signed graph: A graph where each edge is assigned a sign

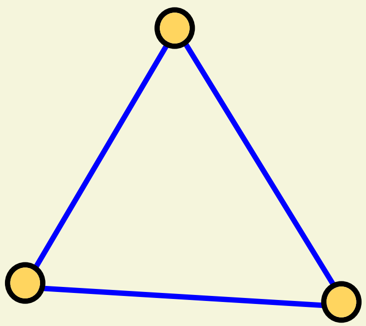
+ positive   - negative

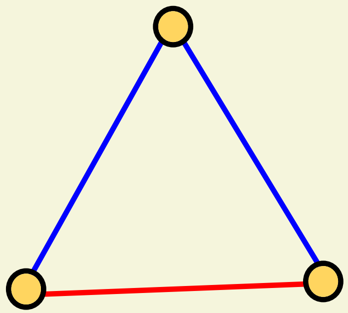
Notation: (G, σ)

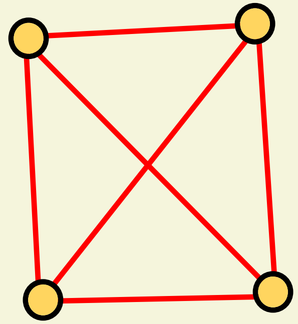
signature

graph

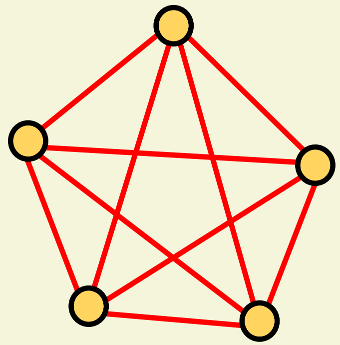
Examples:







$(K_4, -)$



$(K_5, -)$

Main terminology: positive cycle negative cycle

(balanced cycle) (unbalanced cycle)

reflecting the fact that the rule of "friend of a friend is a friend" applies.

positive/negative closed walk

Balanced signed graph: signed graph with no negative cycle

Antibalanced signed graph: where odd cycles are negative
even cycles are positive.

Switching (what makes it different from 2-edge-colored graphs).

To multiply signs of edges incident to a vertex to $-$

Switching at all vertices of a subset X of vertices:

To multiply signs of all edges in the edge cut $(X, V-X)$ to a $-$

Parallel terminologies

~~Resigning~~ switching

~~odd~~ negative

~~odd cycle~~ negative cycle

~~bipartite~~ balanced

(G, Σ) in place of (G, σ) where Σ is the set of negative edges

(i.e. $\Sigma = \sigma^{-1}(-)$)

$\cancel{\Sigma = (G, \sigma)}$, $\hat{G} := (G, \sigma)$

From classic graph theory

- chromatic number
- fractional chromatic number
- circular chromatic number

- edge-chromatic number
- fractional edge-chromatic number

- minor
- homomorphism

Motivating results and conjectures

to study coloring and homomorphisms of signed graphs:

The four-color theorem: every planar (simple) graph can be properly 4-colored.

[Conjectured: Guthrie, 1852]
[Proved: Appel, Haken, 1976]

Reformulation of it: Every bridgeless cubic planar (multi) graph can be 3-edge-colored.

[Tait, 1890]

Hadwiger's conjecture: Every graph with no K_k-minor is $(k-1)$-colorable.

Jaeger conjecture: Every $f(k)$-connected graph admits a circular $(2k+1)$-flow.

(dual in planar case) Every planar graph of girth at least $f(k)$ maps to C_{2k+1}

Original conjecture $f(k)=4k$ works (disproved). [Han, Li, Wu, Zhang, 2018]

planar case: $f(k)=4k$ remains open.

odd-girth $4k+1$ instead of girth is proposed by C.Q. Zhang.

Best result so far: $f(k)=6k$ works.

[Lovasz, Thomassen, Wu, Zhang, 2013]

How may theory of signed graphs help?

1. Stronger results

Theorem. Every K_4-minor-free graph is 3-colorable.

easy to verify

NP-hard for general graphs

how to generalize so to include bipartite graphs?

How may theory of signed graphs help?

2. Fill the gap in theories.

Example. $T_{2k-1}(G)$: obtained from G by replacing each edge with a path of length $2k-1$.

