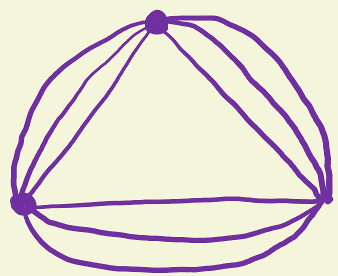
Vizing theorem. $\chi'(G) \leq \Delta + 1$ $\big(\chi'(G) \geq \Delta \big)$

simple

For multigraphs we have two inequalities:

1. $\chi'(G) \leq \Delta + \mu$ → maximum multiplicity

2. $\chi'(G) \leq \frac{3}{2}\Delta$ extermal case:

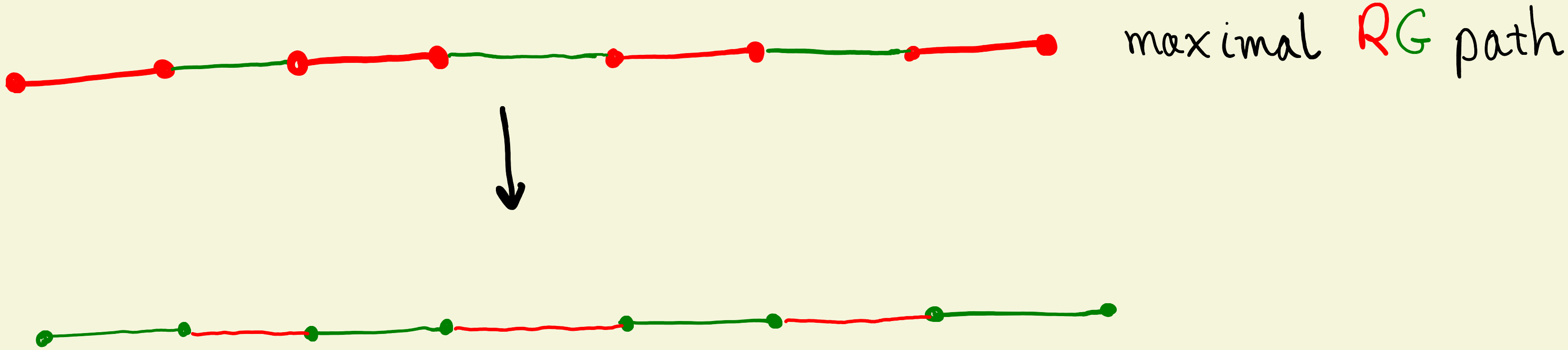


Homework:

Determine $\chi'(K_n)$

Proof.

Main technique: Kempe chain.



We follow a proof by Ehrenfeucht, Fabor, Kierstead.

Stronger claim to prove:

Assum: 1) $d(v) \leq k$

2) for every $u \sim v$, $d(u) \leq k$

3) for at most one $u \sim v$, $d(u) = k$

4) $G-v$ is k-edge-colorable.

Then G is k-edge-colorable.

We follow a proof by Ehrenfeucht, Fabor, Kierstead.

Stronger claim to prove:

Assum: 1) $d(v) \leq k$

2) for every $u \sim v$, $d(u) \leq k$

3) for at most one $u \sim v$, $d(u) \neq k$

4) $G-v$ is k-edge-colorable.

Then G is k-edge-colorable.

$\Longrightarrow$ $\chi'(G) \leq \Delta(G)+1$, moreover, if vertices of degree $\Delta(G)$ induce a forest (no cycle), then $\chi'(G) = \Delta(G)$

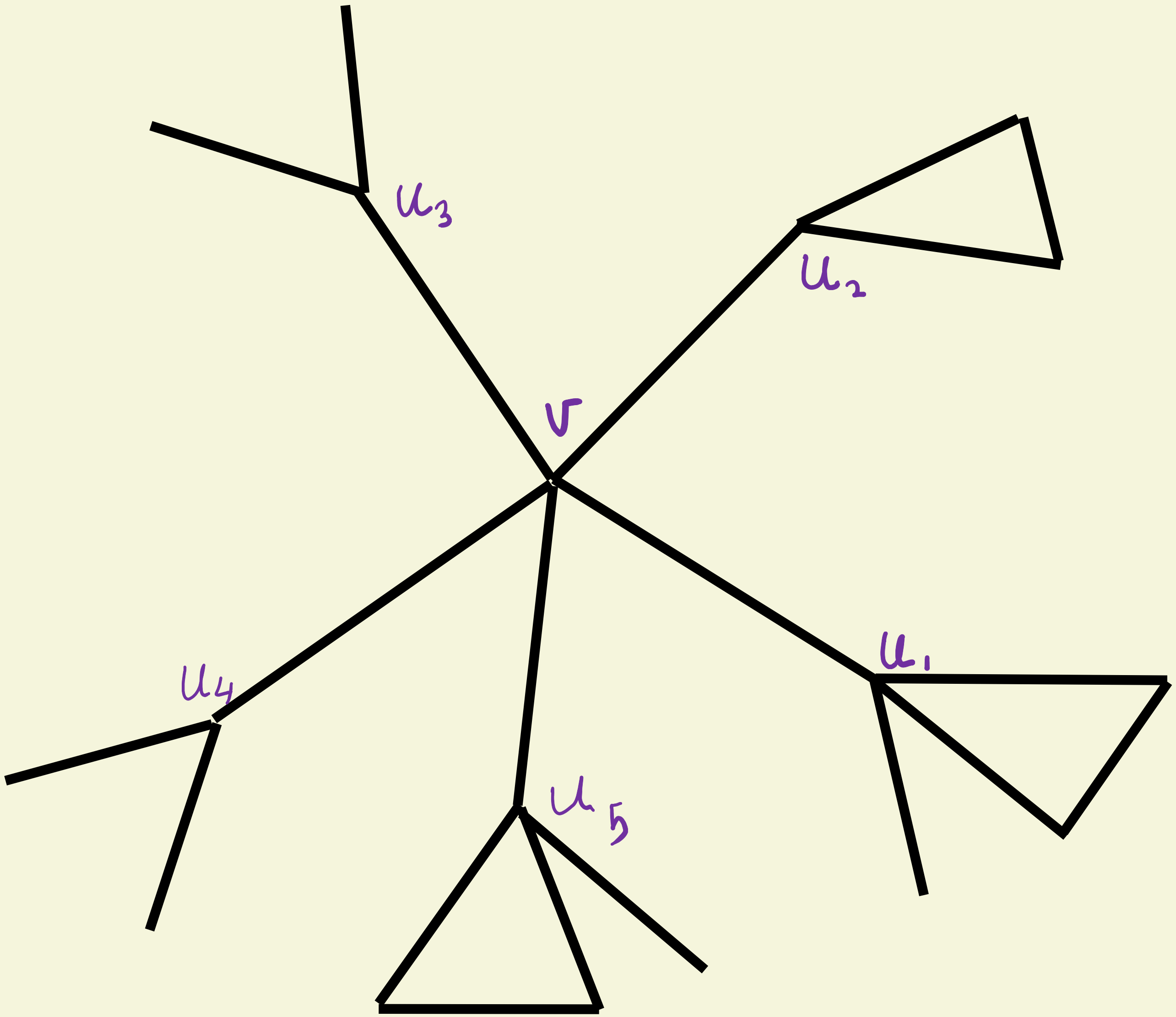
Proof of the stronger claim by induction on K.

