Homomorphism.

↳ between structures of a similar nature

A mapping of the ground set to the ground set which preserves the main structures.

Example. 3-SAT

$(\{x_1,\bar{x}_1\}, \dots \{x_n,\bar{x}_n\}, (x_1,\bar{x}_2,x_4) \dots (\quad)) \to (\{T,F\}, (T,T,T), \dots (\quad))$

↳ clauses

all triples but (F,F,F)

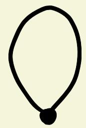
$$\begin{tikzcd} G \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & H \arrow[r, "g"] & \bigcirc \end{tikzcd}$$

$$\chi(G) \leq \chi(H)$$

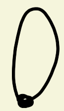
Homomorphisms of graphs ($G \longrightarrow H$)

$$f : V(G) \longrightarrow V(H)$$

$$x \sim y \Rightarrow f(x) \sim f(y)$$

- every graph maps to 

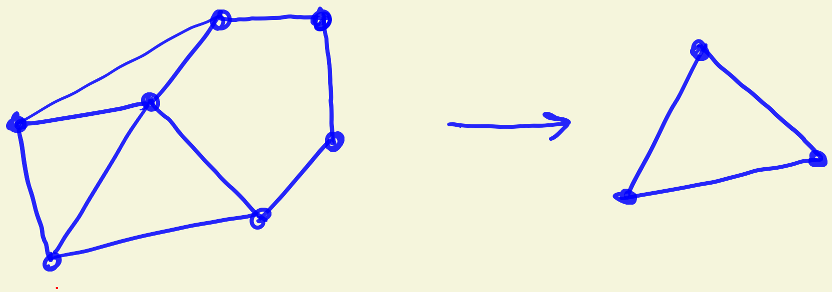
- for loop-free, $\chi(G)$: smallest order of a loop-free homomorphic image.

- $C_{2k+1} \underset{\not\leftarrow}{\overset{}{\longrightarrow}} C_{2k-1} \underset{\not\leftarrow}{\longrightarrow} C_{2k-3} \cdots \longrightarrow C_5 \underset{\not\leftarrow}{\longrightarrow} C_3 \underset{\not\leftarrow}{\longrightarrow}$ 

Core of G:

(The) smallest subgraph G' of G such that $G \to G'$

Example.