Theorem. $\chi(G) \le 2k+1 \iff T_{2k-1}(G) \to C_{2k+1}$

[indicator construction, Hell & Nesetril]

Question. How to capture 2k-coloring?

How may theory of signed graphs help?

3. Developing proof techniques that are not possible for graphs.

We plan to present two such examples in these lectures.

Homework. Given a graph G how many non-equivalent signatures we have on G?

Hint. Number of connected components is important.

Question. Given two signatures σ_1 & σ_2 on a graph G how can we decide if $(G,\sigma_1)\equiv(G,\sigma_2)$?

A "NO" answer: if a cycle C is positive in one and negative in the other.

Question. Given two signatures σ_1 & σ_2 on a graph G how can we decide if $(G,\sigma_1) \equiv (G,\sigma_2)$?

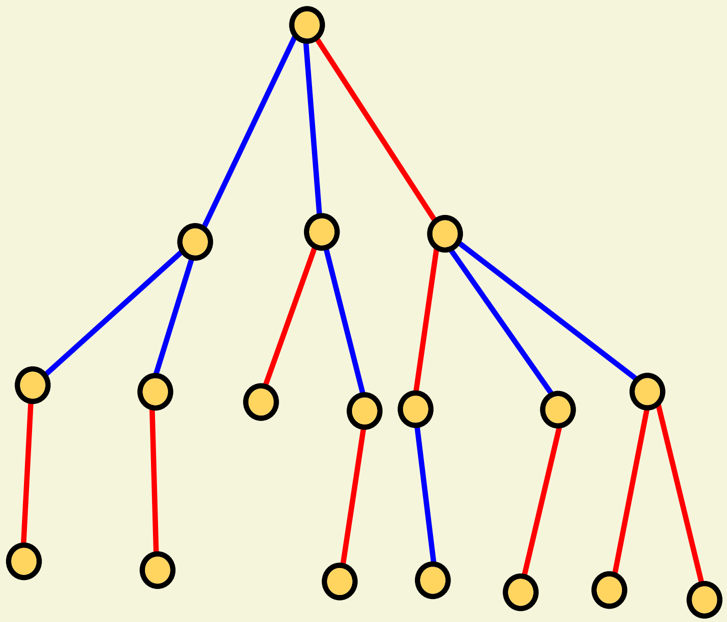
A "NO" answer: if a cycle C is positive in one and negative in the other.

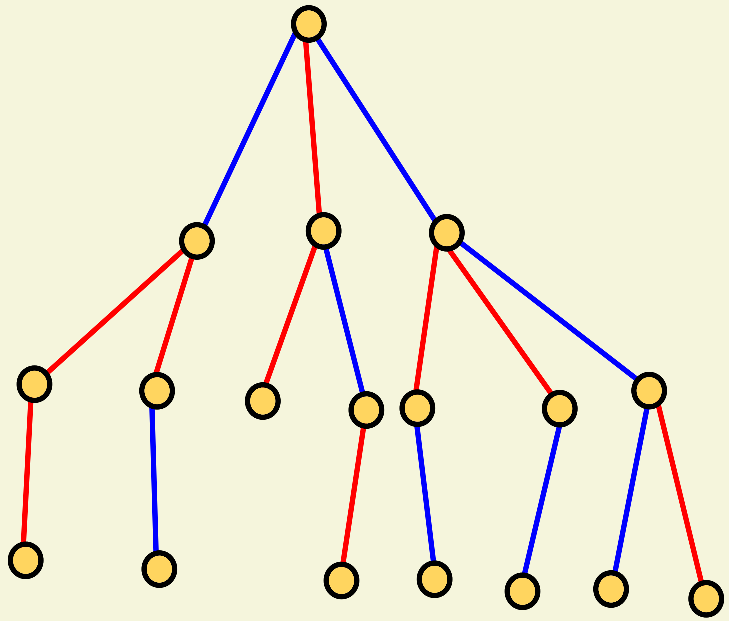
A "YES" answer: otherwise, ie. when every cycle has a same signe in both signatures.

Question: Do we need to check all the cycles?