Based on induction: $k=0$ ✓ $k=1$ ✓

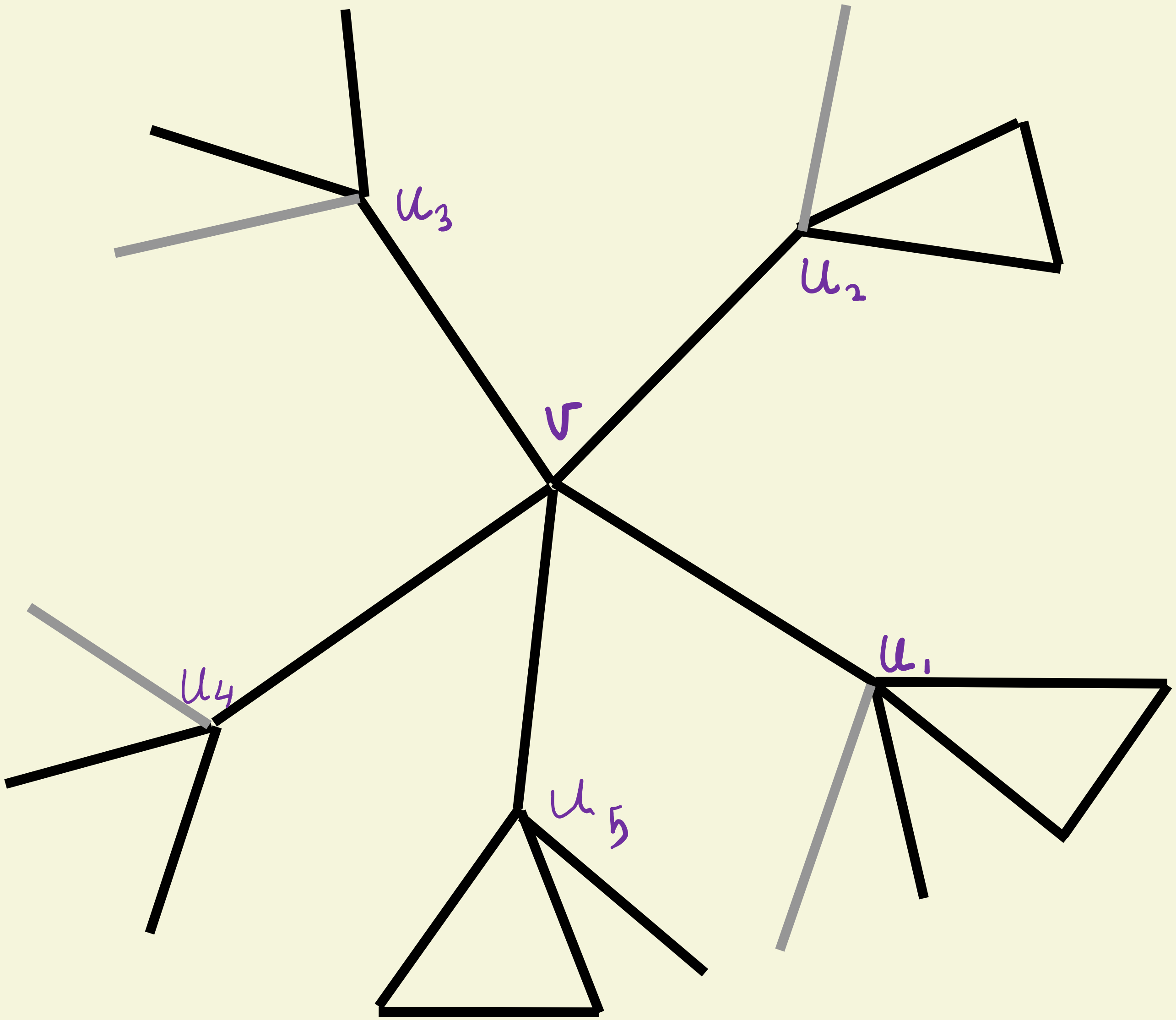
Inductive assumption: the statement holds for K-1.

Our task: to prove it for K.

Let G be a graph together with a vertex v such that G and v satisfy the four conditions.

