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Circular Colouring and Orientation of Graphs

Author: Xuding Zhu

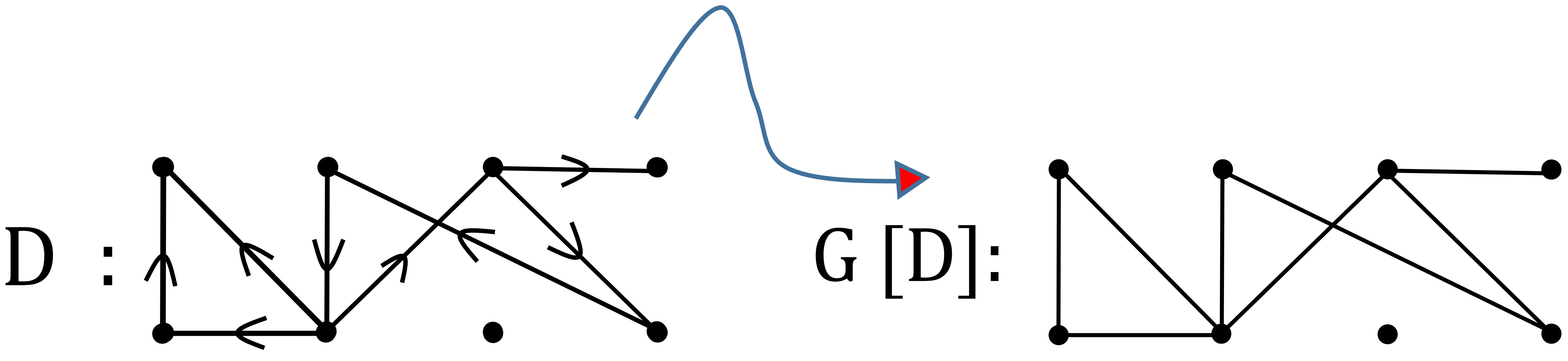
Presented by Mouhamad El Joubbeh

Abstract: This paper proves that if a graph $\boldsymbol{G}$ has an orientation $\boldsymbol{D}$ such that for each cycle $\boldsymbol{C}$ with $\mathbf{d}.|\mathbf{C}|(\mathbf{mod}\ \boldsymbol{k}) \in \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{2}\boldsymbol{d} - \mathbf{1}\}$ we have $\frac{|C|}{|C^+|} \leq \frac{k}{d}$ and $\frac{|C|}{|C^-|} \leq \frac{k}{d}$, then $\boldsymbol{G}$ has a $(\boldsymbol{k}, \boldsymbol{d})$-colouring, and hence $\chi_c(G) \leq \frac{k}{d}$.

Outline:

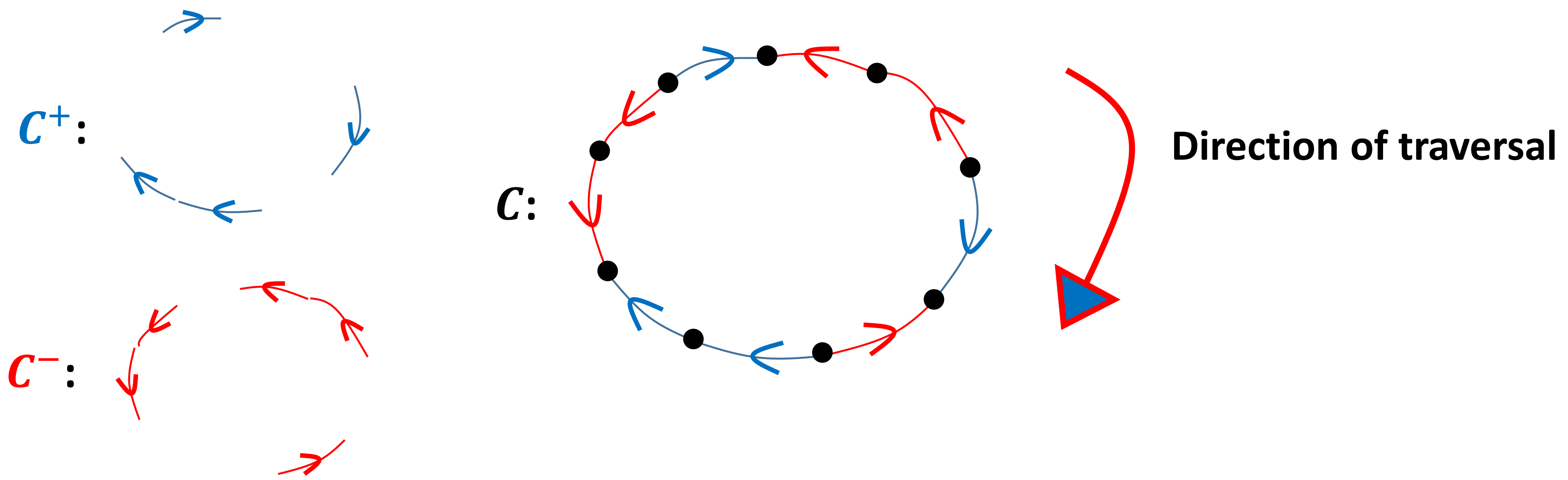
- Basic Definitions
- Some Results
- Xuding Zhu’s Result

