Circular Colouring and Orientation of Graphs

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Abstract: This paper proves that if a graph $G$ has an orientation $D$ such that for each cycle $C$ with $d \cdot |C| \pmod{k} \in \{1, 2, \ldots, 2d - 1\}$ we have $|C| \leq \frac{k}{d}$ and $|C| \leq \frac{k}{d}$, then $G$ has a $(k, d)$-colouring, and hence $\chi_c(G) \leq \frac{k}{d}$. 
Outline:

- Basic Definitions
- Some Results
- Xuding Zhu’s Result
A graph $G$ is a couple $(V(G), E(G))$ where $V(G)$ is a finite, non-empty set of elements, called vertices of $G$, and $E(G)$ is set of pairs of vertices, called edges of $G$.

A digraph $D$ is a couple $(V(D), E(D))$ where $V(D)$ is a finite nonempty set of elements, called vertices of $D$, and $E(D)$ is subset of the set of ordered pairs of distinct elements of $V(D)$, called arcs of $D$.

For a digraph $D = (V(D), E(D))$, the **Underlying Graph** of $D$, denoted by $G[D]$, is the undirected graph created using all of the vertices in $V(D)$, and replacing all arcs in $E(D)$ with undirected edges.

An orientation of a graph $G = (V(G), E(G))$ is a digraph $D = (V(D), E(D))$ such that $G[D] = G$. 
Basic Definitions

For a cycle $C$ of $D$ with a chosen direction of traversal (each cycle has two different directions for traversal), let $C^+$ be the set of positive edges of $C$ (i.e., whose direction coincide with the direction of the traversal) and let $C^-$ be the set of negative edges of $C$.

The parameter $\tau(C) = \max\{|C^+|, |C^-|\}$ measure the “imbalance” of the cycle $C$.

Let $\xi(D) = \max\{\tau(C) : C \text{ is a cycle of } D\}$. Then $\xi(D)$ is a measure of the imbalance of the orientation $D$ of $G$. 
Basic Definitions

For any integer \( n; n \pmod{k} \) denotes the unique integer \( n' \) such that \( 0 \leq n' \leq k - 1 \) and \( n \equiv n'(mod\ k) \).

Suppose \( G = (V, E) \) is a graph and \( 1 \leq 2d \leq k \) are integers. A \((k, d)\)-coloring of \( G \) is a mapping \( f : V \rightarrow \{0, 1, \ldots, k - 1\} \) such that for every edge \( xy \) of \( G \):

\[
d \leq |f(x) - f(y)| \leq k - d
\]

The circular chromatic number \( \chi_c(G) \) of \( G \) is defined as:

\[
\chi_c(G) = \min \left\{ \frac{k}{d} : \text{there exists a \((k, d)\)-coloring of } G \right\}.
\]
Outline:

- Basic Definitions
- Some Results
- Xuding Zhu’s Result
Some Results

- Theorem 1: (Minty [2])

For any finite graph $G$,

$$\chi(G) = \min\{[\xi(D)]: D \text{ is an orientation of } G\}$$

The non-trivial direction of Minty’s result asserts that if $G$ has an orientation $D$ with $\xi(D) \leq k$ where $k$ is an integer then $G$ is $k$-colorable.

Tuza’s result says that to obtain the same conclusion, instead of requiring $\xi(D) \leq k$ which is equivalent to require that $\tau(C) \leq k$ for every cycle $C$, it suffices to require that $\tau(C) \leq k$ for those cycles $C$ with $|C|(\text{mod } k) = 1$.

Goddyn, Tarsi and Zhang’s result says that if $\xi(D) \leq \frac{k}{d}$ for some fraction $\frac{k}{d}$ then $G$ is $(k, d)$-colorable.
Outline:

- Basic Definitions
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Xuding Zhu’s Result

Xuding Zhu in his paper proves that if a graph $G$ has an orientation $D$ such that for each cycle $C$ with $d \, |C| (\text{mod} \, k) \in \{1, 2, \ldots, 2d - 1\}$ we have $\frac{|C|}{|C^+|} \leq \frac{k}{d}$ and $\frac{|C|}{|C^-|} \leq \frac{k}{d}$, then $G$ has a $(k, d)$-colouring, and hence $\chi_c(G) \leq \frac{k}{d}$.

