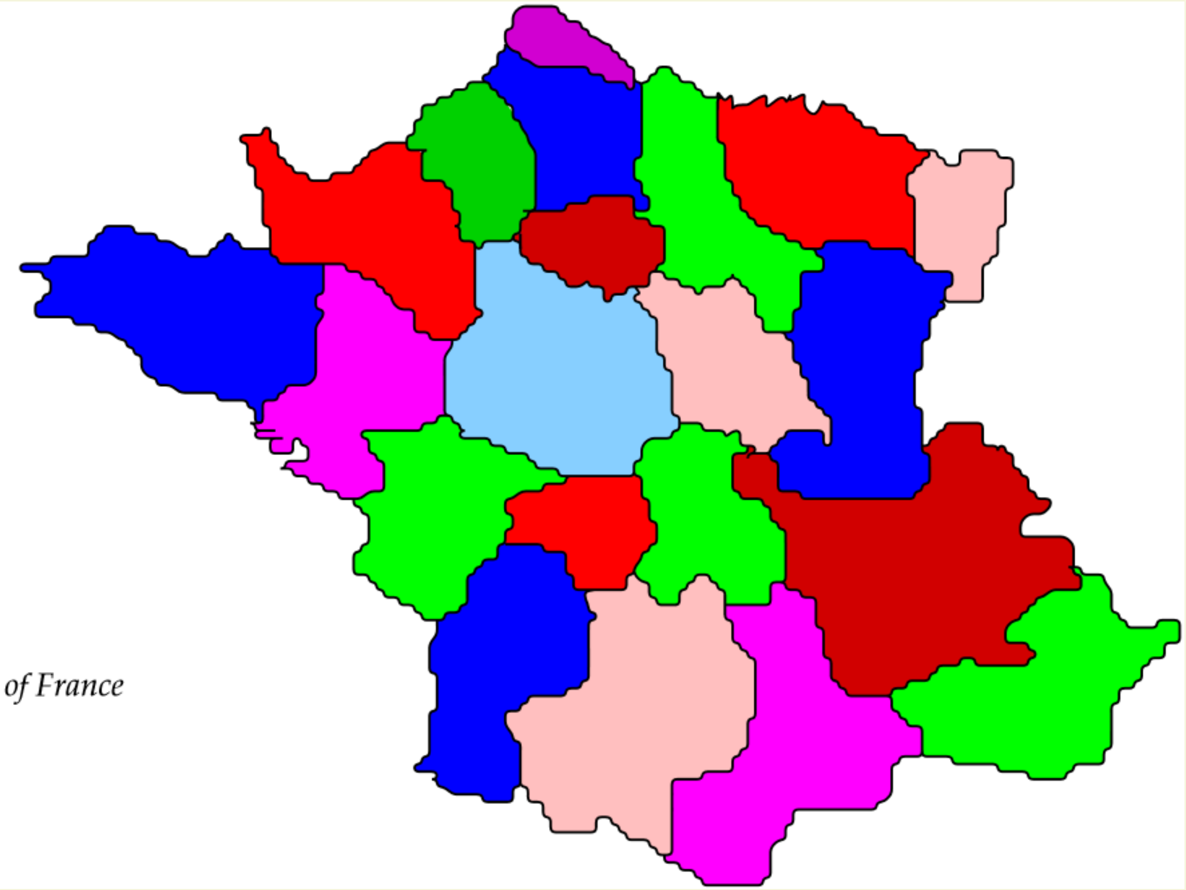
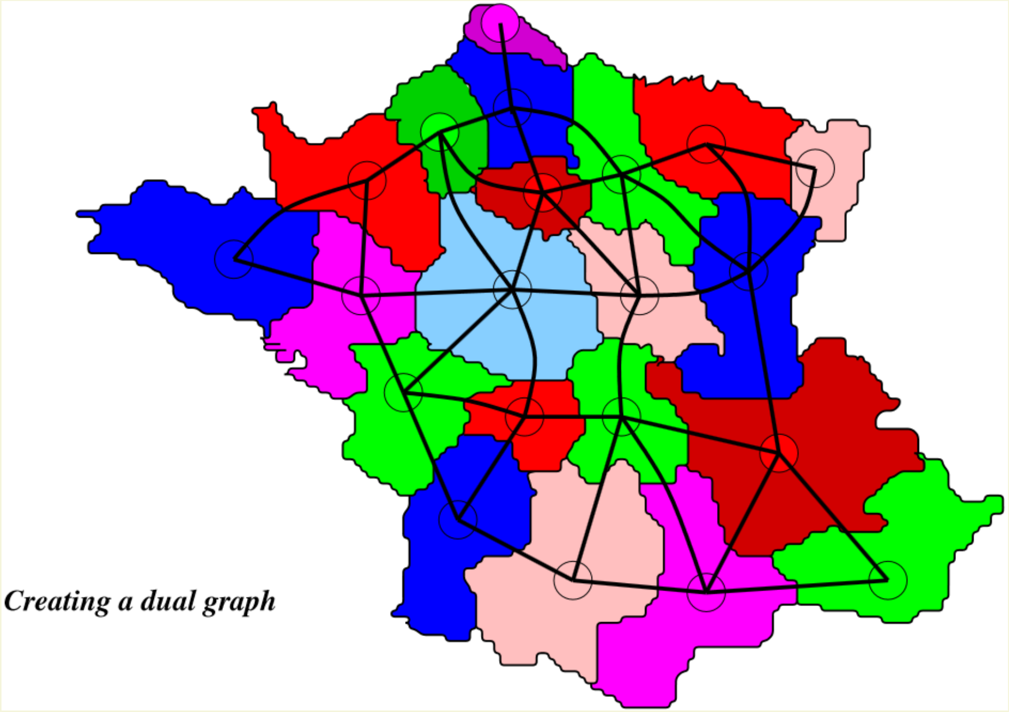