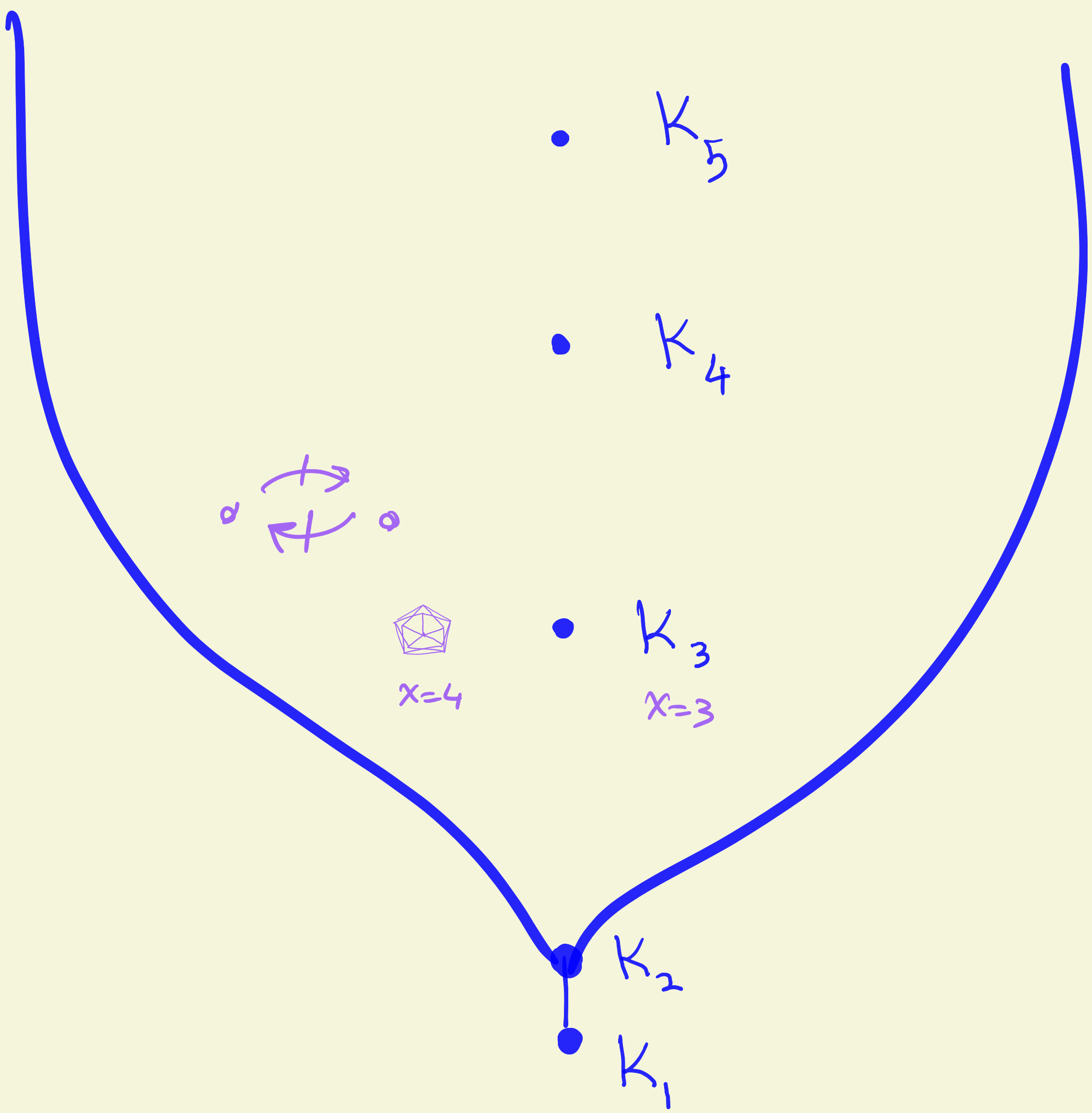


Homework. Prove that, up to isomorphism, the core of a graph is unique.