This result is a generalization of Tuza’s result (if $d = 1$) and is an improvement of Goddyn et al’s result.
Proof

The proof of this result is parallel to Tuza’s proof of that special case. Let \( r \) be a fixed vertex of \( D \). Each spanning tree \( T \) of \( D \) is considered as rooted at \( r \). Given such a spanning tree \( T \), we define the weight \( w_T(x) \) of a vertex \( x \) of \( D \) (with respect to \( T \)) recursively as follows:

- \( w_T(r) = 0 \);
- If \( xy \) is an edge of \( T \) oriented from \( x \) to \( y \) and \( w_T(x) \) has already been defined, then \( w_T(y) = w_T(x) - k + d \);
- If \( xy \) is an edge of \( T \) oriented from \( x \) to \( y \) and \( w_T(y) \) has already been defined, then \( w_T(x) = w_T(y) + d \);

Since \( T \) is a spanning tree, for each vertex \( x \) of \( D \), \( w_T(x) \) is uniquely defined. Then we define the weight \( w(T) \) of \( T \) as

\[
w(T) = \sum_{x \in V(D)} W_T(x)
\]

Choose a rooted spanning tree \( T \) of \( D \) which has the maximum \textit{weight}. Let \( f : V \to \{0,1,\ldots,k-1\} \) be defined as \( f(x) = w_T(x) \mod k \): We shall show that \( f \) is a \((k,d)\)-colouring of \( G \).
Let $xy \in E(G)$. Without loss of generality we assume that the edge $xy$ is oriented from $x$ to $y$ in $D$.

We will study three different cases:

- If $x$ is not on the $y$–$r$-path of $T$ and $y$ is not on the $x$–$r$-path of $T$;
- If $y$ is on the $x$–$r$-path of $T$;
- If $x$ is on the $y$–$r$-path of $T$. 

![Diagram with three cases labeled 1, 2, and 3, illustrating the different paths and orientations of edges and vertices.](image-url)
Case 1:

If $w_T(x) - w_T(y) < d$; then we delete the edge of $T$ connecting $x$ to its father, and add the edge $xy$.

Then we obtain a spanning tree $T'$ for which $w_{T'}(v) \geq w_T(v)$ for each $v$; and $w_{T'}(v) > w_T(v)$ for every descendents of $x$; including $x$ itself. So $w(T') > w(T)$, contrary to our choice of $T$. Thus $w_T(x) - w_T(y) \geq d$.

Similarly we can prove that $w_T(x) - w_T(y) \leq k - d$.

Those, $d \leq w_T(x) - w_T(y) \leq k - d$. 
Case 1:

Those, $d \leq w_T(x) - w_T(y) \leq k - d$.

which implies that $d \leq |f(x) - f(y)| \leq k - d$ (as $f(x) - f(y) = w_T(x) - w_T(y) \mod k$), a contradiction.
Case 1:

Those, \(d \leq w_T(x) - w_T(y) \leq k - d\).

Then, \(d \leq w_T(x) - w_T(y) \pmod{k} \leq k - d\), and so:

\[
d \leq |f(x) - f(y)| \leq k - d
\]

As \(f(x) - f(y) = w_T(x) - w_T(y) \pmod{k}\).
Case 2:

Assume \( w_T(x) - w_T(y) = ak + j \) for some integers \( a; j \) such that \( 0 \leq j \leq k - 1 \):

Then, \( f(x) - f(y) = j \),

And hence \( j \in \{0, \ldots, d - 1\} \cup \{k - d + 1, \ldots, k - 1\} \).
Case 2:

Let $p$ be the number of edges on the $x$–$y$-path of $T$ oriented toward the root, and let $q$ be the number of edges on this path oriented away from the root. By the definition of the weight, we know that $w_T(x) - w_T(y) = pd - q(k - d)$. Therefore, we have $(pd + qd)(mod k) = j$; and $(p + q + 1).d(mod k) = j + d(mod k) \in \{1,2,\ldots,2d - 1\}$ note that the cycle $C$ consisting the $x$–$y$–path and the edge $xy$ is a cycle of length $|C| = p + q + 1$: Hence $d|C|(mod k) \in \{1,2,\ldots,2d - 1\}$. By our assumption, $\frac{|C|}{|C^-|} \leq \frac{k}{d}$, which implies $\frac{|C^+|}{|C^-|} \leq \frac{(k-d)}{d}$ here we choose the direction of traversal of $C$ so that those edge on the $x$–$y$–path oriented towards $r$ belongs to $C^+$: Hence $|C^+| = p$ and $|C^-| = q + 1$; which implies that $pd - q(k - d) \leq k - d$: Hence $w_T(x) - w_T(y) = pd - q(k - d) \leq k - d$. Therefore, $d \leq w_T(x) - w_T(y) \leq k - d$, a contradiction.
Case 3:

Similarly to the Case 3.
Thank you