signed graph: A graph where each edge is assigned a sign
+ positive  - negative

Notation: \((G, \sigma)\)
\(G\) \(\rightarrow\) signature

Examples:

\((K_{4; -})\)
\((K_{5; -})\)
Main terminology:

- **positive cycle**  
  (balanced cycle)

- **negative cycle**  
  (unbalanced cycle)

    reflecting the fact that the rule of  
    "friend of a friend is a friend"  
    applies.

POSITIVE / NEGATIVE closed walk

Balanced signed graph: signed graph with no negative cycle

Antibalanced signed graph: where odd cycles are negative  
 even cycles are positive.
Switching (what makes it different from 2-edge-colored graphs).

To multiply signs of edges incident to a vertex to -

Switching at all vertices of a subset $X$ of vertices:
To multiply signs of all edges in the edge cut $(X, V-X)$ to a -
Parallel terminologies

Resigning  switching
odd  negative
odd cycle  negative cycle
bipartite  balanced

\((G, \Sigma)\) in place of \((G, \delta)\) where \(\Sigma\) is the set of negative edges

(i.e. \(\Sigma = \delta^{-1}(\neg)\))

\(\Sigma = (G, \delta)\), \(\hat{G} := (G, \delta')\)
From classic graph theory

- chromatic number
- fractional chromatic number
- circular chromatic number
- edge-chromatic number
- fractional edge-chromatic number

- minor
- homomorphism
Motivating results and conjectures
to study coloring and homomorphisms of signed graphs:

The four-color theorem: every planar (simple) graph can be
[Conjectured: Guthrie, 1852] 

Reformulation of it: Every bridgeless cubic planar (multi)graph
[Tait, 1890] can be 3-edge-colored.

Hadwiger’s conjecture: Every graph with no $K_k$-minor is (k-1)-colorable.

Jaeger conjecture: Every f(k)-connected graph admits a circular (2k+1)-flow.
(dual in planar case) Every planar graph of girth at least f(k) maps to $C_{2k+1}$

Original conjecture $f(k) = 4k$ works (disproved). [Han, Li, Wu, Zhang, 2018]
planar case: $f(k) = 4k$ remains open.
odd-girth $4k+1$ instead of girth is proposed by C.Q. Zhang.

Best result so far: $f(K) = 6k$ works.
[Lovász, Thomassen, Wu, Zhang, 2013]
How may theory of signed graphs help?

1. Stronger results

Theorem. Every $K_4$-minor-free graph is 3-colorable.

- easy to verify
- NP-hard for general graphs

how to generalize so to include bipartite graphs?
How may theory of signed graphs help?

2. Fill the gap in theories.

Example. $T_{2k-1}(G)$: obtained from $G$ by replacing each edge with a path of length $2k-1$.

\[
\text{Theorem. } \chi(G) \leq 2K+1 \iff T_{2k-1}(G) \rightarrow C_{2^{1+1}}\]

[indicator construction, Hell & Nesetril]

Question. How to capture $2k$-coloring?
How may theory of signed graphs help?

3. Developing proof techniques that are not possible for graphs.

We plan to present two such examples in these lectures.
Homework. Given a graph $G$ how many non-equivalent signatures we have on $G$?

Hint. Number of connected components is important.
Question. Given two signatures $\sigma_1$ & $\sigma_2$ on a graph $G$, how can we decide if $(G, \sigma_1) \equiv (G, \sigma_2)$?

A "No" answer: if a cycle $C$ is positive in one and negative in the other.
Question. Given two signatures $\sigma_1$ & $\sigma_2$ on a graph $G$, how can we decide if $(G,\sigma_1) = (G,\sigma_2)$?

A "NO" answer: if a cycle $C$ is positive in one and negative in the other.

A "YES" answer: otherwise, i.e. when every cycle has a same sign in both signatures.

Question: Do we need to check all the cycles?
Observation.

Any two signatures on a tree are equivalent.
T: spanning tree of G

T in \((G, \sigma_1)\)

T in \((G, \sigma_2)\)
Harary. \((G, \sigma)\) is switch-equivalent to \((G, +)\) if and only if it is balanced. no negative cycle

Zaslavsky. \((G, \sigma_1)\) and \((G, \sigma_2)\) are switch-equivalent if and only if every cycle of \(G\) has a same sign in both.
Two theories to develop on singlyed graphs

- theory of minor (a brief mention)

- coloring and homomorphism (our main focus)
A minor of $(G, \emptyset)$ is obtained by

- deleting *(vertices or edges)*
- contracting a positive edge
- switching
A minor of \((G, \varepsilon)\) is obtained by

- **deleting** (vertices or edges) can kill a cycle but would not change its sign
- **contracting a positive edge** does not change sign of cycle
- **switching** also does not
Homework. Determine the class of signed graphs which do not have a $K_3$-minor.

Which members of the previous class can be switched to all negative edges?

What can you say about the chromatic number of the graphs in this subclass?
An odd $K_4$:

A subdivision of a (plane) $K_4$ where each face is an odd cycle.

An example
Homework. Show that a graph $G$ has no odd $K_4$ (as subgraph) if and only if $(G, -)$ has no $(K_4, -)$-minor.