Homomorphism order

$[G] = \{H : G \to H,\ G \leftarrow H\}$

↳ core



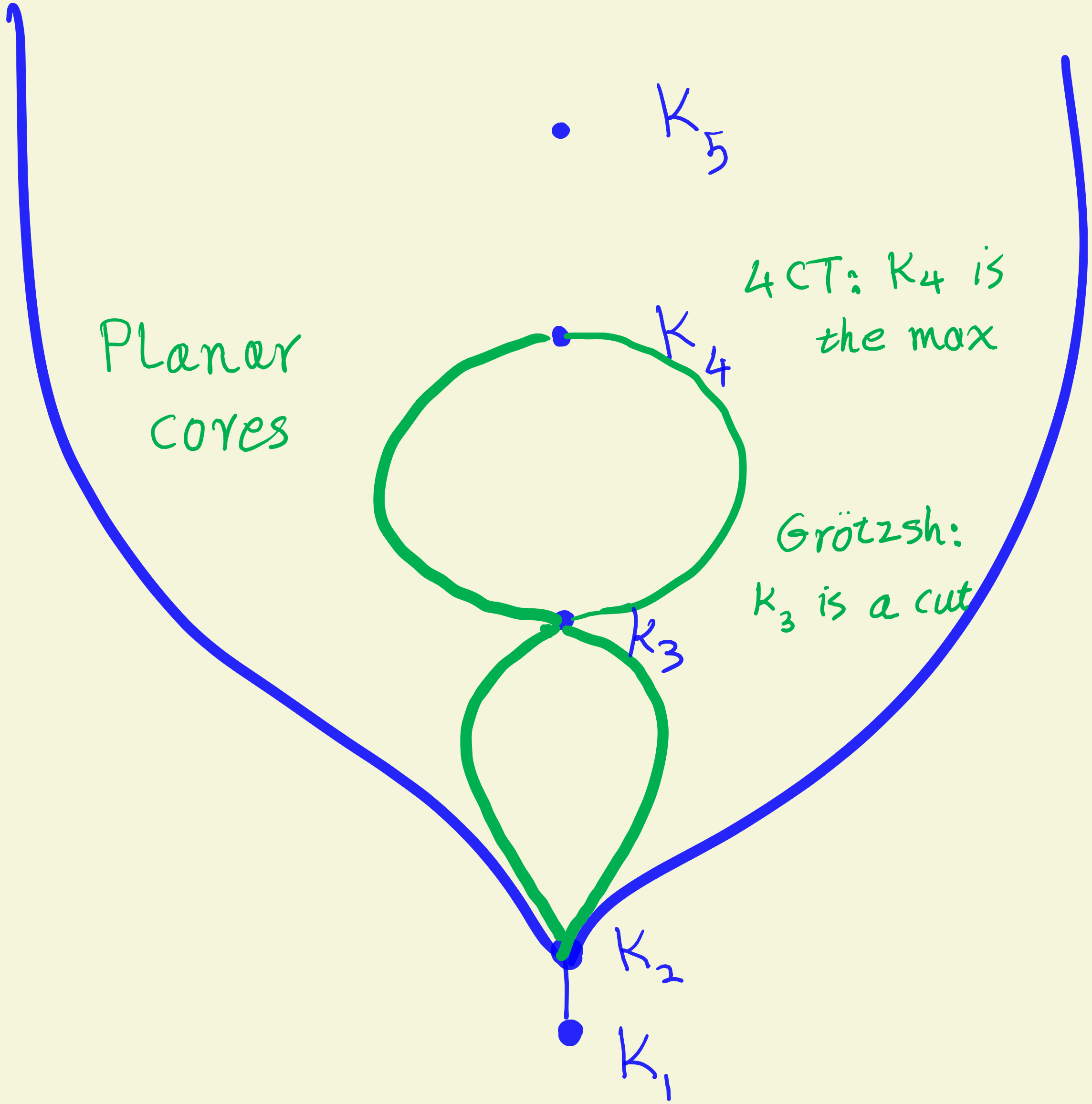
Questions:

1. Present a pair of incomparable elements.

Theorem. Every countable order has an isomorphic copy in this order.

2. What is the most natural embedding of $\mathbb{Z}^+$?
   What does $\lceil G \rceil$ and $\lfloor G \rfloor$ present?

The four-color theorem and the Grötzsch theorem presented in homomorphism order.



Duality of homomorphism & minor

signed graph: A graph where each edge is assigned a sign

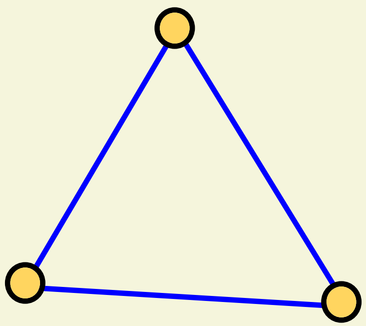
+ positive - negative

Notation: (G, σ)

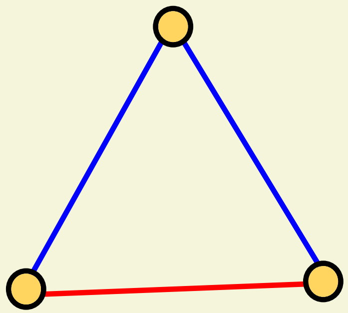
signature

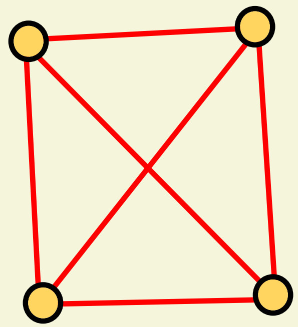
graph

Examples:

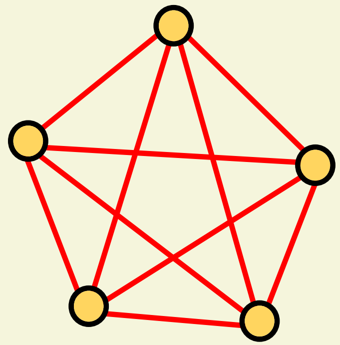


$+$





$(K_4, -)$



$(K_5, -)$

Main terminology: positive cycle (balanced cycle) negative cycle (unbalanced cycle)

reflecting the fact that the rule of "friend of a friend is a friend" applies.

positive/negative closed walk

Balanced signed graph: signed graph with no negative cycle

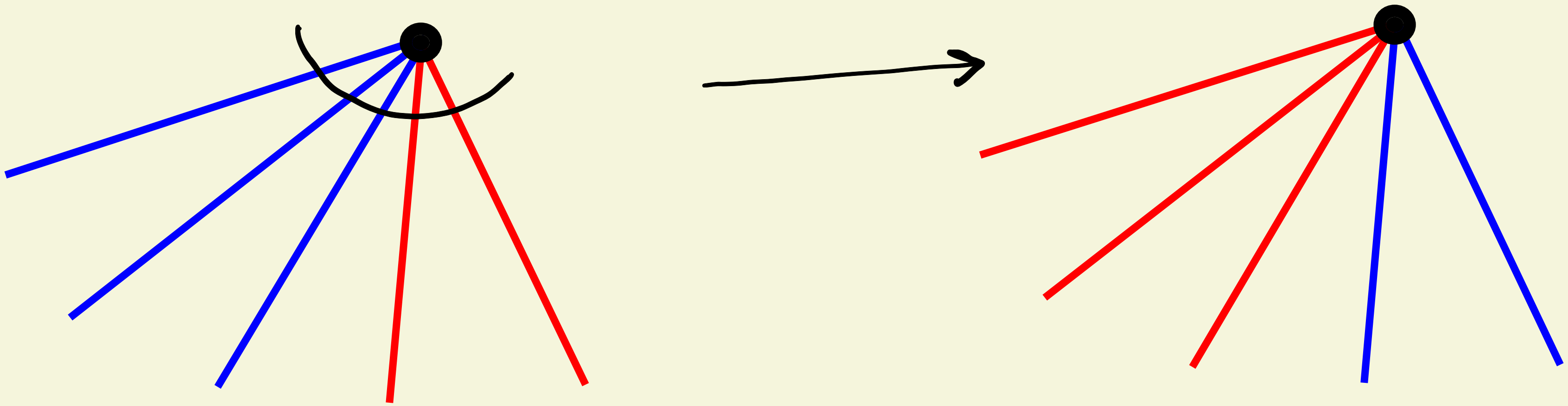
Antibalanced signed graph: where odd cycles are negative
even cycles are positive.

Switching (what makes it different from 2-edge-colored graphs).

To multiply signs of edges incident to a vertex to $-$ $\qquad G = (V, E_B\, E_R)$

Switching at all vertices of a subset X of vertices:

to multiply signs of all edges in the edge cut $(X, V-X)$ to a $-$



Parallel terminologies

~~Resigning~~	switching
~~odd~~	negative
~~odd cycle~~	negative cycle
~~bipartite~~	balanced

(G, Σ) in place of (G, σ) where Σ is the set of negative edges

(i.e. $\Sigma = \sigma^{-1}(-)$)

$\cancel{\Sigma = (G, \sigma)}$, $\quad \hat{G} := (G, \sigma)$

How may theory of signed graphs help?

