Multiplicative graphs

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Outline

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In particular, K_3 is multiplicative. [El-Zahar, Sauer, 1985]

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$f : V(C) \to V(K)$ (not necessarily a homomorphism).

- A *fixed point* of f is a vertex v_i such that $f(v_{i-1}), f(v_i), f(v_{i+1})$ are three consecutive vertices of K.
- f is odd or even if f has an odd number or an even number of fixed points.

Odd cycles are multiplicative

Assume G, H are connected graphs,
C, D are odd cycles of G, H, respectively.
$f : (G \times D) \cup (C \times H) \to K$ is a homomorphism.
We need to show that $G \to K$ or $H \to K$.

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For $v \in V(D)$, $f_v : V(G) \to V(K)$ is defined as $f_v(x) = f(x, v)$.

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Assume $C = (v_1, v_2, \ldots, v_p)$.
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So either

$$f_v(v_i) = f_v(v_{i+2})$$

or

$f_v(v_i), f_v(v_{i+2})$ have distance 2 (if K is not a triangle).

- $vv' \in E(H)$ implies that the restrictions of f_v and $f_{v'}$ to C have the same parity,
- For $v \in V(D), u \in V(C)$, f_u, f_v have different parities.

Assume for $v \in V(D)$ and $u \in V(C)$,

- f_u is odd.
- f_v is even.

Since G and H are connected, for any odd cycle C' of G, for any vertex u of D, the restriction of f_u to C' is odd,
for any vertex v of C, for any cycle D' of H, the restriction of f_v to D' is even.

Let u be a vertex of D and let

$$X = \{v \in V(G) : \exists vv' \in E(G), f(v,u)f(v',u) \notin E(K)\}.$$

Lemma

X contains no odd cycle.

Assume X contains an odd cycle C'.

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So $f(v_i, u') f(v_i', u) \notin E(K)$, but $(v_i, u')(v_i', u) \in E(G \times D)$.

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If $xx' \in E(G)$, then either $h(x) = f(x,u) \neq f(x',u') = h(x')$, or $h(x) = f(x,u), h(x') = f(x',u), x \notin X$ and hence

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$$f(x,u)f(x',u) \in E(K).$$

So h is a homomorphism from G to K. Therefore K is strongly multiplicative.

Theorem (Tardif (2005))

For any positive integers $2q \leq p < 4q$, $K_{p;q}$ *is multiplicative.*

Graph operations

- $\Gamma_{2k+1}(G)$ (also denoted by $P_{2k+1}(G)$):
  - Vertex set $V(G)$.
  - Join two vertices of G by an edge if they are connected by a walk of length $2k+1$.
- $\Omega_{2k+1}(G)$ (also denoted by $P^{-1}_{2k+1}(G)$):
  - Vertices:

$$\{(A_0, A_1, \ldots, A_k) : A_i \subseteq V(G), |A_0| = 1, A_1 \neq \emptyset, A_i \bowtie A_{i+1}\}.$$

  - Edges: $A = (A_0, A_1, \ldots, A_k)$ adjacent to $B = (B_0, B_1, \ldots, B_k)$ if

$$A_i \subseteq B_{i+1}, B_i \subseteq A_{i+1}, A_k \bowtie B_k.$$

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6. $\Omega_k(G \times H) \to \Omega_k(G) \times \Omega_k(H)$ *and* $\Omega_k(G) \times \Omega_k(H) \to \Omega_k(G \times H)$.

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$(\Gamma_k(G \times D)) \cup (\Gamma_k(C \times H))$ is a subgraph of $\Gamma_k((G \times D) \cup (C \times H))$.

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$\Omega_3(\Gamma_3(K_{p;q}))$ and $K_{p;q}$ are homomorphically equivalent.

Thus for $p/q < 12/5$, if $K_{p;q}$ is strongly mulplticative, then $\Gamma_3(K_{p;q}) = K_{k;3q-p}$ is strongly multplicative.

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Obtain a dense sequence

$$P = \{\frac{n+1}{n-(3^i-1)/2} : n \geq 1, 0 \leq i \leq \log_3 n\}$$

$K_{p;q}$ are strongly multiplicative for $p/q \in P$.

Theorem (Wrochna (2017))

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The graph $(G \times D) \cup (C \times H)$ has many 4-cycles.

Then one of the following holds:

- All cycles in $\phi((G \times D) \cup (C \times H))$ winds around a same closed walk C^* in K.
- For $u \in V(D)$, under the projection ϕ_u, all cycles in G are mapped to trivial closed walks in K, i.e., can be reduced to the empty walk by recursively removing backtracking arcs of the form $(u, v)(v, u)$. Or symmetrically, for $v \in V(C)$, under the projection ϕ_v, all cycles in H are mapped to trivial closed walks in K.

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As a consequence of Theorem 11, graphs of girth at least 5 are strongly multiplicative. It was further proved in [**?**] that if K has girth at least 13, then $\Omega_3(\Gamma_3(K))$ and K are homorphically equivalent. Then the same argument as above shows that for graphs K of girth at least 13, $\Gamma_3(K)$ are strongly multiplicative.

Theorem (Wrochna and Tardif)

If every edge of K is contained in at most one C_4, then K is strongly multiplicative.