DC(G, 6): Double Cover of (G, 6)

vertices: two copies of $V(G)$ \[ V_L(G) \cup V_R(G) \]

edges: * If $ab$ is a positive edge, then
   \[ X_L \sim Y_L \quad \& \quad X_R \sim Y_R \]
   * If $ab$ is a negative edge, then
   \[ X_L \sim Y_R \quad \& \quad X_R \sim Y_L \]

Example

- Small graph
- Large graph
$(G, \phi)$

switch at a vertex $X$

$\text{DC}(G, \phi)$

switch the place of $X_L$ with $X_R$

Example
EDC(\(G, \phi\)): Extended Double Cover of \((G, \phi)\)

vertices: two copies of \(V(G)\) \(V_L(G) \cup V_R(G)\)

positive edges: * If an edge is a positive edge, then
\(X_L \sim Y_L \& X_R \sim Y_R\)

* If an edge is a negative edge, then
\(X_L \sim Y_R \& X_R \sim Y_L\)

negative edges: \(X_L \sim X_R\)
Example
Special examples
First definition of $\text{SPC}(d)$, the Signed Projective Cube of dimension $d$

$\text{SPC}(0)$: the projective cube of dimension 0.

$\text{SPC}(1)$: the projective cube of dimension 1.

$\text{SPC}(d) = \text{EDC}(\text{SPC}(d-1))$
Second definition. (where the name comes from)

\( H_d \): Hypercube of dimension \( d \)

\((z^d, e_1, e_2, e_3, \ldots, e_d^d)\)

\( \sigma_i \): - sign to the edges corresponding to \( E_i \) + sign to all other edges

\( SPC(d) \) is obtained from \( (H_{d+1}, \sigma_i) \) by identifying antipodal pairs of vertices.
Forth definition (as a Cayley signed graph)

\[ \text{SPC}(d) := (\mathbb{Z}_2^d, \{e_1, e_2, e_3, \ldots, e_d, J^3\}) \]

\( \{e_1, e_2, e_3, \ldots, e_d, J^3\} \) can be replaced with any basis as positive elements and their sum as the negative element.

Homework. Prove that all these definitions result in a same signed graph.
Properties of SPC(d):

* Its negative girth is d+1.
* Every negative cycle is of same parity as d+1.
* Every positive cycle is even.

Proofs are left as homework.
Theorem. A signed graph \((G, \sigma)\) maps to \(\text{SPC}(d)\) if and only if \(E(G)\) can be partitioned into \(E_1, E_2, \ldots, E_{d+1}\) such that \((G, E_i)\), for each \(i\), is switch equivalent to \((G, \sigma)\).

Proof
Theorem. A signed graph $(G, \delta)$ maps to $\text{SPC}(d)$ if and only if $E(G)$ can be partitioned into $E_1, E_2, \ldots, E_{d+1}$ such that $(G, E_i)$, for each $i$, is switch equivalent to $(G, \delta)$.

Corollary. A graph $G$ is 4-colorable if and only if $E(G)$ can be partitioned into 3 sets $E_1, E_2, E_3$ such that:

"each $E_i$ contains an odd number of edges of each odd cycle and an even number of edges of each even cycle."
Conjecture. Given a signed planar graph \((G, \phi)\), if \([\text{Naserasr, Guenin}]\)
\[ g_{ij}(G, \phi) \geq g_{ij}(\text{SPC}(d)) \text{ for every } ij \in \mathbb{Z}^2, \]
then \((G, \phi) \rightarrow \text{SPC} (d)\).
Conjecture. Given a signed planar graph \((G, \delta)\), if
\[
g_{ij}(G, \delta) \geq g_{ij}(SPC(d)) \text{ for every } i,j \in \mathbb{Z}_2,
\]
then \((G, \delta) \rightarrow SPC(d)\).

Comments.

1. To formulate the conjecture for odd values of \(d\), B. Guenin introduced the notion of "switch" homomorphisms of signed graphs.