1. Stronger results

Theorem. Every $\underbrace{K_4\text{-minor-free}}_{\text{easy to verify}}$ graph is $\underbrace{\text{3-colorable}}_{\text{NP-hard for general graphs}}$.

how to generalize so to include bipartite graphs?

How may theory of signed graphs help?

2. Fill the gap in theories.

Example. $T_{2k-1}(G)$: obtained from G by replacing each edge with a path of length $2k-1$.

Theorem. $\chi(G) \leq 2k+1 \iff T_{2k-1}(G) \to C_{2k+1}$

Question. How to capture $2k$-coloring?

How may theory of signed graphs help?

3. Developing proof techniques that are not possible for graphs.

We plan to present one such an example in today's lecture.

Homework. Given a graph G how many non-equivalent signatures we have on G?

Hint. Number of connected components is important.

Question. Given two signatures σ_1 & σ_2 on a graph G how can we decide if $(G,\sigma_1)\equiv(G,\sigma_2)$?

A "NO" answer: if a cycle C is positive in one and negative in the other.

Question. Given two signatures σ_1 & σ_2 on a graph G

how can we decide if $(G,\sigma_1) \equiv (G,\sigma_2)$?

A "NO" answer: if a cycle C is positive in one and

negative in the other.

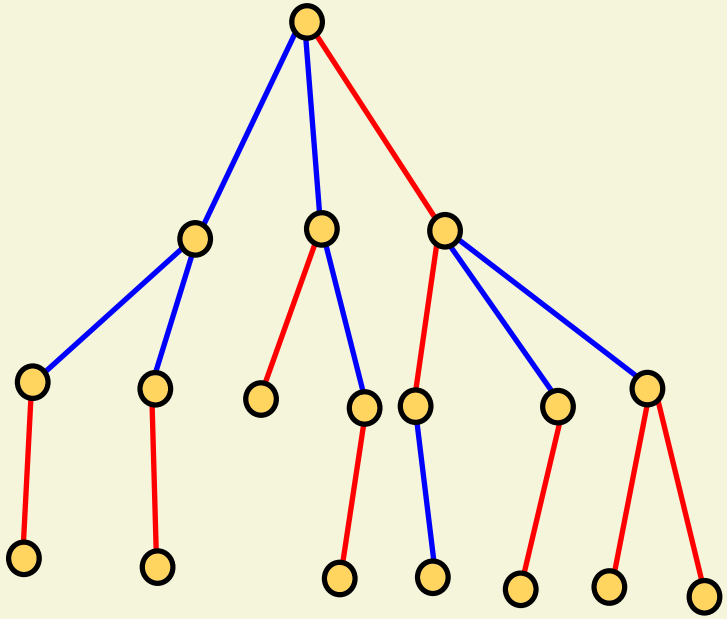
A "YES" answer: otherwise, ie. when every cycle has a

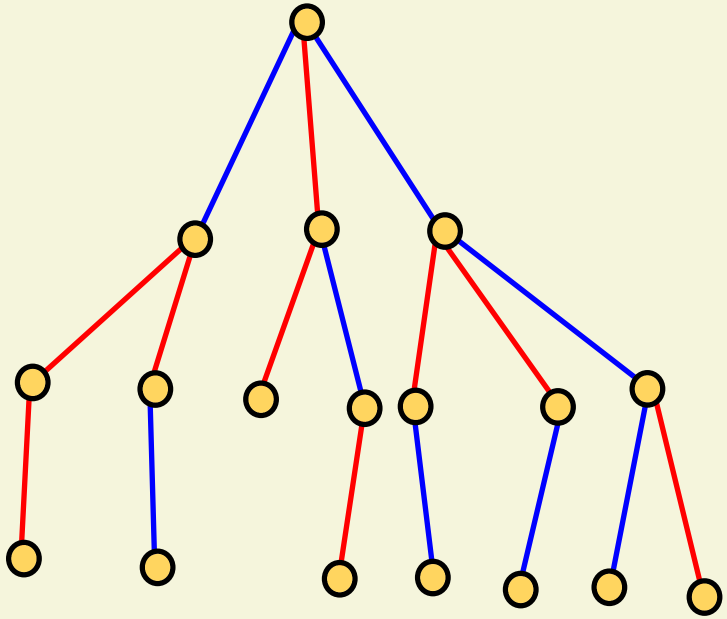
same signe in both signature.