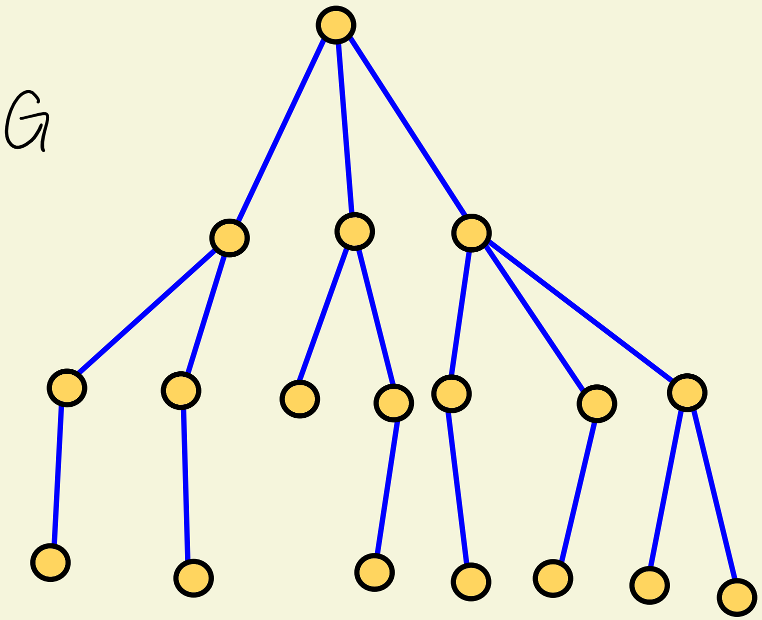
Observation.

Any two signatures on a tree are equivalent.