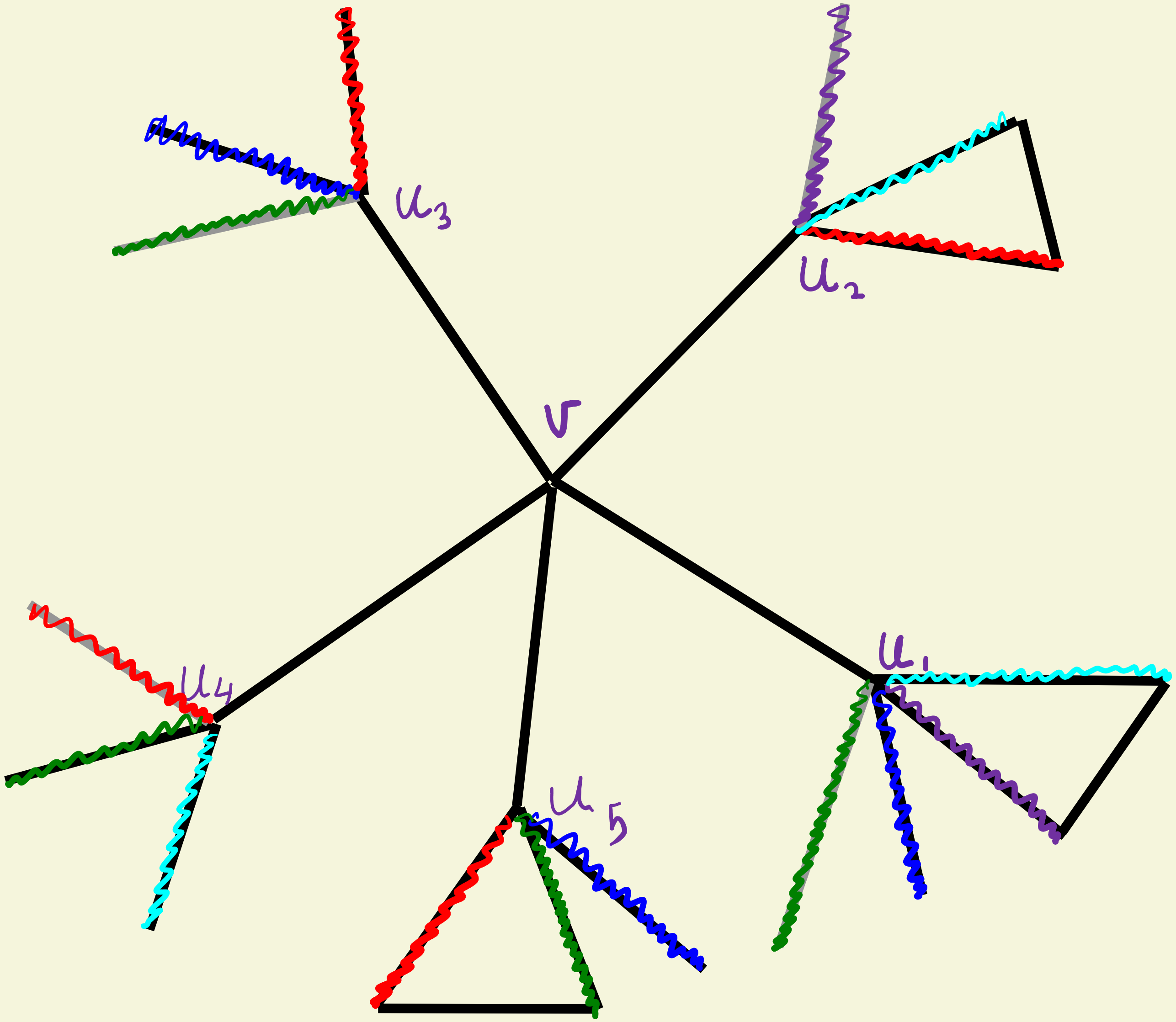


u_3
u_2
v
u_1
u_4
u_5

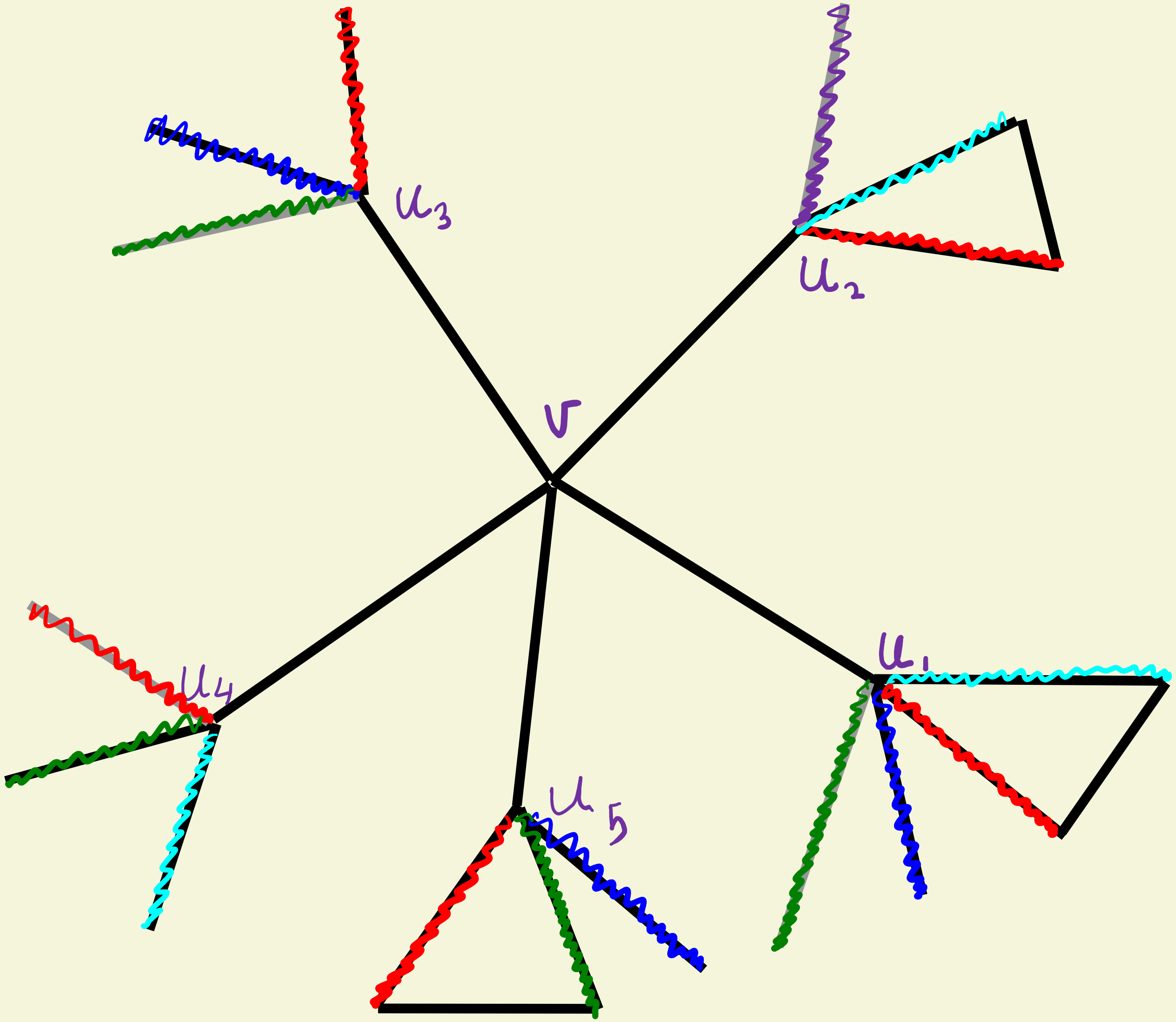


Step 1. Add pendant edges so that:

$$d(u_1) = k, \quad d(u_2) = d(u_2) = \cdots = d(u_k) = k-1.$$



Step 2. Color edges of G-v using k colors.



Step 3. Define X_i:

vertices u_i missing color i

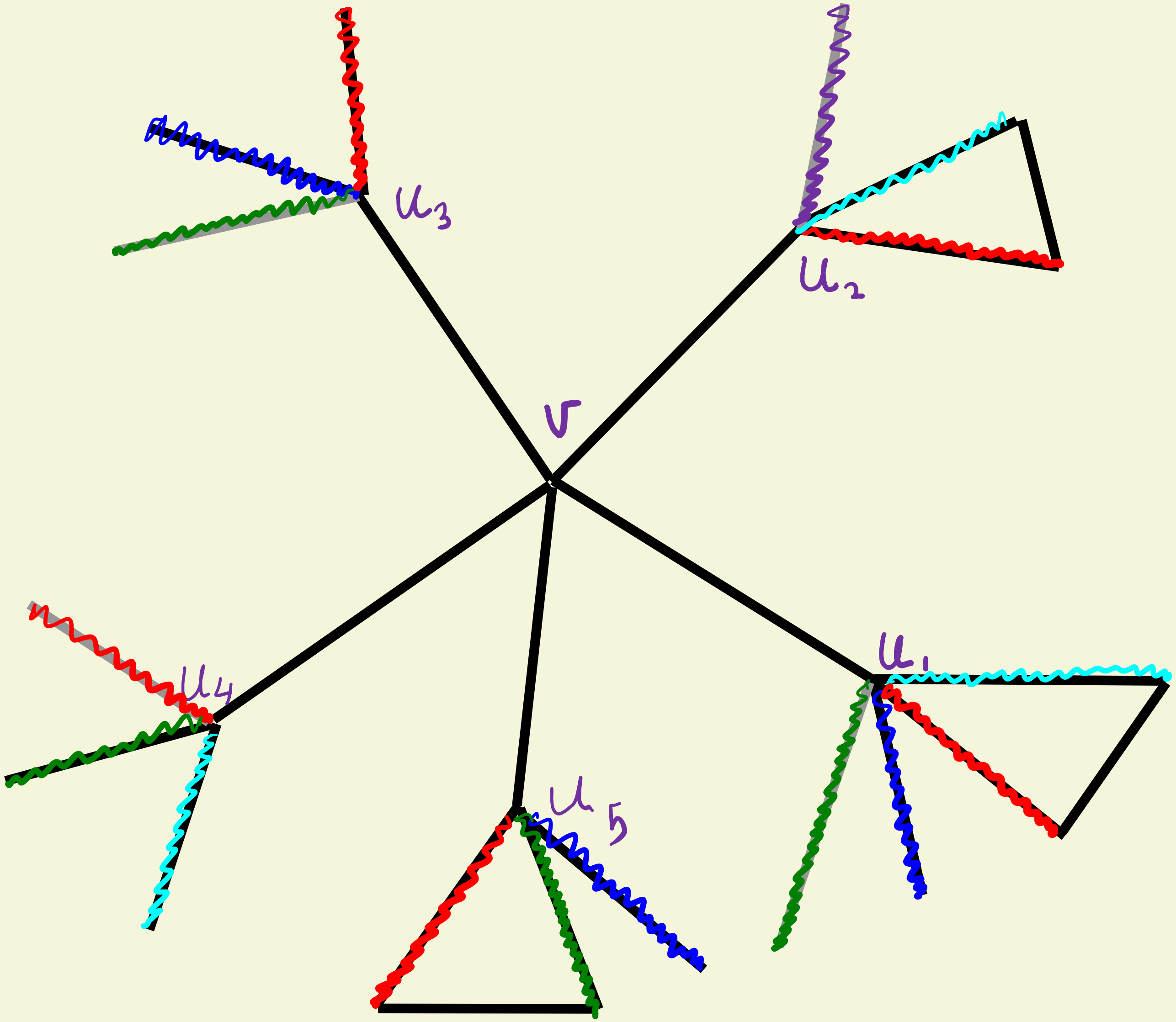
$X_1 = \emptyset$

$X_2 = \{u_2\}$

$X_3 = \{u_2, u_4\}$

$X_4 = \{u_1, u_3, u_4, u_5\}$

$X_5 = \{u_3, u_5\}$



Step 3. Define X_i:

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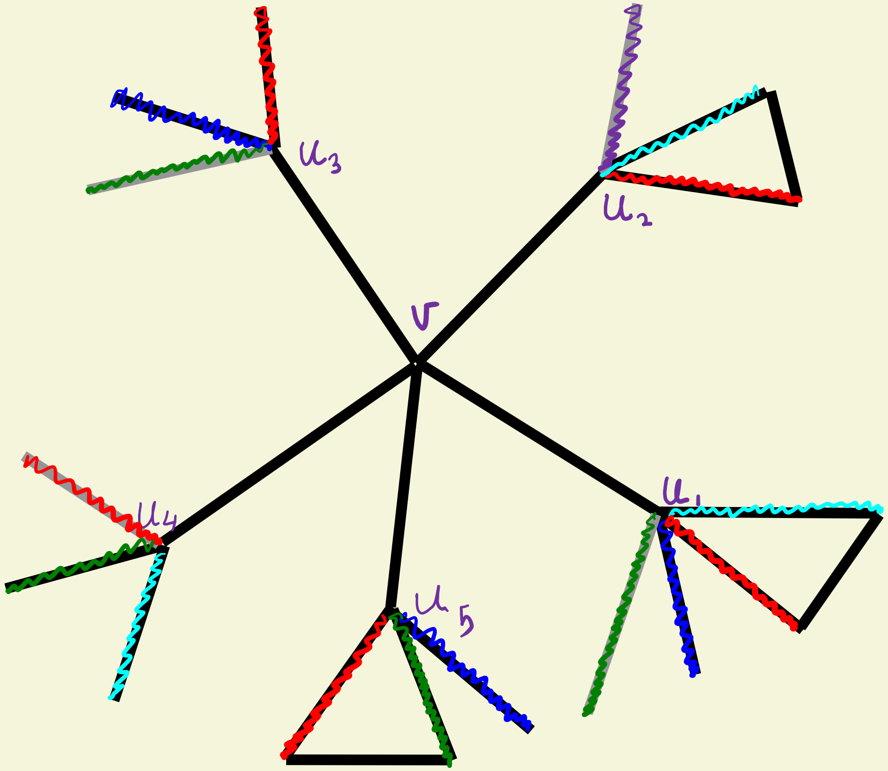
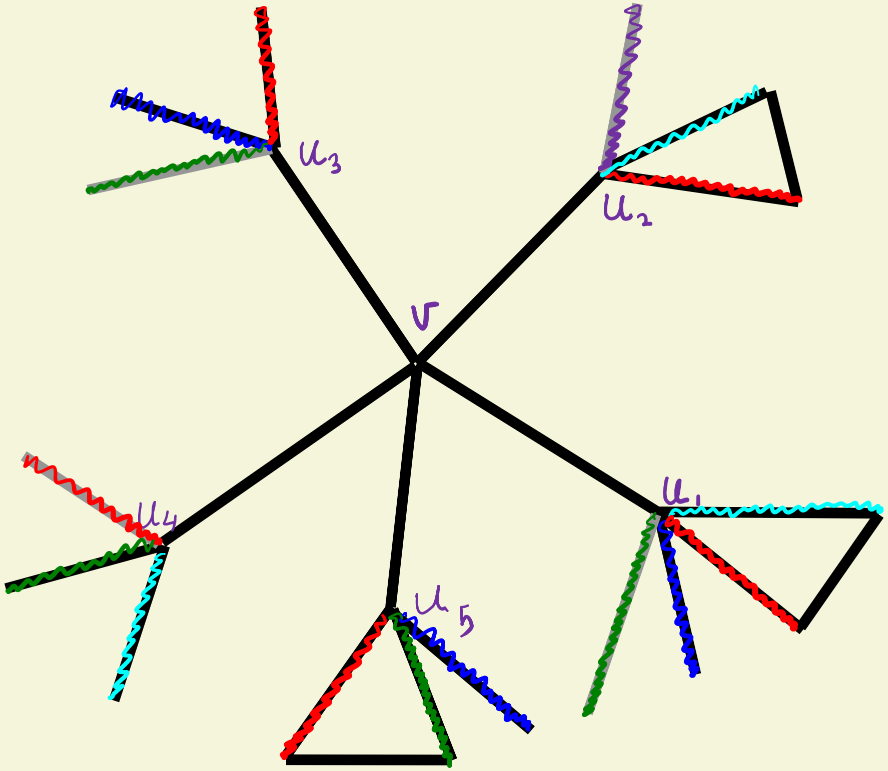


Observation:

$$\sum_{i=1}^{k} |X_i| = 2k-1$$

We wish to make these sets of nearly equal size.

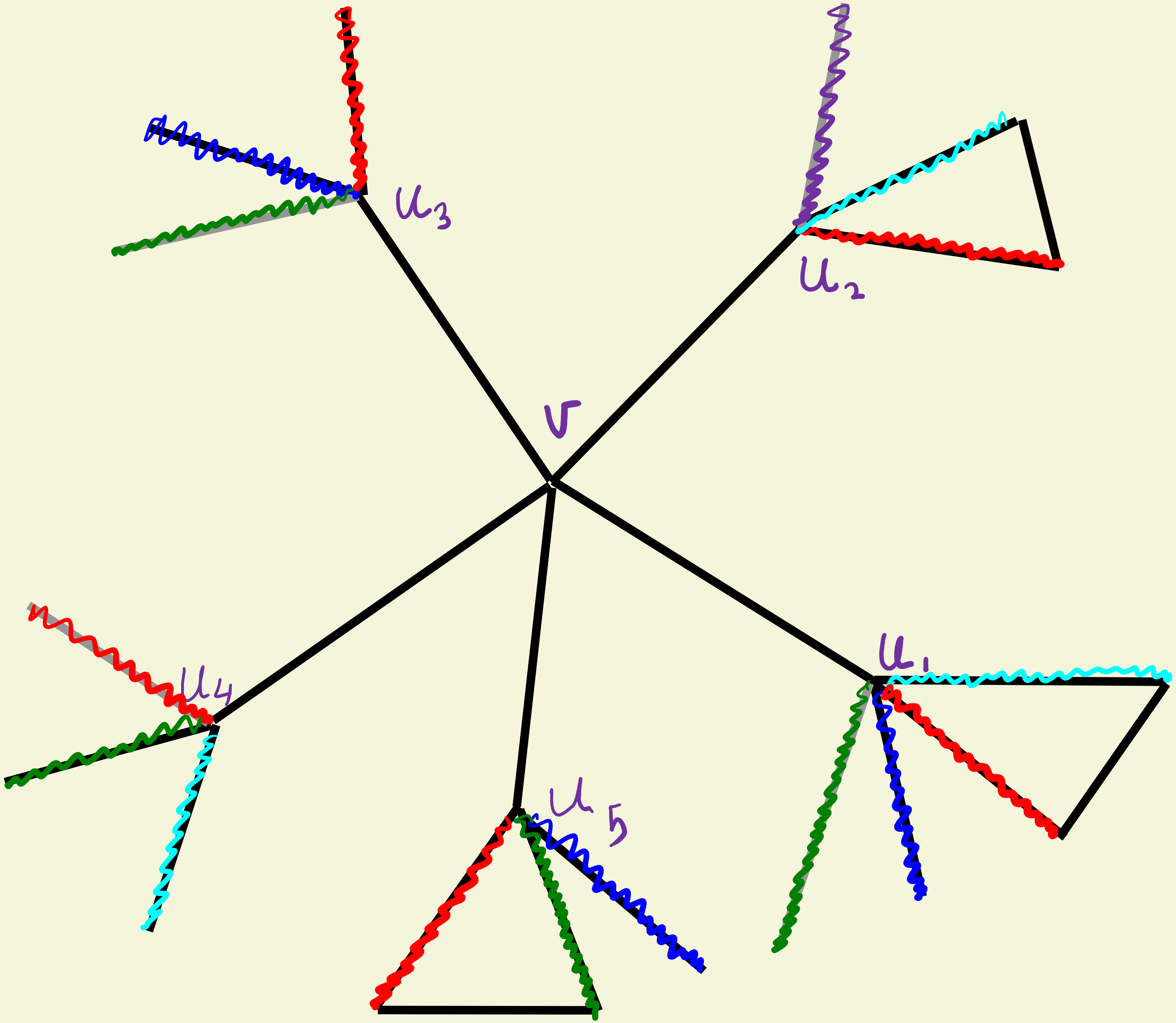
How to do that?