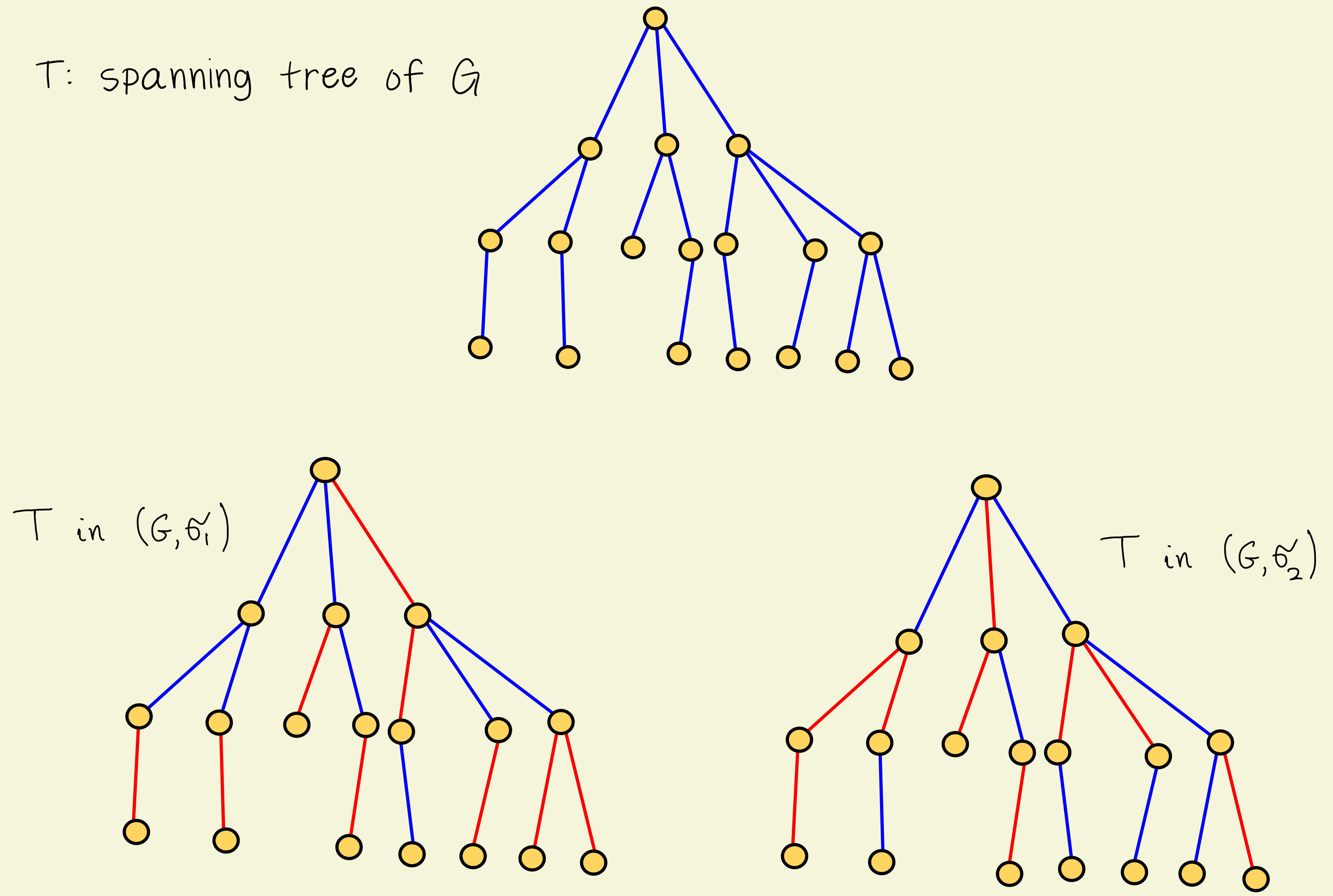
Question: Do we need to check all the cycles?

