The 4-color theorem: 

Every normal map can be properly 4-colored.
The 4-color theorem:

Every normal map can be properly 4-colored.
Grötzsch theorem:
Planar + no triangle
⇒ \( x \leq 3. \)
History of the 4-color theorem:

First asked by Francis Guthrie in October 1852.

Kempe "proved" it in 1870.

Heawood found a hole in the proof.

Furthermore, he showed Kempe's method would not always work!

P.G. Tait 1890, introduced a reformulation and claimed a proof based on not proven claim.

- every cubic bridgless (planar) graph is Hamiltonian"

- every cubic bridgless graph is 3-edge-colorable"

  Disproved by Petersen 1890

  Disproved by Tutte 1946
Tait's reformulation of the 4-color theorem:

"Every cubic bridgless planar graph is 3-edge-colorable"

Proof that the two claims are equivalent:
Handshake Lemma:

The number odd-degree vertices of any graph is even.

→ number of vertices of a cubic graph is even.

→ If G is cubic & \( \chi'(G) = 3 \), then G is bridgeless.

Observation. G cubic planar \( \iff \) \( G^D \) dual of G bridgeless \( \iff \) no loop.

G plane triangulation \( \iff \) \( G^D \) cubic bridgeless
Given vertex coloring using 00, 01, 10, 11 we change it to edge-coloring of the dual, and conversely given an edge-coloring we change it to vertex coloring but here we have to show that our coloring is well-defined.
Notions to study:

- Chromatic number,
- Edge-chromatic number,
- Nowhere-zero flows,
- Hamiltonicity
Discharging technique:

Example. Given a planar graph $G$, either there exists a vertex satisfying $d(v) \leq 4$, or there exists an edge $uv$ satisfying $d(u) + d(v) \leq 11$.

\[ \text{A proof for triangulations} \]

Original charge: \[ c(v) = d(v) - 6 \]
\[ \Rightarrow \sum c(v) \leq -12 < 0 \] ( euler formula).

Discharging rule $u \sim v$, $d(u) \geq 7$, $d(v) = 5 \Rightarrow u$ give a charge \[ \frac{1}{2} \to v \].

If $G$ is a counterexample, then after this process all vertices have nonnegative charges, contradicting $\sum c(v) < 0$.

If $G$ is not a triangulation, by adding an edge $uv$ we have $d(u) + d(v) \geq (5+1) + (5+1) = 12$, so the solution is an original edge.
Discharging technique:

Example. Given a planar graph $G$, either there exists a vertex satisfying $d(v) \leq 4$, or there exists an edge $uv$ satisfying $d(u) + d(v) \leq 11$. 
Discharging technique:

Example. Given a planar graph $G$, either
there exists a vertex satisfying $d(v) \leq 4$, or
there exists an edge $uv$ satisfying $d(u) + d(v) \leq 11$.

Introduced by Wernicke 1904.
Further developed by Heesch to attack 4-color problem.
The technique was successfully used to prove the 4-CT
(1976) by Appel & Haken (1482 forbidden configurations).
Roberstson, Sanders, Seymour & Thomas (1996) provided alternative
proof with 633 forbidden configurations.