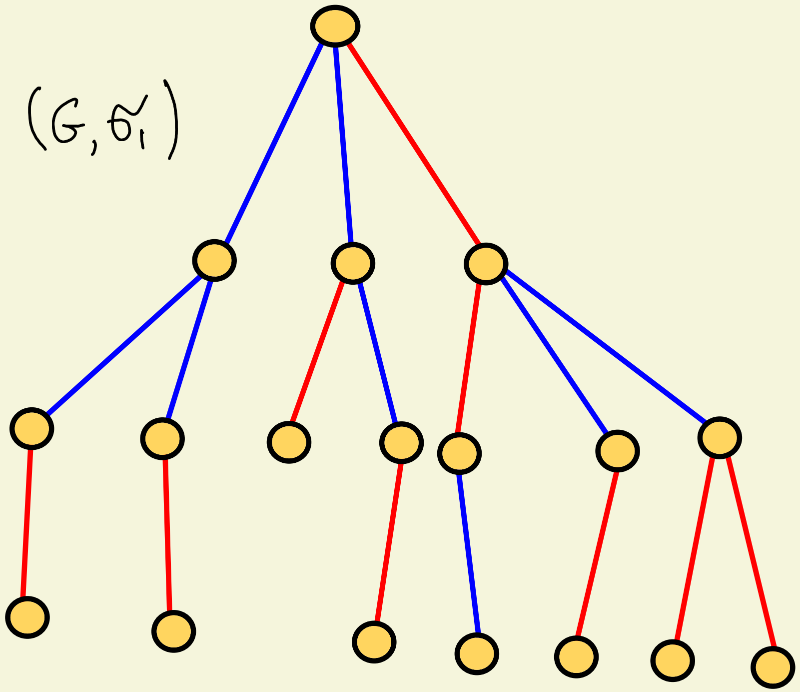




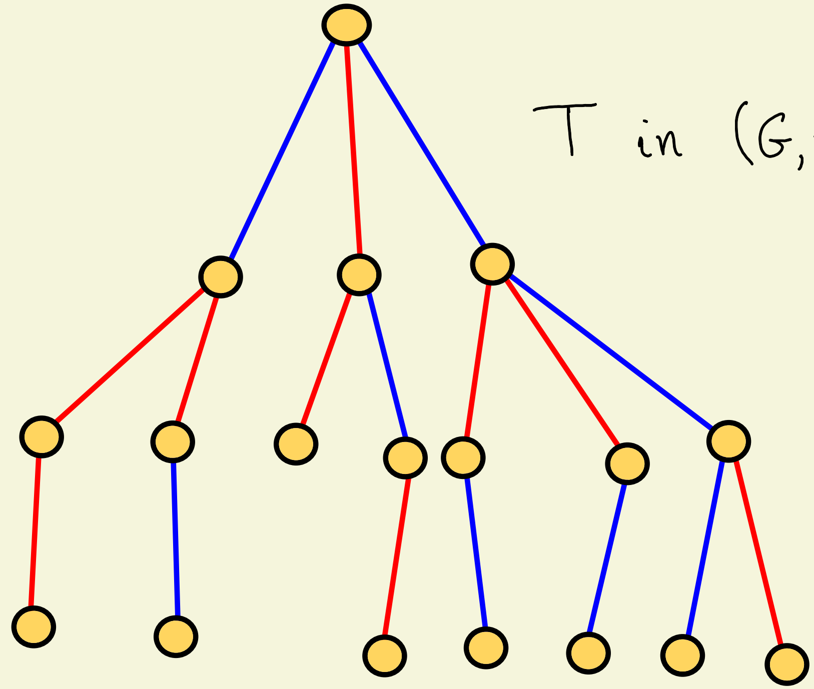
T: spanning tree of G



T in $(G, \tilde{\sigma}_1)$



T in $(G, \tilde{\sigma}_2)$



Harary. (G, σ) is switch-equivalent to $(G, +)$ if and only if it is balanced.

no negative cycle

Zaslavsky. (G, σ_1) and (G, σ_2) are switch-equivalent if and only if every cycle of G has a same sign in both.

Two theories to develop on singed graphs

- theory of minor (a brief mention)

- coloring and homomorphism (our main focus)

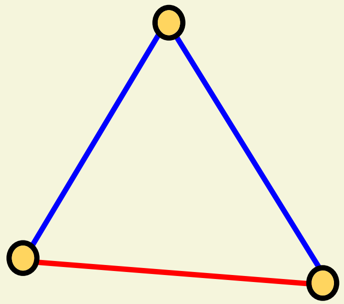
A minor of (G, σ) is obtained by

- deleting (vertices or edges)
- contracting a positive edge
- switching

A minor of (G, σ) is obtained by

- deleting (vertices or edges)
  - can kill a cycle but woud not change its sign
- contracting a positive edge
  - does not change sign of cycle
- switching
  - also does not

Homework. Determine the class of signed graphs which

do not have a  -minor.

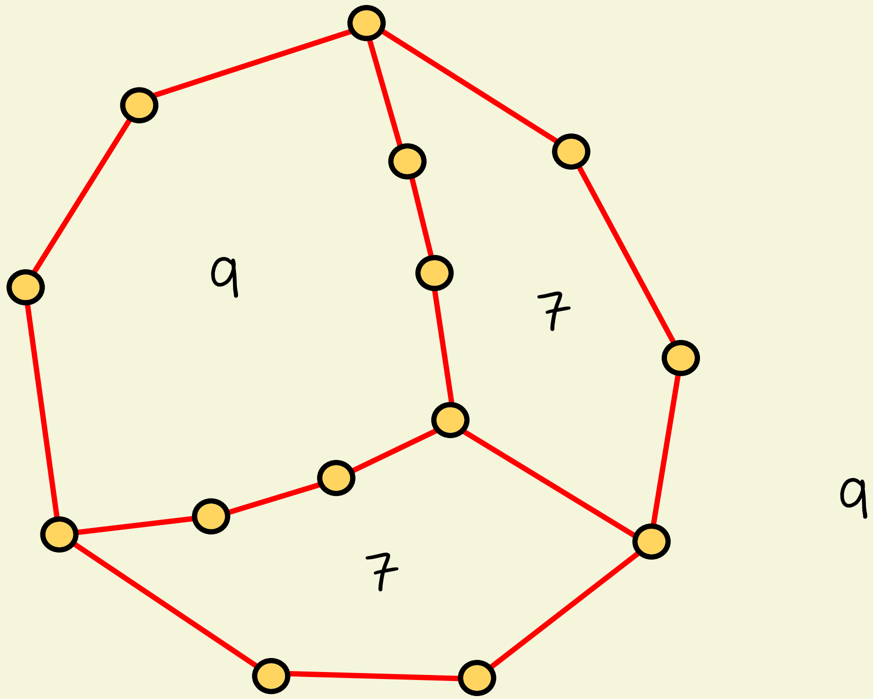
Which members of the previous class can be switched to all negative edges?