Observation.

Any two signatures on a tree are equivalent.







T: spanning tree of G
T in $(G, \tilde{\sigma}_1)$
T in $(G, \tilde{\sigma}_2)$

Harary. (G, σ) is switch-equivalent to $(G, +)$ if and only if it is balanced.

no negative cycle

Zaslarsky. (G, σ_1) and (G, σ_2) are switch-equivalent if and only if every cycle of G has a same sign in both.

Two theories to develop on singed graphs

-theory of minor (a brief mention)

-coloring and homomorphism (our main focus)

Homomorphism & Homeomorphism

Given two structures of same type a mapping of ground elements of one to the other where the main structures are preserved.

Example.

Group Homomorphism

$$(\Gamma_1, +) \xrightarrow{f} (\Gamma_2, *)$$

$$f(x+y) = f(x) * f(y)$$

Topology Homeomorphism

$$(T_1, O_1) \xrightarrow{f} (T_2, O_2)$$

$$A \in O_1 \Longrightarrow f(A) \in O_2$$

Homomorphisms of graphs

$$G \to H$$

$$f: V(G) \to V(H)$$

$$x \sim y \Rightarrow f(x) \sim f(y)$$

Homework. $\chi(G)=$ smallest number of vertices of a homomorphic image of G where there exists no loop.

Homomorphisms of signed graphs:

Before formulating a definition must decide what are the main structures.

Vertices form the ground sets. Edges are main part of the structure.

- View 1. Sign of edges are part of the main structure.

- View2. Signs of cycles and closed walks are part of the main structure.

  (in this view two switch equivalent signed graphs are regarded to be identical).

View 1 leads to the notion of homomorphisms of 2-edge-colored graphs

(- red, + blue)

(studied since 1980's)

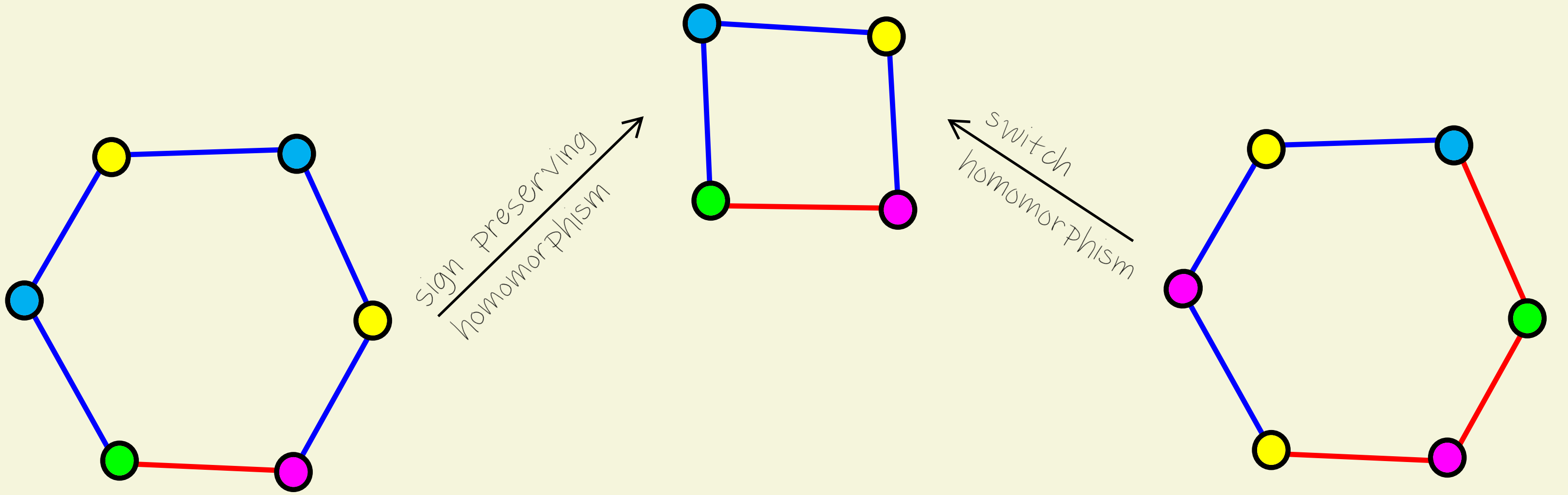
Our main interest is based on the View 2, but there is a strong connection to View 1.