Observation:

$$\sum_{i=1}^{k} |X_i| = 2k - 1$$

Show that at least one $|X_i| = 1$.

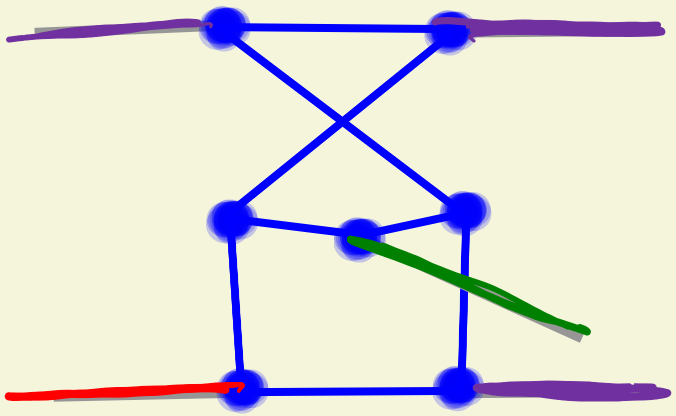


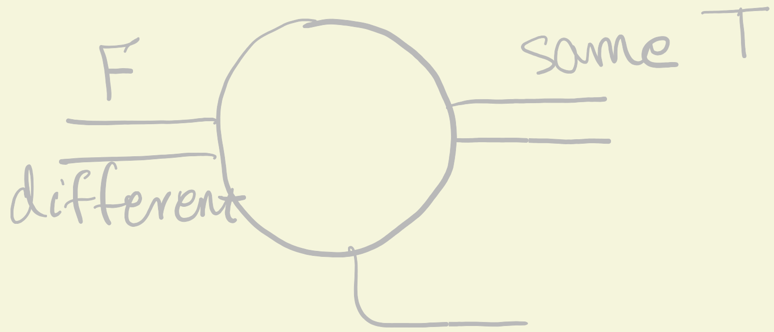
v
u1
u2
u3
u4
u5

Construction of graphs of maximum degree 3 which are not 3-edge-colorable.

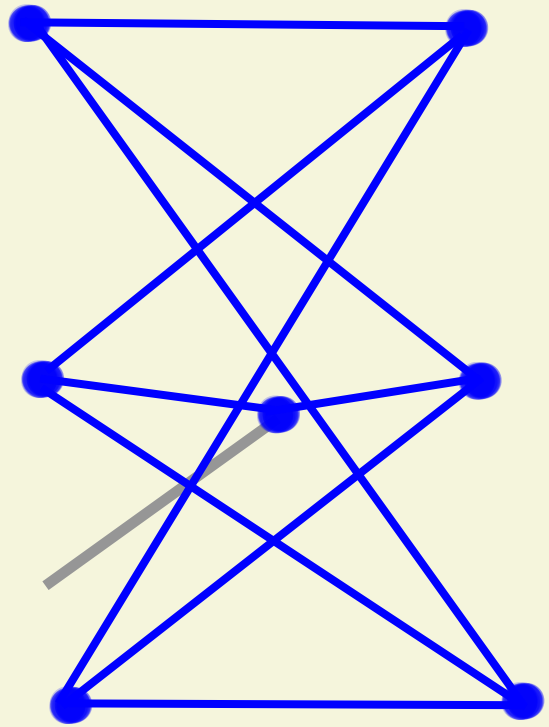
A gadget:

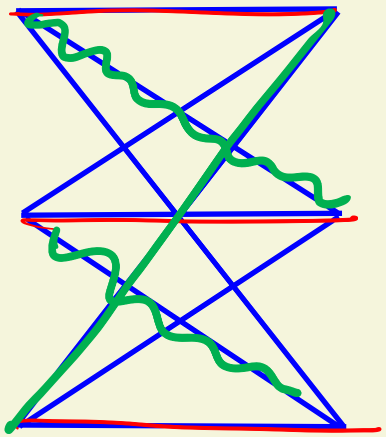
In every 3-edge-coloring of the gadget parallel edges of one side receive a same color and the three other pendant edges receive 3 different colors.

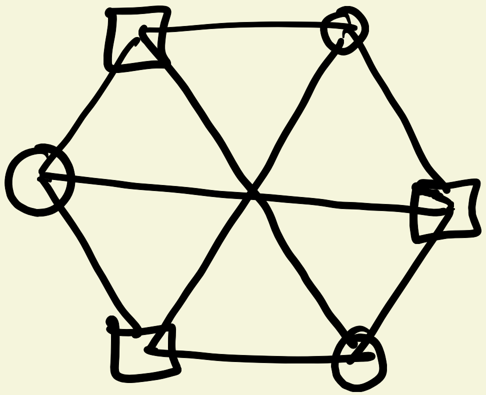




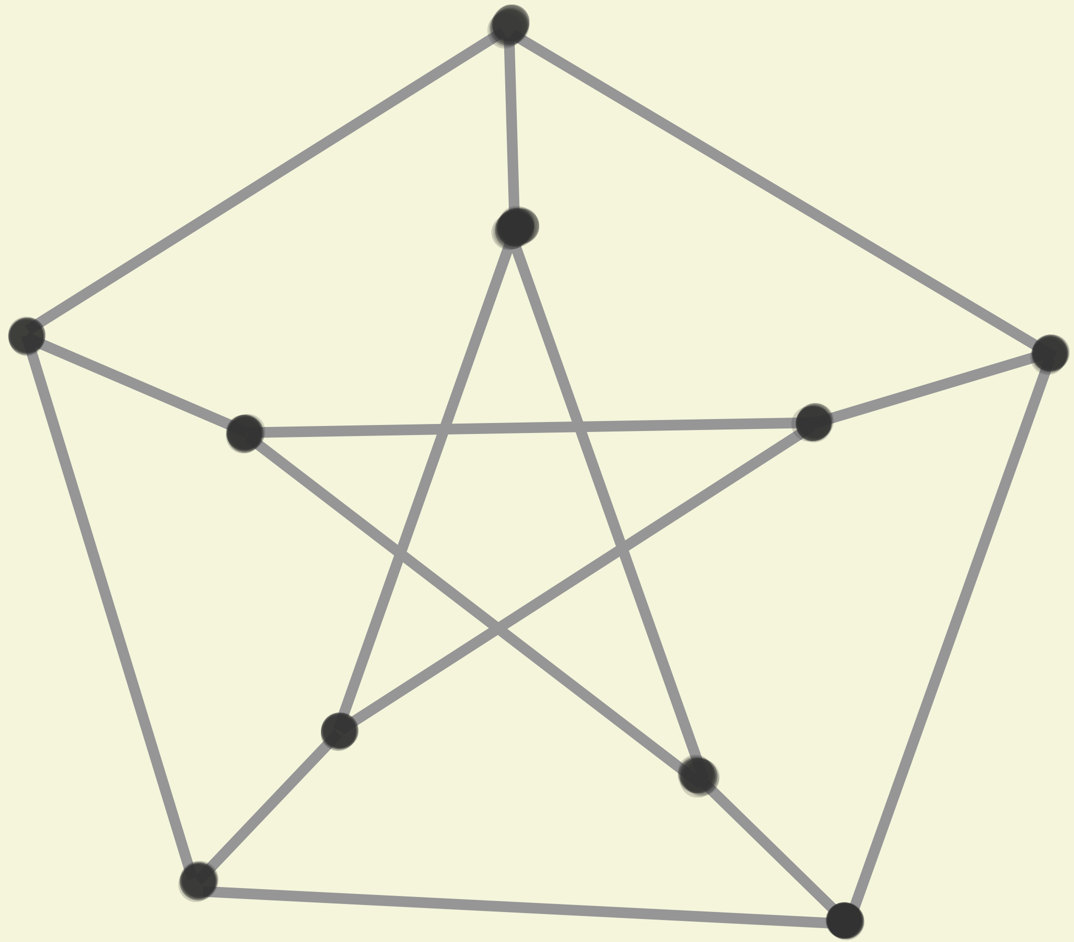
Corollary. The graph obtained from $K_{3,3}$ by subdividing one edge is not 3-edge-colorable.