Basic Definitions



- A graph G is a couple $(V(G), E(G))$ where $V(G)$ is a finite, non empty set of elements, called vertices of G, and $E(G)$ is set of pairs of vertices, called edges of G.
- A digraph D is a couple $(V(D), E(D))$ where $V(D)$ is a finite nonempty set of elements, called vertices of D, and $E(D)$ is subset of the set of ordered pairs of distinct elements of $V(D)$, called arcs of D .
- For a digraph $D = (V(D), E(D))$, the **Underlying Graph** of D, denoted by $G[D]$, is the undirected graph created using all of the vertices in $V(D)$, and replacing all arcs in $E(D)$ with undirected edges.
- An orientation of a graph $G = (V(G), E(G))$ is a digraph $D = (V(D), E(D))$ such that $G[D] = G$.

Basic Definitions



For a cycle C of D with a chosen direction of traversal (each cycle has two different directions for traversal), let C^+ be the set of positive edges of C (i.e., whose direction coincide with the direction of the traversal) and let C^- be the set of negative edges of C.

The parameter $\tau(C) = \max\{\frac{|C|}{|C^+|}; \frac{|C|}{|C^-|}\}$ measure the "**imbalance**" of the cycle C.

Let $\xi(D) = \max\{\tau(C)$: C is a cycle of $D\}$. Then $\xi(D)$ is a measure of the imbalance of the orientation D of G.

Basic Definitions

For any integer n; $n\ (mod\ k)$ denotes the unique integer n' such $that\ 0 \leq n' \leq k-1$ and $n \equiv\ n'(mod\ k)$.

Suppose $G = (V, E)$ is a graph and $1 \leq 2d \leq k$ are integers. A (k, d)-coloring of G is a mapping $f : V \rightarrow \{0,1, \ldots, k-1\}$ such that for every edge xy of G:

$$d \leq |f(x) - f(y)| \leq k - d$$

The circular chromatic number $\chi_c(G)$ of G is defined as:

$$\chi_c(G) = \min\{\frac{k}{d} : \text{there exists a } (k, d)\text{-coloring of } G\}.$$

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Some Results

- Theorem 1: (Minty [2]) For any finite graph G,

$$\chi(G) = \min\{\lceil \xi(D) \rceil : D \text{ is an orientation of } G\}$$

The non-trivial direction of Minty's result asserts that if G has an orientation D with $\xi(D) \leq k$ where k is an integer then G is k-colorable.

Tuza's result says that to obtain the same conclusion, instead of requiring $\xi(D) \leq k$ which is equivalent to require that $\tau(C) \leq k$ for every cycle C, it suffices to require that $\tau(C) \leq k$ for those cycles C with $|C|(mod\ k) = 1$.

Goddyn, Tarsi and Zhang's result says that if $\xi(D) \leq \frac{k}{d}$ for some fraction $\frac{k}{d}$ then G is (k, d)-colorable.