Homomorphisms of signed graphs.

Definition. Given signed graphs (G, σ) & (H, π) a mapping of $V(G)$ to $V(H)$ (and $E(G)$ to $E(H)$) is said to be a homomorphism of (G, σ) to (H, π) if it preserves djacencies, (incidences) and signs of closed walks.

It is said to be edge-sign prserving homomorphism if it furthermore, preserves signs of edges.

Examples

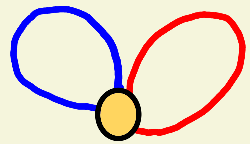


Comment. The edge mapping is implied unless (H, π) contains a digon.

Theorem. Signed graph (G, σ) admits a homomorphism to signed graph (H, π) if and only if for some switching (G, σ') there exists an edge-sign preserving homomorphism of (G, σ') to (H, π).

Observation

Any signed graph admits a homomorphism to $K_1^{\pm}$.



A notion of chromatic number:

Given $K_1^{\pm}$-free signed graph (G,σ) we define $\chi_h(G,\sigma)$ to be the minimum number of vertices in a $K_1^{\pm}$-free homomorphic image of (G,σ).

Question. Given integer $n>1$, what is the largest $\chi_h(G,\sigma)$ over all possible signatures?

Comment. The answer is closely related to the Ramsey number $R(P,P)$.

Observation. In a mapping of (G,σ) to (H,π) the image of every closed walk is a closed walk which has a same (parity) of length and a same sign.

This leads to four notion of girth:

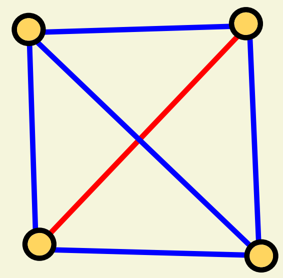
$g_{00}(G,\sigma)$: length of shortest positive even closed walk,

$g_{10}(G,\sigma)$: length of shortest negative even closed walk,

$g_{01}(G,\sigma)$: length of shortest positive odd closed walk,

$g_{11}(G,\sigma)$: length of shortest negative odd closed walk.

Example.



$g_{00}(G,\sigma)=2$, $g_{10}(G,\sigma)=4$, $g_{01}(G,\sigma)=3$, $g_{11}(G,\sigma)=3$

The main no homomorphism Lemma.

If $(G,\sigma) \longrightarrow (H,\pi)$ then,

$$g_{ij}(G,\sigma) \geq g_{ij}(H,\pi)$$

for every $ij \in Z_2^2$

A main question:

When do the conditions of the no-homomorphism Lemma (or similar but stronger conditions) become sufficient?

Example

For $(H, \pi) = (K_4, -)$ and all planar signed graphs.

Example

For $(H, \pi) = (K_3, -)$ and all planar signed graphs satisfying stronger condition of

$$g_{ij}(G, \sigma) \geq g_{ij}(H, \pi) + 2 .$$