Homework. The Petersen graph is not 3-edge-colorable.

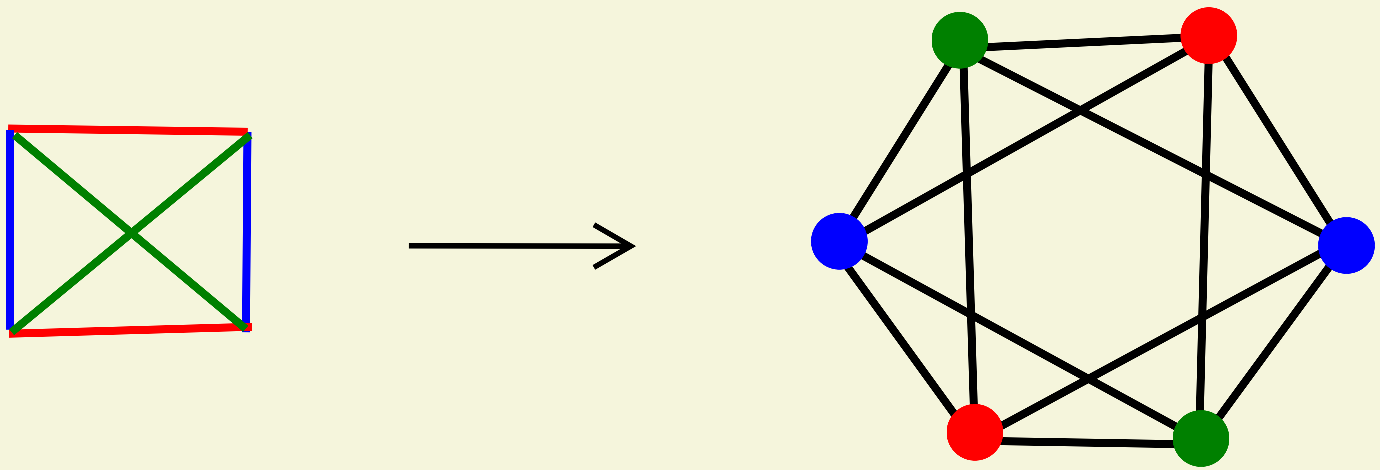


Edge-coloring $\longrightarrow$ vertex-coloring

G $\qquad\qquad$ $L(G)$

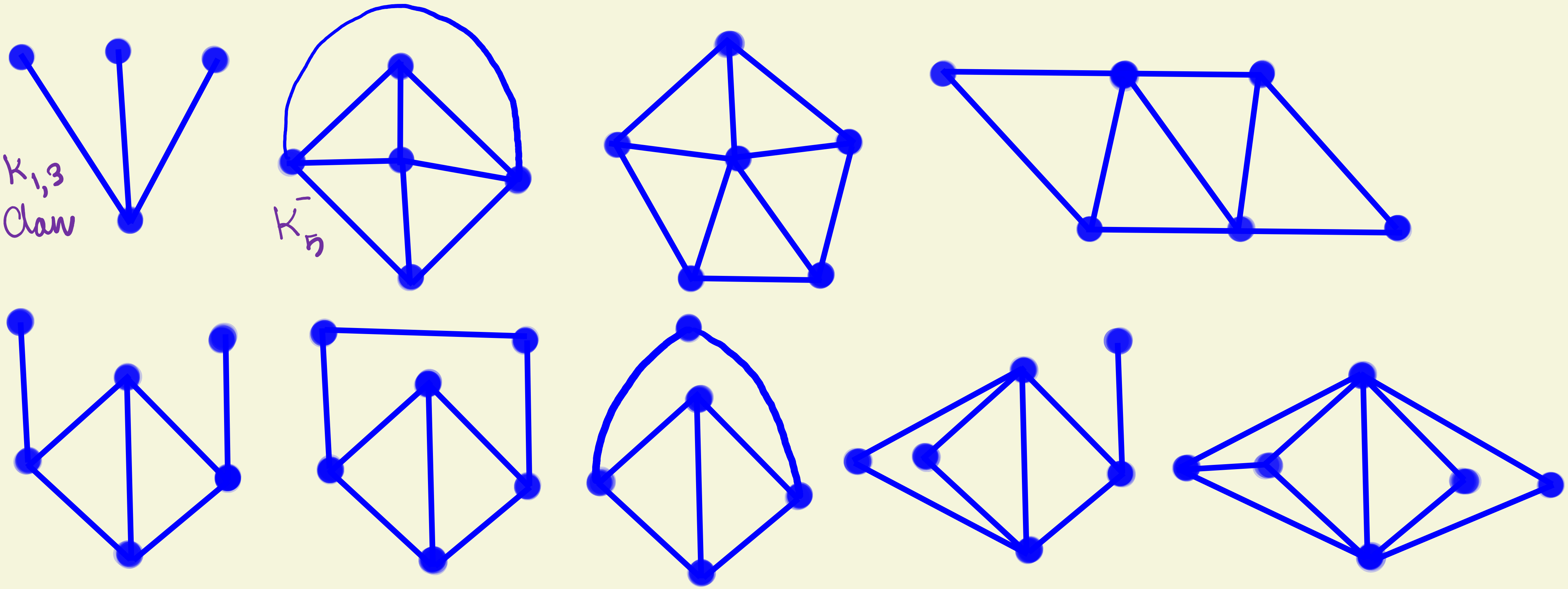
$\Delta = k \longrightarrow \Delta \leqslant 2k-2$

?



Beineke theorem: $(L(G)) < \chi(L(G)) < (L(G)) + 1$

A graph H is the line graph of a graph G if and only if it does not have any of the following 9 graphs as a subgraph.



Strenthening Vizing theorem (Kierstead)

If H is a graph with no induced $K_{1,3}$ or K_5^-

then $\chi(H) \in \{\omega(G), \omega(G)+1\}$.