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Xuding Zhu's Result

Xuding Zhu in his paper proves that if a graph $\boldsymbol{G}$ has an orientation $\boldsymbol{D}$ such that for each cycle $\boldsymbol{C}$ with $\mathbf{d}.|\mathbf{C}|(\mathbf{mod}\ \boldsymbol{k}) \in \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{2}\boldsymbol{d}\ - \mathbf{1}\}$ we have $\frac{|\boldsymbol{C}|}{|\boldsymbol{C}^+|} \leq \frac{\boldsymbol{k}}{\boldsymbol{d}}$ and $\frac{|\boldsymbol{C}|}{|\boldsymbol{C}^-|} \leq \frac{\boldsymbol{k}}{\boldsymbol{d}}$, then $\boldsymbol{G}$ has a $(\boldsymbol{k}, \boldsymbol{d})$-colouring, and hence $\boldsymbol{\chi_c}(\boldsymbol{G}) \leq \frac{\boldsymbol{k}}{\boldsymbol{d}}$.

This result is a generalization of Tuza's result (if $\boldsymbol{d} = \mathbf{1}$) and is an improvement of Goddyn et al's result.

Proof

The proof of this result is parallel to Tuza's proof of that special case.
Let r be a fixed vertex of D. Each spanning tree T of D is considered as rooted at r.
Given such a spanning tree T, we define the weight $w_T\ (x)$ of a vertex x of D
(with respect to T) recursively as follows:

- $w_T\ (r) = 0$;
- If xy is an edge of T oriented from x to y and $w_T\ (x)$ has already been defined, then $w_T(y) = w_T\ (x) - k + d$;
- If xy is an edge of T oriented from x to y and $w_T\ (y)$ has already been defined, then $w_T(x) = w_T\ (y) + d$;

Since T is a spanning tree, for each vertex x of D; $w_T\ (x)$ is uniquely defined. Then we define the weight $w(T)$ of T as

$$\boldsymbol{w(T)} = \sum_{\boldsymbol{x \in V(D)}} \boldsymbol{W_T(x)}$$

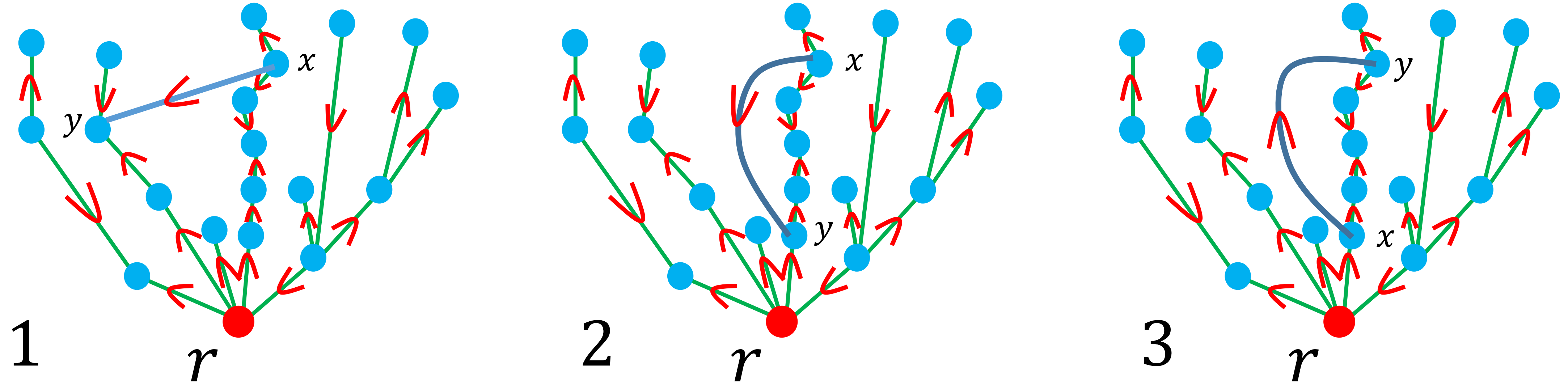
Choose a rooted spanning tree T of D which has the maximum $weight$. Let
$f : V \rightarrow \{0,1,\ldots,k-1\}$ be defined as $f(x) = w_T\ (x)\ mod\ k$: We shall show that f is a (k,d)-colouring of G.