What can you say about the chromatic number of the graphs in this subclass?

An odd K_4:

A subdivision of a (plane) K_4 where each face is an odd cycle.

An example



Homework. Show that a graph G has no odd K_4 (as subgraph) if and only if $(G,-)$ has no $(K_4,-)$-minor.

Theorem (Catlin). Graphs with no odd K_4 are 3-colorable.

Restate the theorem in the language of signed graphs.

Propose a generalization.

Find a statement of "odd Hadwiger conjecture" and compare it to your generalization.

(H, π)-minor problem

Input: A signed graph (G, σ).

Output:
- YES if (H, π) is a minor of (G, σ).
- No otherwise.

Polynomial time solvable as shown in PhD. thesis of Tony Chi Thong Huynh

Graph-minor theorem (by Seymour & Robertson)

In any infinite set of graphs there are two graphs of which one is a minor of the other.

Extension to special classes of Matroids, which includes signed graphs, is announced in 2011, by J. Geelen, B. Gerards, G. Whittle.

Homomorphism & Homeomorphism

Given two structures of same type a mapping of ground elements of one to the other where the main structures are preserved.

Example.

Group

Homomorphism

$$(\Gamma_1, +) \xrightarrow{f} (\Gamma_2, *)$$

$$f(x+y) = f(x) * f(y)$$

Topology

Homeomorphism

$$(T_1, O_1) \xrightarrow{f} (T_2, O_2)$$

$$A \in O_1 \Longrightarrow f(A) \in O_2$$

Homomorphisms of graphs

$$G \to H$$

$$f: V(G) \to V(H)$$

$$x \sim y \Rightarrow f(x) \sim f(y)$$

Homework. $\chi(G)$ = smallest number of vertices of a homomorphic image of G where there exists no loop.

Homomorphisms of signed graphs:

Before formulating a definition must decide what are the main structures.

Vertices form the ground sets. Edges are main part of the structure.

- View 1. Sign of edges are part of the main structure.

- View2. Signs of cycles and closed walks are part of the main structure.

  (in this view two switch equivalent signed graphs are regarded to be identical).

View 1 leads to the notion of homomorphisms of 2-edge-colored graphs

(- red, + blue)

(studied since 1980's)

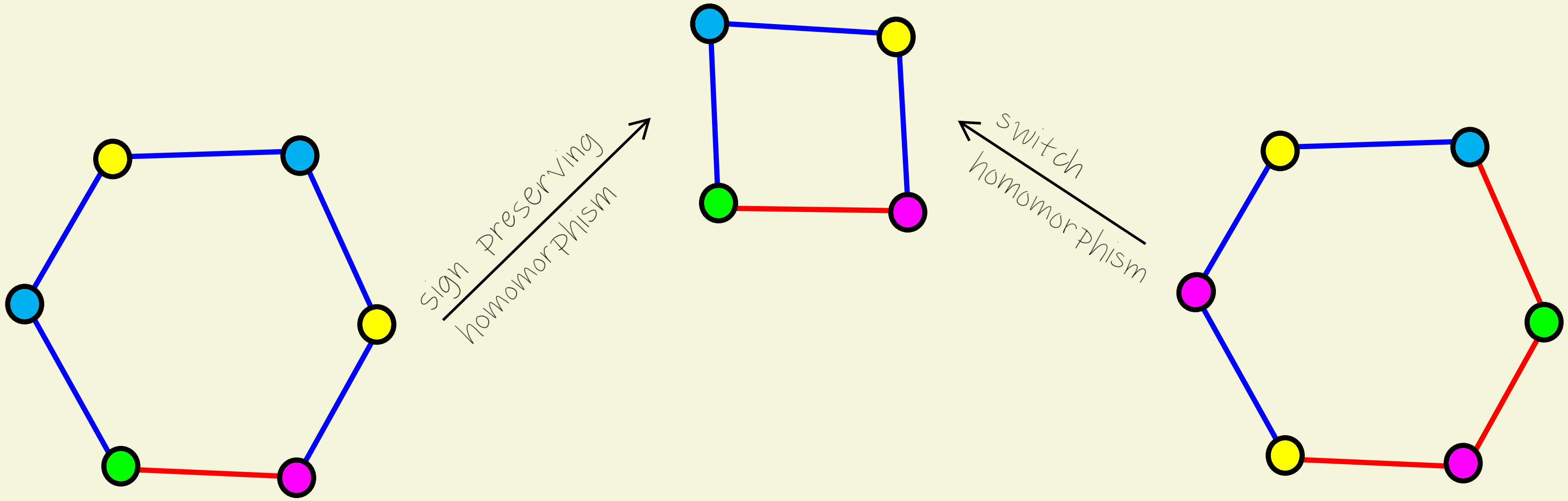
Our main interest is based on the View 2, but there is a strong connection to View 1.