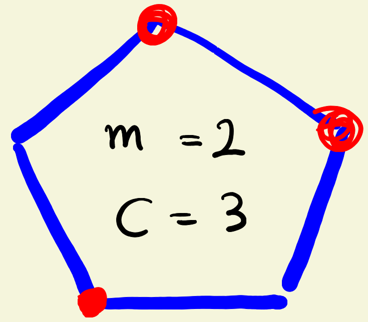
Improving Vizing theorem for bipartite graphs:

Theorem. If G is a bipartite graph, then

$$\chi'(G) = \Delta(G)$$

We will prove it using a min-max theorem.

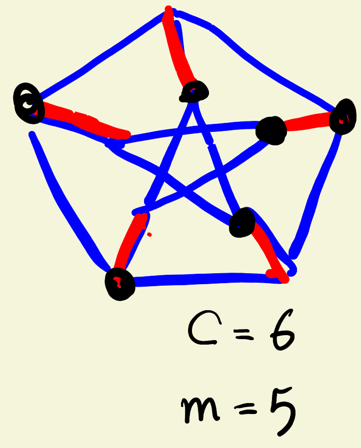
Definition:



$m(G)$: size of maximum matching of G

$\downarrow$

A set of edges with no common vertex



$c(G)$: order of a minimum cover of G

$\downarrow$

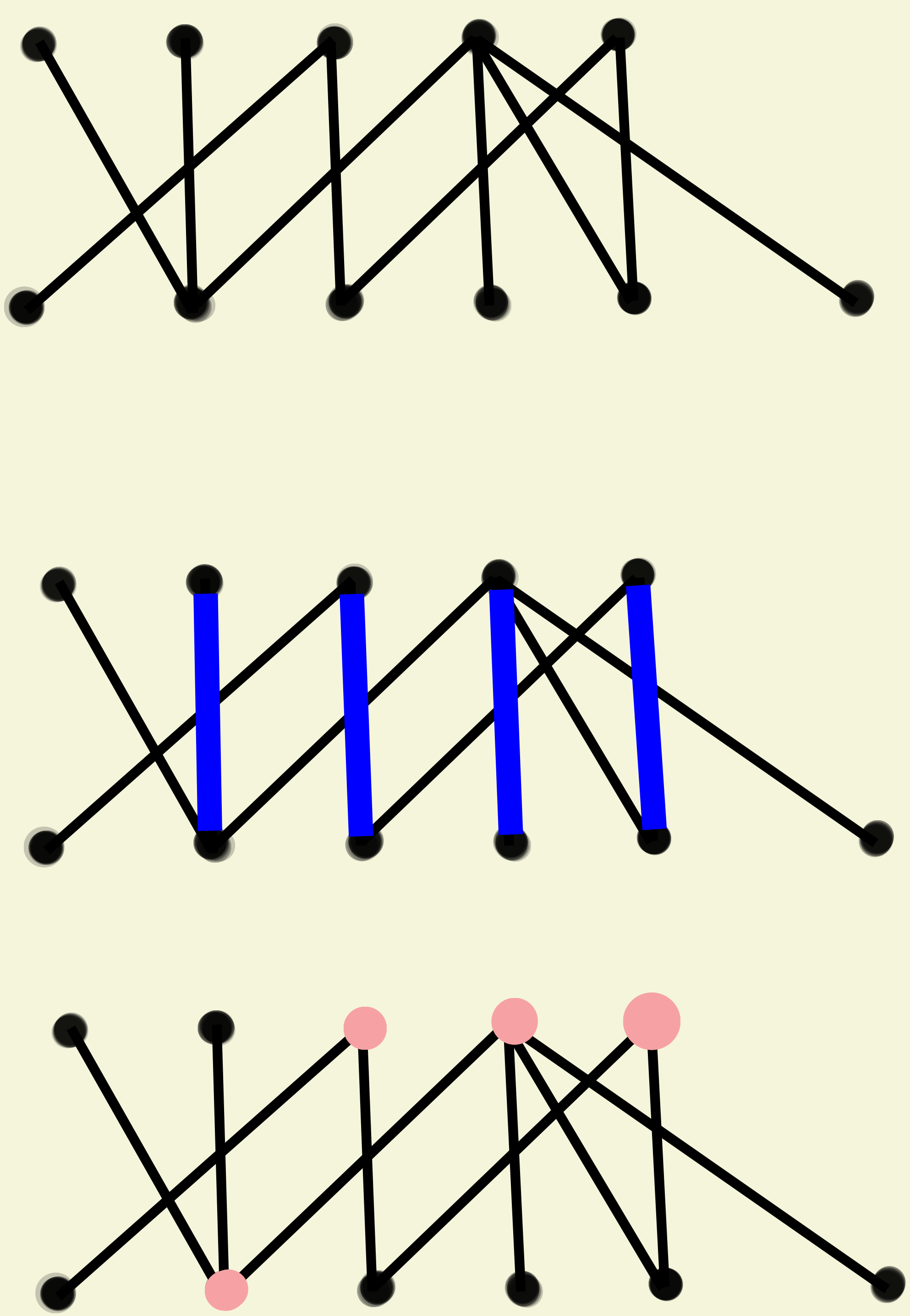
A set vertices that cover all edges

Theorem. If G is a bipartite graph, then

$$m(G) = c(G)$$

Example

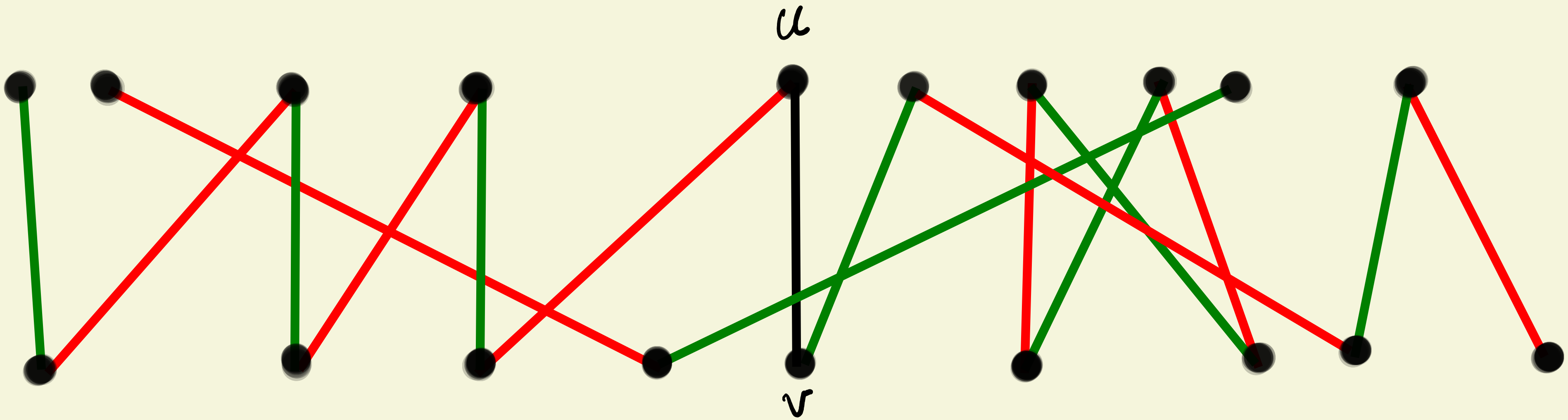
$c = m = 4$



Lemma. If G is a bipartite graph with at least one edge, then it has a vertex u which belongs to every maximum matching.