Let $xy \in E(G)$. Without loss of generality we assume that the edge xy is oriented from x to y in D.

We will study three different cases:

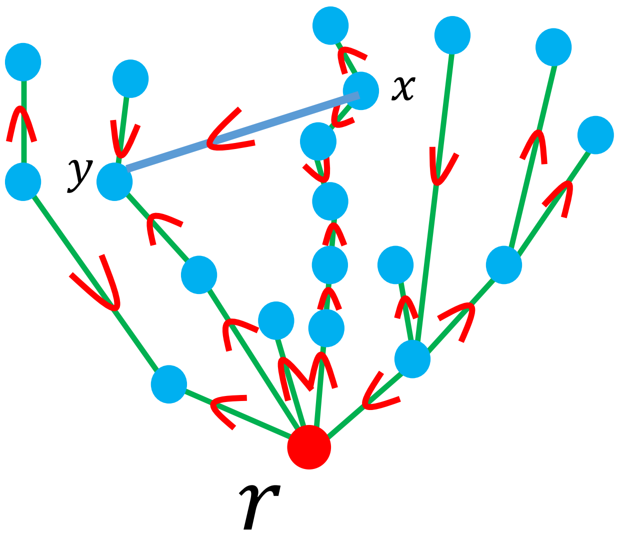
- If x is not on the y–r-path of T and y is not on the x–r-path of T;
- If y is on the x–r-path of T;
- If x is on the y–r-path of T.

T:



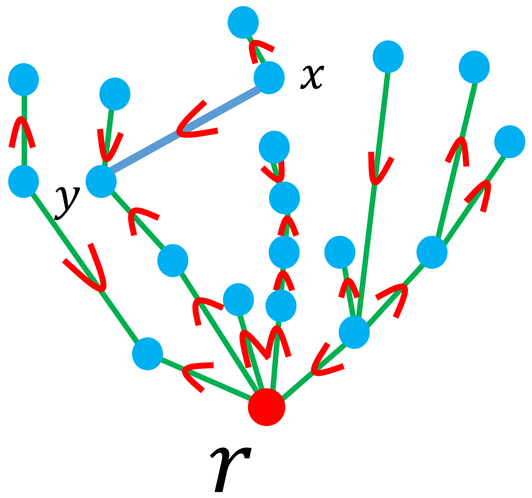
Case 1:

T:



If $w_T(x) - w_T\,(y) < d$; then we delete the edge of T connecting x to its father, and add the edge xy

T':



$$w_{T'}\,(x) = w_{T'}(y) + d = w_T(y) + \mathrm{d} > w_T(x)$$

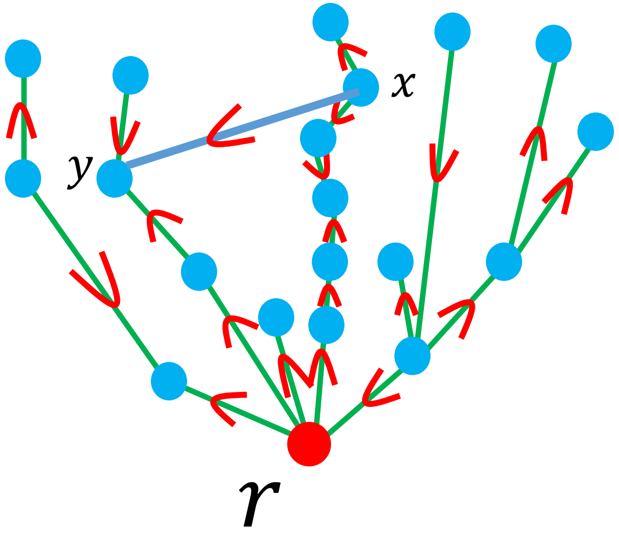
Then we obtain a spanning tree $T\,'$ for which $w_{T'}(v) \geq w_T(v)$ for each v; and $w_{T'}\,(v)\ >\ w_T\,(v)$ for every descendents of x; including x itself. So $w(T') >\ w(T\ )$, contrary to our choice of T . Thus $w_T\,(x) - w_T\,(y) \geq d$.