Coloured map of France

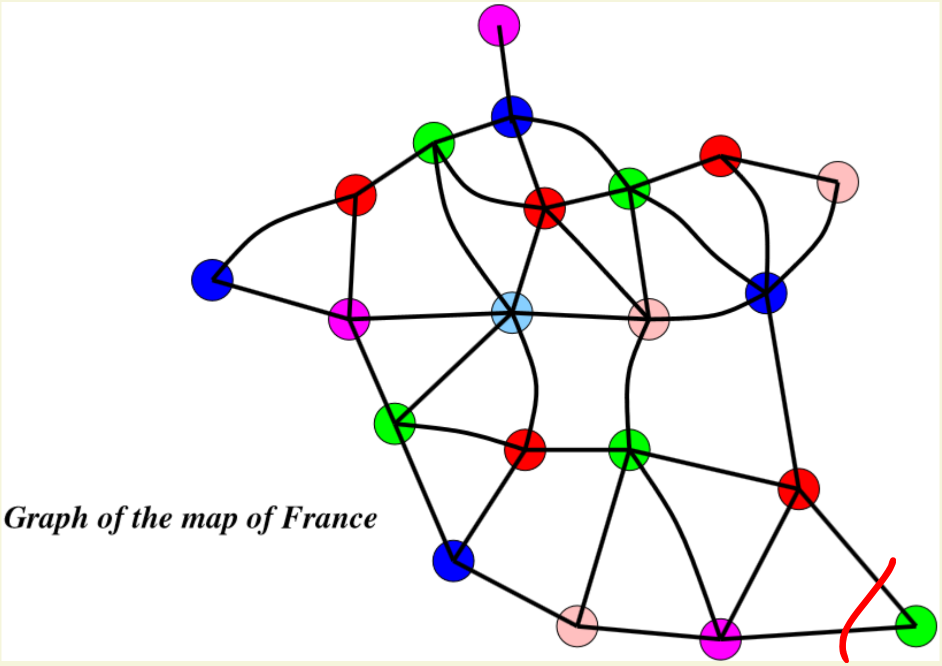


Creating a dual graph

Grötzsch theorem:

Planar + no triangle

$\Rightarrow \chi \leq 3.$



Graph of the map of France

History of the 4-color theorem:

First asked by Francis Guthrie in October 1852.

Kempe "proved" it in 1880.

Heawood found a hole in the proof.

Furthermore, he showed Kempe's method would not always work!

P.G. Tait 1890, introduced a reformulation and claimed a proof based on not proven claim.

- every cubic bridgless (planar) graph is Hamiltonian"
  - every cubic bridgless graph is 3-edge-colorable"
    - → Disproved by Petersen 1890
  - → Disproved by Tutte 1946

Tait's reformulation of the 4-color theorem:

"Every cubic bridgeless planar graph is 3-edge-colorable"

Proof that the two claims are equivalent:

Handshake Lemma:

The number odd-degree vertices of any graph is even.

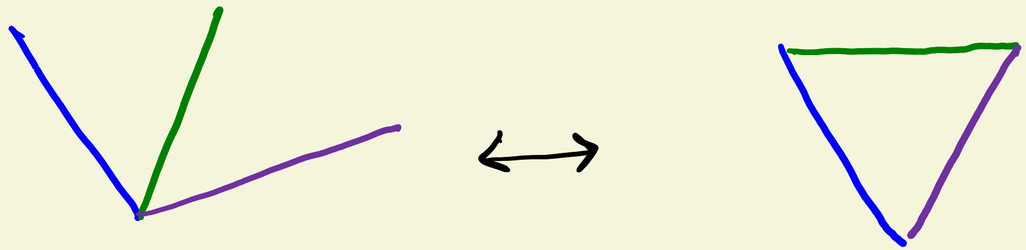
$\to$ number of vertices of a cubic graph is even.