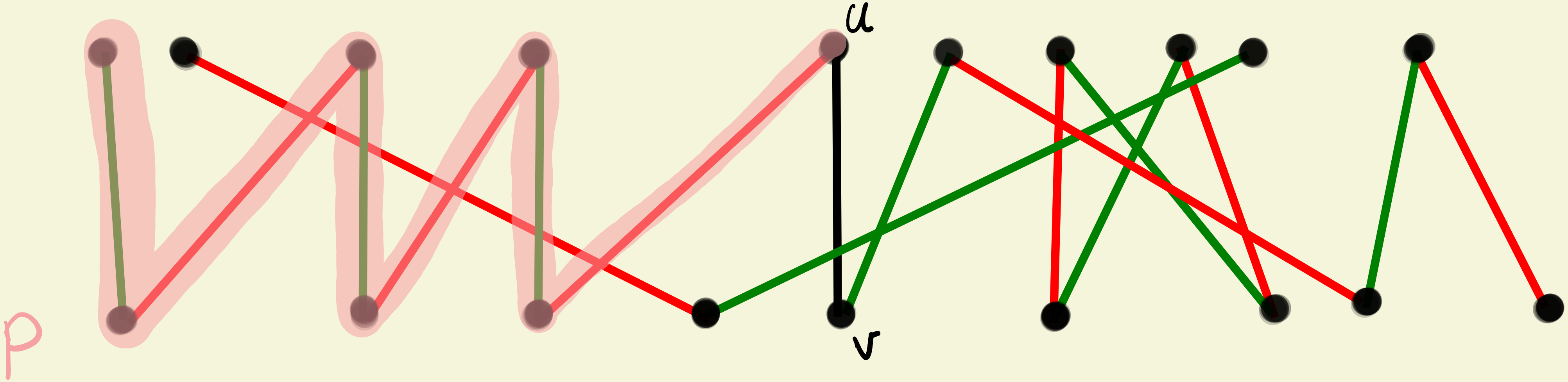
Proof. In fact we prove that for every edge uv one of u or v satisfies the condition of the lemma.

Toward a contradiction, suppose for an edge uv there are matchings M_u and M_v, each of maximum size where M_u misses the vertex u and M_v misses the vertex v.



Consider the subgraph induced by $m_u \cup m_v$

Let P be the connected component of this subgraph that contains the vertex u.



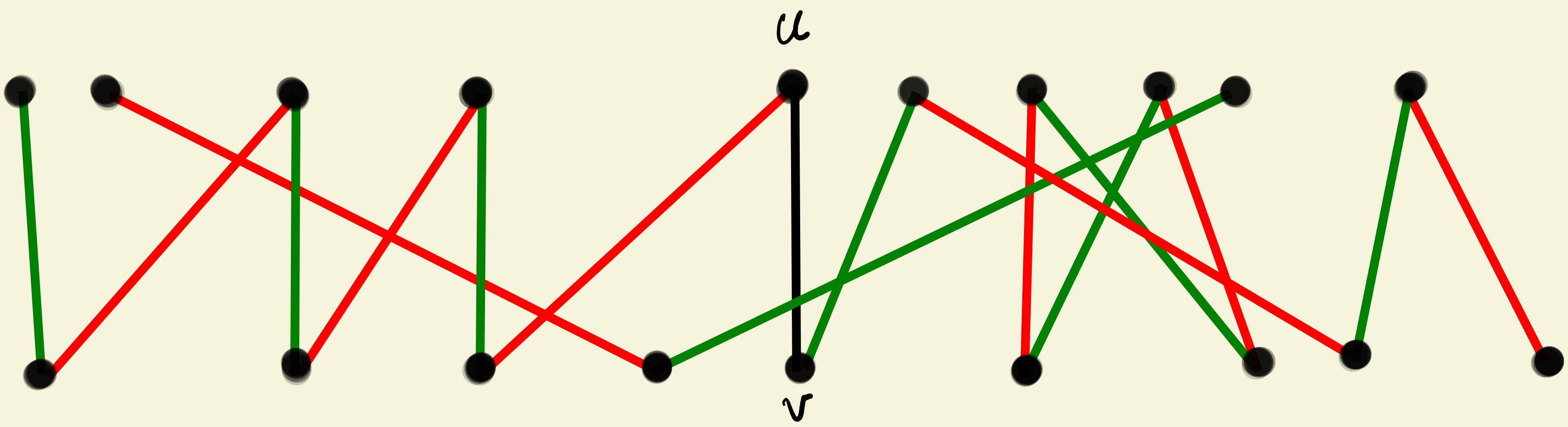
Claim 1. P is a path (not a cycle).

Claim 2. P starts with a red edge and ends with a green one.

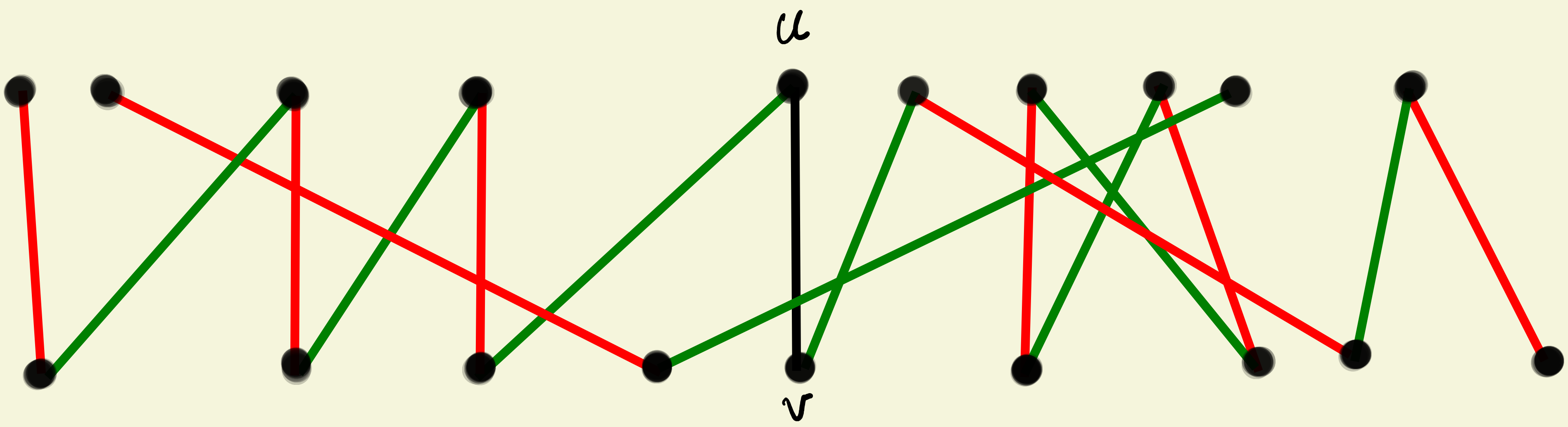
Claim 3. The vertex v does not belong to P.

Because G is bipartite, last vertex of P is in the same part as u,

but v is in the other part.



In P switch red and green.



Theorem. If G is a bipartite graph, then $m(G) = c(G)$

Proof. By induction on $m(G)$.

If $m(G)=0$ then ✓

Also if $m(G)=1$ then ✓

Assume that the claim is valid for $m(G) \leq k-1$ and let G be a bipartite graph with $m(G)=k$.

Theorem. If G is a bipartite graph, then $m(G) = c(G)$

Let u be a vertex which is in every maximum matching.

Consider $G-u$.

Corollary. If G is a k-regular bipartite graph, then $\chi'(G) = k$

Corollary. For every bipartite graph G we have

$$\chi'(G) = \Delta(G)$$

Corollary. Line graph of every bipartite graph G satisfies:

$$\chi(L(G)) = \omega(L(G))$$

Valid for every induced subgraph of $L(G)$

Perfect graph: A graph G where every induced subgraph H satisfies

$$\chi(H) = \omega(H)$$

Homework.

1. Show that for $k \geq 2$ the odd cycle C_{2k+1} and its complement are not perfect.

2*. Show that there are graphs with $\omega(G) = 2$ and $\chi(G) = k$ (for every k).

3**. Show that for every g and k there exists a graph G which has no cycle of length smaller than g and has chromatic number k.