Similarly we can prove taht $w_T\,(x) - w_T\,(y) \leq k - d$.

Those, $d \leq w_T(x) - w_T\,(y) \leq k - d$.

Case 1:

T:

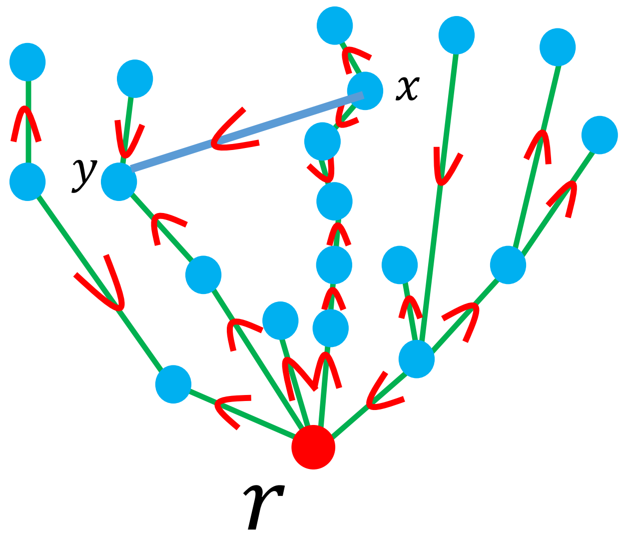


Those, $d \leq w_T(x) - w_T\ (y) \leq k - d$.

which implies that $d \leq |f(x) - f\ (y)| \leq k\ - d$ (as $f(x) - f\ (y) =\ w_T\ (x) - w_T\ (y)\ mod\ k$), a contradiction.

Case 1:

T:



Those, $d \leq w_T(x) - w_T\ (y) \leq k - d$.

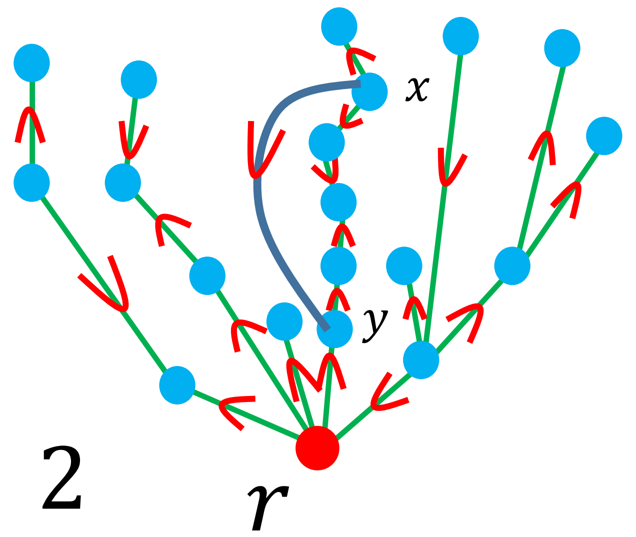
Then , $d \leq w_T(x) - w_T\ (y)(modk) \leq k - d$, and so:

$$\boldsymbol{d \leq |f(x) - f(y)| \leq k - d}$$

As $f(x) - f\ (y) = \ w_T\ (x) - w_T\ (y)\ mod\ k$.

