MPRI - Graph Algorithms

Final Exam 2021-22

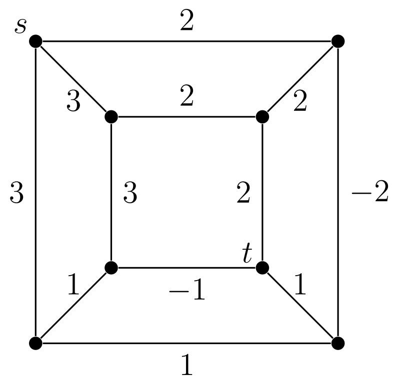
No document other than the lecture notes are authorized. The exercises are independent and are not sorted by increasing difficulty.

Exercise 1 : List chromatic number
List coloring is a natural concept that appears when we want to extend coloring of some vertices of the graph to the uncolored vertices. The proper definition is as follows:
Let G be a graph and for each vertex v of G suppose there is a list $L(v)$ of colors which is the set of available colors for the vertex v.
The graph G is said to be k-list colorable if for any list assignment with each list being of size k there is a proper coloring of G where color of v is in $L(v)$.

1. Show that even cycles are 2-list colorable.
2. Show that odd cycles are not 2-list-colorable.
3. Show that $K_{3,3}$ is not 2-list colorable, but it is 3-list colorable.
4. [*] Show that for any k there is a bipartite graph which is not k-list colorable.
5. [Bonus] Can you extend Brooks theorem to list-coloring?

Exercise 2 : Edge-coloring question
Let G be a graph obtained from $K_{3,3}$ by subdividing an edge once.
1. Show that G is 2-connected.
2. Show that G is not 3-edge-colorable.
3. What is the edge-chromatic number of this graph?
4. Write the integer programming for the edge-chromatic number of this graph.
5. [*] By solving the linear version of this program, or by any other technique, find the fractional edge-chromatic number of this graph.

Exercise 3 : Shortest path
Using T-joins, find a shortest path between s and t in the following graph. Explain the different steps.



Exercise 4 : Matchings in regular bipartite graphs
Let $G = (V, E)$ be a d-regular bipartite graph (all vertices are of degree $d \geq 1$) on n vertices, with bipartition $\{A, B\}$.

Let $X \subseteq V$, and denote by a the number of edges between $A \cap X$ and $B \setminus X$, and by b the number of edges between $B \cap X$ and $A \setminus X$.

1. Show that $d|A \cap X| - a = d|B \cap X| - b$.

2. Deduce that if $a + b < d$, then $|A \cap X| = |B \cap X|$.

3. Deduce that if $|X|$ is odd, then the cut $\delta(X)$ consists of at least d edges.

4. Using Seymour's theorem on T-joins and T-cuts in bipartite graphs, deduce that G has a perfect matching.

Exercise 5 : Application of matching algorithms
Given a weighted graph G (where each edge has a nonnegative weight), algorithm A can find, in polynomial time, a maximum weighted matching of G. That is a matching M of G whose sum of weights is maximum possible.

Let H be a graph (without weights) and let u and v be two vertices of H.

1. Using algorithm A find a polynomial time algorithm that determines if there is an even path connecting u and v and if so, finds a shortest such a path.
Hint: Consider two copies of H where in one copy u is a special vertex and in the other v. Build a new weighted graph where maximum matching of the new graph corresponds to such a path in H.

Exercise 6 : Cycle-and-triangles theorem
The aim of this exercise is to prove the following result, sometimes referred to as the *cycle-and-triangles theorem*:

> Given a cycle C_{3m} of length $3m$, and a partition of its vertex set into sets $T_1, T_2, \ldots, T_m$, each of size 3, there is an independent set S of C_{3m} containing exactly one vertex of each set T_i.

A short proof can be given using the following strengthening of the Lovász–Kneser theorem:

> The subgraph $\mathrm{SG}(n, k)$ of the Kneser graph $\mathrm{KG}(n, k)$ induced by all the vertices containing no pairs of consecutive elements (modulo n) has chromatic number $n - 2k + 2$.

For instance, $\mathrm{SG}(4, 2)$ has two vertices ($\{1, 3\}$ and $\{2, 4\}$), whereas $\mathrm{KG}(4, 2)$ has six vertices ($\{1, 3\}$, $\{2, 4\}$, $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$ and $\{1, 4\}$).

Assume C_{3m} and $T_1, T_2, \ldots, T_m$ are as in the statement of the cycle-and-triangles theorem. Consider the following colouring of $\mathrm{SG}(3m, m)$: if an independent set S of C_{3m} intersects some T_i in at least two vertices, colour S using the least such i; otherwise, leave S uncoloured.

1. Deduce the cycle-and-triangles theorem.

2. Explain why the following stronger statement holds: there exist two disjoint independent sets S, S' of C_{3m}, each containing exactly one vertex of each set T_i.