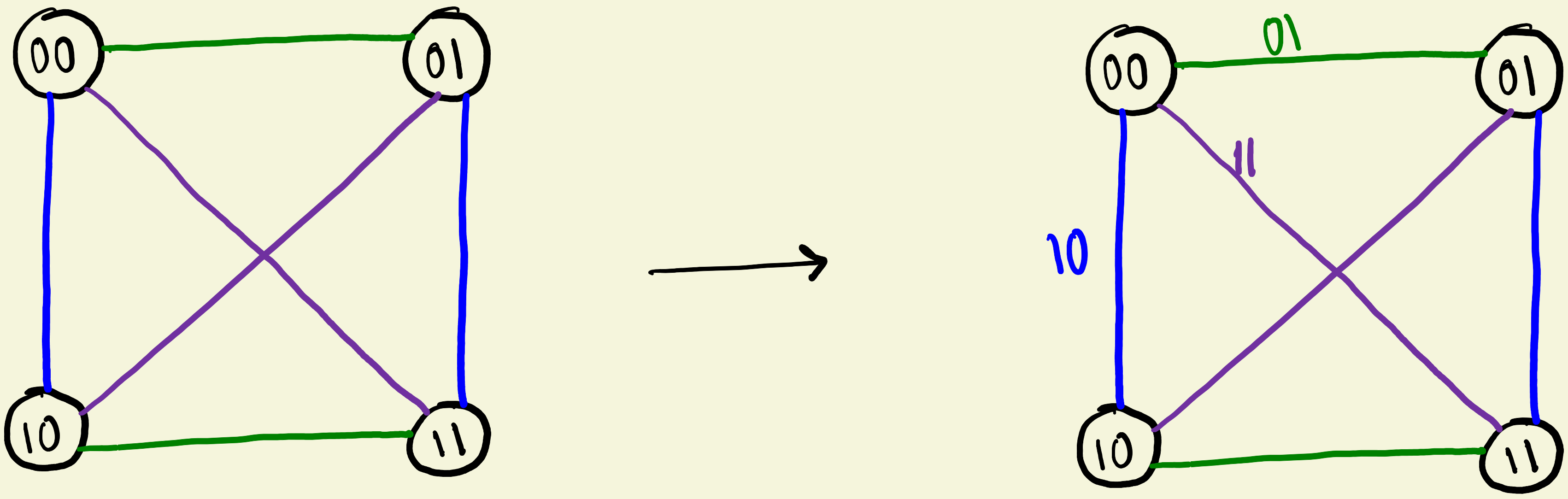
$\to$ If G is cubic & $\chi'(G) = 3$, then G is bridgeless.

Observation. G cubic planar $\qquad$ G^D dual of G

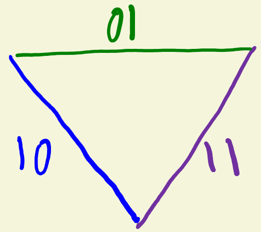
bridgeless $\Longleftrightarrow$ no loop.

G plane triangulation $\leftrightarrow$ G^D cubic bridgless





Given vertex coloring using 00, 01, 10, 11 we change it to edge-coloring of the dual, and conversely given an edge-coloring we change it to vertex coloring but here we have to show that our coloring is well-defined.



$$\boxed{01+10+11=00}$$

Notions to study:

chromatic number,

edge-chromatic number,

nowhere-zero flows,

Hamiltonicity

Discharging technique:

Example. Given a planar graph G, either there exists a vertex statisfying $d(v) \leq 4$, or there exists an edge uv staistisfying $d(u)+d(v) \leq 11$.

A proof for triangulations

Original charge: $c(v) = d(v) - 6$

$\Rightarrow \sum c(v) \leq -12 < 0$ (euler formula).

Discharging rule $u \sim v$, $d(u) \geq 7$, $d(v) = 5 \Rightarrow$ u give a charge of $\frac{1}{5}$ to v.

If G is a counterexample, then after this process all vertices have nonnegative charges, contradicting $\sum c(v) < 0$.

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Introduced by Wernicke 1904.

Further developed by Heesch to attack 4-color problem.

The technique was successfully used to prove the 4-CT

(1976) by Appel & Haken (1482 forbidden configurations).

Roberstson, Sanders, Seymour & Thomas (1996) provided alternative

proof with 633 forbidden configurations.