Case 2:

T:



- If y is on the x–r-path of T:

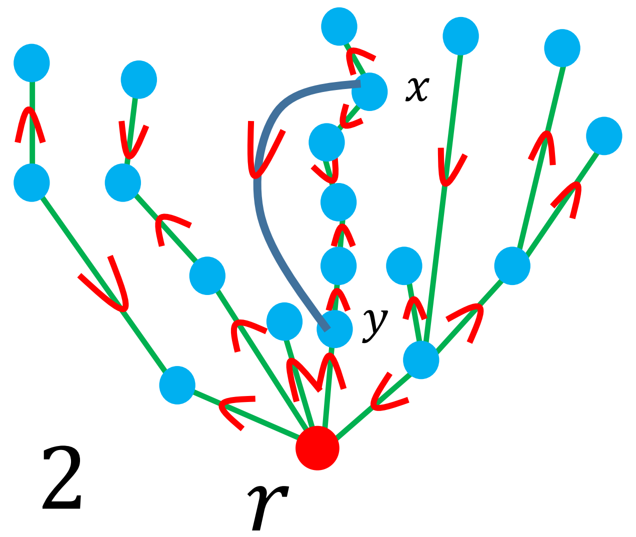
Assume $w_T\,(x) - w_T\,(y) = \; ak + \; j$ for some integers a; j such that $0 \leq j \leq k \; - 1$:

Then, $f(x) - f(y) = j$,

And hence $j \in \; \{0,..,d-1\} \cup \; \{k-d+1,..,k-1\}$.

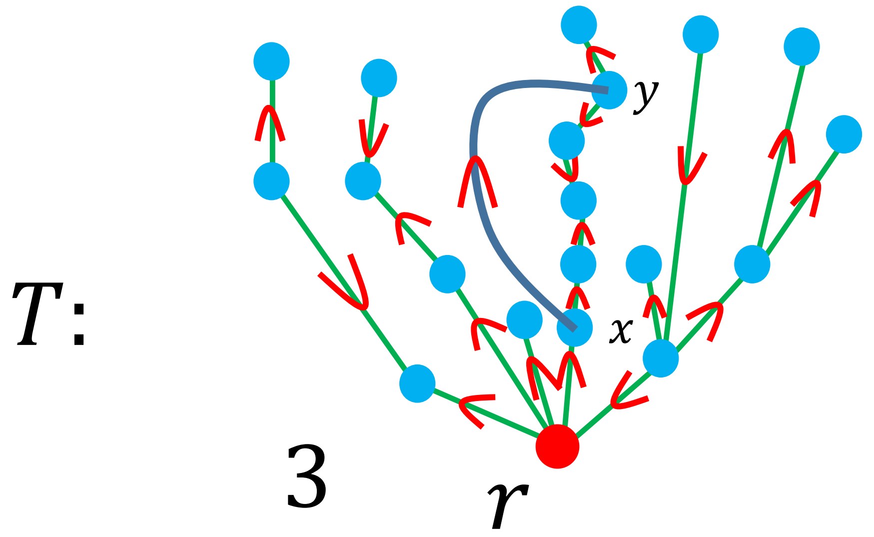
Case 2:

T:



Let p be the number of edges on the $x - y$-path of T oriented toward the root, and let q be the number of edges on this path oriented away from the root. By the definition of the weight, we know that $w_T\,(x) - w_T\,(y) = pd - q(k - d)$.Therefore, we have $(pd + qd)(mod\ k) = j;\ and\ (p + q + 1).\,d(mod\ k) = j + d(\,mod\ k) \in \{1,2,\ldots,2d - 1\}$ note that the cycle C consisting the $x - y$ −path and the edge xy is a cycle of length $|C| = p + q + 1$: Hence $d|C|(modk\ ) \in \{1,2,\ldots,2d - 1\}$. By our assumption, $\frac{|C|}{|C^-|} \leq \frac{k}{d}$; which implies $\frac{|C^+|}{|C^-|} \leq \frac{(k-d)}{d}$ here we choose the direction of traversal $of\ C$ so that those edge on the $x - y$ −path oriented towards r belongs to C^+: Hence $|C^+| = p$ and $|C^-| = q + 1$; which implies that $pd - q(k - d) \leq k - d$: Hence $w_T\,(x) - w_T\,(y) = pd - q(k - d) \leq k - d$. **Therefore,** $d \leq w_T\,(x) - w_T\,(y) \leq k - d$, **a contradiction.**

Case 3:



Similarly to the Case 3.

Thank you