Theorem (Catlin). Graphs with no odd $K_4$ are 3-colorable.

Restate the theorem in the language of signed graphs.

Propose a generalization.

Find a statement of "odd Hadwiger conjecture" and compare it to your generalization.
\((H, \pi)\)-minor problem

Input: A signed graph \((G, \sigma)\).

Output:  
- YES if \((H, \pi)\) is a minor of \((G, \sigma)\).
- No otherwise.

Polynomial time solvable as shown in PhD. thesis of Tony Chi Thong Huynh
Graph-minor theorem (by Seymour & Robertson)

In any infinite set of graphs there are two graphs of which one is a minor of the other.

Extension to special classes of Matroids, which includes signed graphs, is announced in 2011, by J. Geelen, B. Gerards, G. Whittle.
Homomorphism & Homeomorphism

Given two structures of same type a mapping of ground elements of one to the other where the main structures are preserved.

Example.  

Group  
Homomorphism  
$(\Gamma_1, +) \xrightarrow{f} (\Gamma_2, *)$  
$f(x+y)=f(x)*f(y)$

Topology  
Homeomorphism  
$(T_1, O_1) \xrightarrow{f} (T_2, O_2)$  
$A \in O_1 \implies f(A) \in O_2$
Homomorphisms of graphs

\[ G \rightarrow H \]

\[ f: \ V(G) \rightarrow V(H) \]

\[ x \sim y \Rightarrow f(x) \sim f(y) \]

Homework. \( x(G) = \) smallest number of vertices of a homomorphic image of \( G \) where there exists no loop.
Homomorphisms of signed graphs:

Before formulating a definition must decide what are the main structures.

Vertices form the ground sets. Edges are main part of the structure.

- View 1. Sign of edges are part of the main structure.

- View 2. Signs of cycles and closed walks are part of the main structure.
  
  (in this view two switch equivalent signed graphs are regarded to be identical).

View 1 leads to the notion of homomorphisms of 2-edge-colored graphs

\((-\text{red}, +\text{blue})\)

(studied since 1980's)

Our main interest is based on the View 2, but there is a strong connection to View 1.
Homomorphisms of signed graphs.

Definition. Given signed graphs $(G, \sigma)$ & $(H, \pi)$ a mapping of $V(G)$ to $V(H)$ (and $E(G)$ to $E(H)$) is said to be a **homomorphism** of $(G, \sigma)$ to $(H, \pi)$ if it preserves **adjacencies**, (incidences) and signs of closed walks.

It is said to be **edge-sign preserving homomorphism** if it furthermore, preserves signs of edges.

**Examples**
Comment. The edge mapping is implied unless \((H, \pi)\) contains a digon.

Theorem. Signed graph \((G, \sigma)\) admits a homomorphism to signed graph \((H, \pi)\) if and only if for some switching \((G, \sigma')\) there exists an edge-sign preserving homomorphism of \((G, \sigma')\) to \((H, \pi)\).
Important note:

Notions of "isomorphism" and "automorphism" depend on our view and choice of homomorphism: edge-sign preserving homomorphism or switch homomorphism. Associated definitions like "vertex transitive" and "edge-transitive" change accordingly.

under edge-sign preserving homomorphism

the only automorphism:

\[ a \Leftrightarrow d \]
\[ b \Leftrightarrow c \]

Dihedral group \( D_4 \) (8 elements)

Thus \( C_4 \) is:

vertex-transitive & edge-transitive.
Definition.

- Core of a signed graph \((G, e)\) is a minimal subgraph of \((G, e)\) to which \((G, e)\) admits a homomorphism.

- A core is a signed graph which is its own core.

Homework.

Any two cores of a signed graph are isomorphic.
(with respect to both notions of homomorphisms).
Homework.

How many non isomorphic signed graphs we can build on the Petersen graph?
Observation. In a mapping of \((G, \sigma)\) to \((H, \pi)\) the image of every closed walk is a closed walk which has a same (parity) of length and a same sign.

This leads to four notions of girth:

\[ g_{00}(G, \sigma) \] length of shortest positive even closed walk,

\[ g_{10}(G, \sigma) \] length of shortest negative even closed walk,

\[ g_{01}(G, \sigma) \] length of shortest positive odd closed walk,

\[ g_{11}(G, \sigma) \] length of shortest negative odd closed walk.

Example.

\[ g_{00}(G, \sigma) = 2, \quad g_{10}(G, \sigma) = 4, \quad g_{01}(G, \sigma) = 3, \quad g_{11}(G, \sigma) = 3 \]
The main no homomorphism Lemma.

If \((G, \sigma) \rightarrow (H, \pi)\) then,

\[ g_{ij}(G, \sigma) \geq g_{ij}(H, \pi) \]

for every \(i, j \in \mathbb{Z}_2\).
A main question:

When do the conditions of the no-homomorphism Lemma (or similar but stronger conditions) become sufficient?
Example [the four-color theorem]

For $(H, \pi) = (K_4, -)$ and all planar signed graphs.
Example [Grotzsch's theorem]

For \((H, \pi) = (K_3, -)\) and all planar signed graphs satisfying stronger condition of

\[ g_{ij}(G, \sigma) \geq g_{ij}(H, \pi) + 2. \]