2. The conjecture is expected to hold on larger classes of signed graphs:
   \[
   \text{planar} \subseteq K_5\text{-minor-free} \subseteq \text{no}(K_5,\text{-})\text{-minor}.
   \]

3. It generalizes the four-color theorem and it is strongly related to some related conjectures.
Comments

Folding Lemma. If $(G, \omega)$ is a signed planar connected graph where $g_{01}(G, \omega) = \infty$, then it has a planar image where every face is of length $g_{-1}(G, \omega)$.

Corollary. In a minimum counterexample to the conjecture for SPC(d) every facial cycle is a negative cycle of length $d+1$. 
Fraction chromatic number.

(n, k) -coloring of G: assignment of colors from \([n]\) to the vertices of G such that 1. each vertex gets \(k\) colors,

2. adjacent vertices have no common color.

\[X_k(G) = \min \{n | G \text{ admits an } (n, k)\text{-coloring}\}\]

\[X_f(G) = \inf \frac{X_k(G)}{k}\]

Note: Given \(\frac{P}{q} > 2\) to decide if an input graph \(G\) satisfies

\[X_f(G) \leq \frac{P}{q}\] is NP-complete.
$X'_k(G)$: minimum number of colors used to assign $k$-colors to each edge such that each color class is a matching.

$$X'_f(G) = \inf \frac{X'_k(G)}{k}$$

Follows from the definition:

Given a color $i$ and a subset $\mathcal{Z}$ of vertices,

at most $\frac{|\mathcal{Z}|^2}{2}$ edges induced by $\mathcal{Z}$ are of color $i$. 
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Given a color $i$ and a subset $\mathcal{Z}$ of vertices, at most $\frac{|\mathcal{Z}|}{2}$ edges induced by $\mathcal{Z}$ are of color $i$.

Thus, the total number of colors is at least:

$$\frac{|E(\mathcal{Z})|}{\frac{|\mathcal{Z}|}{2}}$$

Focusing on odd subsets we have:

$$X_f(G) \geq \frac{2|E(\mathcal{Z})|}{|\mathcal{Z}| - 1}$$

for every odd subset $\mathcal{Z}$ of vertices.
Definition

\[ \Lambda(G) = \max_{z \text{ odd}} \left\{ \frac{2|E(z)|}{|z| - 1} \right\}. \]

Theorem. For any multigraph \( G \) we have

\[ X'_f(G) = \max \{ \Delta(G), \Lambda(G) \}. \]

Corollary. If \( G \) is a \( k \)-regular multigraph, then \( X'_f(G) = k \) if and only for every subset \( Z \) of \( V(G) \) with \( |Z| \) being odd the edge cut \( (Z, V \setminus Z) \) has at least \( k \) edges.

Proof of the corollary is left as a homework.
Conjecture. For every planar multigraph we have [Seymour]

\[ X'(G) = \lceil X_f'(G) \rceil. \]

Note: Restriction to planar cubic graph is the Tait’s statement which is equivalent to the 4CT.

Conjecture. If G is a planar k-regular multigraph where (restrictio to special case)

for every odd subset Z of vertices, the edg cut \((Z, V \setminus Z)\) is of size at least k, then \(X'(G) = k\).
Conjecture. Given a signed planar graph \((G, \sigma)\), if
\[
g_{ij}(G, \sigma) \geq g_{ij}(\text{SPC}(d)) \quad \text{for every } i,j \in \mathbb{Z}_2^2,
\]
then \((G, \sigma) \rightarrow \text{SPC}(d)\).

Conjecture. If \(G\) is a planar \(k\)-regular multigraph where
\[
[B, \text{case } k]
\]
for every odd subset \(Z\) of vertices, the edge cut \((Z, V \setminus Z)\) is of size at least \(k\), then \(\chi'(G) = 1\).

Theorem. For \(d = k\), \(A(d)\) holds if and only if \(B(k)\) holds.