Homomorphisms of signed graphs.

Definition. Given signed graphs (G, σ) & (H, π) a mapping of $V(G)$ to $V(H)$ (and $E(G)$ to $E(H)$) is said to be a homomorphism of (G, σ) to (H, π) if it preserves djacencies, (incidences) and signs of closed walks.

It is said to be edge-sign prserving homomorphism if it furthermore, preserves signs of edges.

Examples



Comment. The edge mapping is implied unless (H, π) contains a digon.

Theorem. Signed graph (G, σ) admits a homomorphism to signed graph (H, π) if and only if for some switching (G, σ') there exists an edge-sign preserving homomorphism of (G, σ') to (H, π).

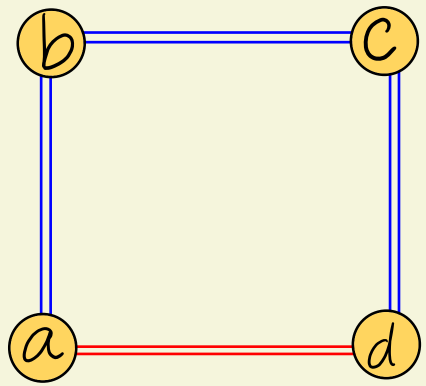
Important note:

Notions of "isomorphism" and "automorphism" depend on our view and choice of homomorphism: edge-sign preserving homomorphism or switch homomorphism. Associated definitions like "vertex transitive" and "edge-transitive" change accordingly.

under edge-sign preserving homomorphism

the only automorphism:

$$a \rightleftarrows d$$
$$b \rightleftarrows c$$



under switch homomorphism

Dihedral group D_4 (8 elements)

Thus C_4 is:

vertex-transitive &

edge-transitive.

Definition.

- Core of a signed graph (G, σ) is a minimal subgraph of (G, σ) to which (G, σ) admits a homomorphism.
- A core is a signed graph which is its own core.

Homework.

Any two cores of a signed graph are isomorphic.
(with respect to both notions of homomorphims).

Homework.*

How many non isomorphic signed graphs we can build on the Petersen graph?

Observation. In a mapping of (G,σ) to (H,π) the image of every closed walk is a closed walk which has a same (parity) of length and a same sign.

This leads to four notions of girth:

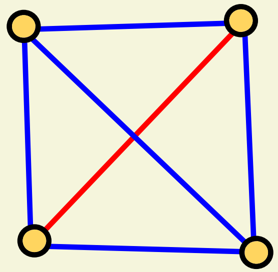
$g_{00}(G,\sigma)$: length of shortest positive even closed walk,

$g_{10}(G,\sigma)$: length of shortest negative even closed walk,

$g_{01}(G,\sigma)$: length of shortest positive odd closed walk,

$g_{11}(G,\sigma)$: length of shortest negative odd closed walk.

Example.



$g_{00}(G,\sigma)=2$, $g_{10}(G,\sigma)=4$, $g_{01}(G,\sigma)=3$, $g_{11}(G,\sigma)=3$

The main no homomorphism Lemma.

If $(G,\sigma) \longrightarrow (H,\pi)$ then,

$$g_{ij}(G,\sigma) \geq g_{ij}(H,\pi)$$

for every $ij \in \mathbb{Z}_2^2$

A main question:

When do the conditions of the no-homomorphism Lemma (or similar but stronger conditions) become sufficient?

Example [the four-color theorem]

For $(H, \pi) = (K_4, -)$ and all planar signed graphs.

Example [Grotzsch's theorem]

For $(H,\pi)=(K_3,-)$ and all planar signed graphs satisfying stronger condition of

$$g_{ij}(G,\sigma) \geq g_{ij}(H,\pi) + 2.$$