$DC(G,\sigma)$: Double Cover of (G,σ)

vertices: two copies of $V(G)$ $\quad V_L(G) \cup V_R(G)$

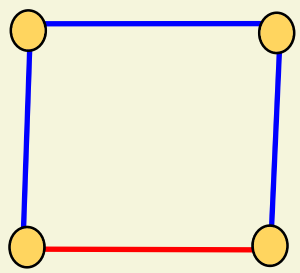
edges: * If xy is a positive edge, then

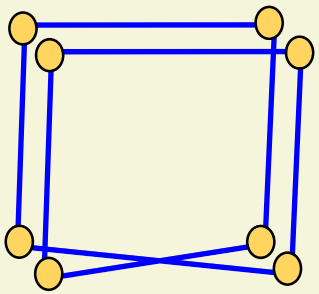
$x_L \sim y_L$ & $x_R \sim y_R$

* If xy is a negative edge, then

$x_L \sim y_R$ & $x_R \sim y_L$

Example





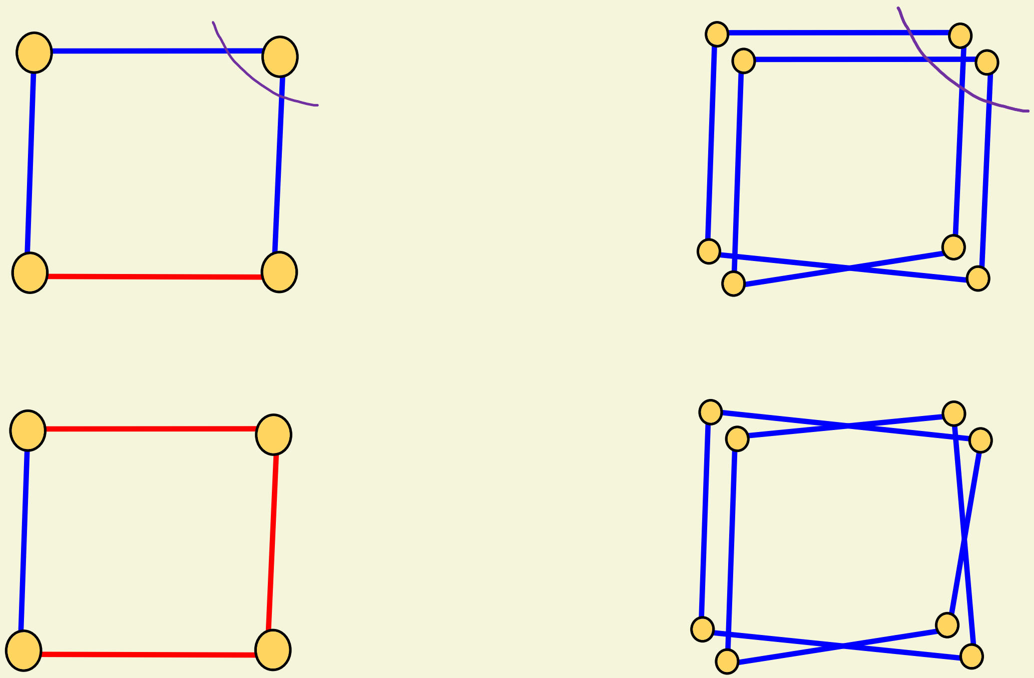
(G, σ)

switch at a vertex x

$DC(G, \sigma)$

switch the place of x_L with x_R

Example



EDC(G,σ): Extended Double Cover of (G,σ)

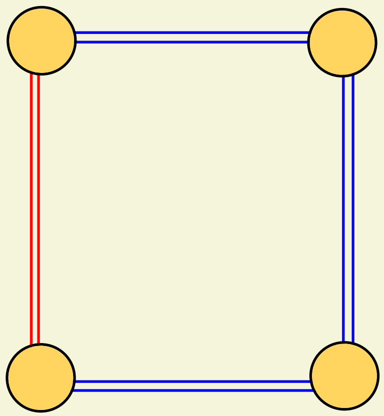
vertices: two copies of $V(G)$ $\quad V_L(G) \cup V_R(G)$

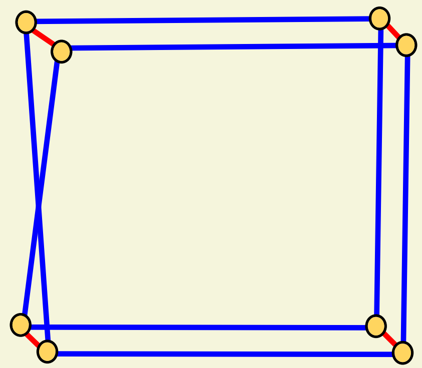
positive edges: * If xy is a positive edge, then
$x_L \sim y_L$ & $x_R \sim y_R$

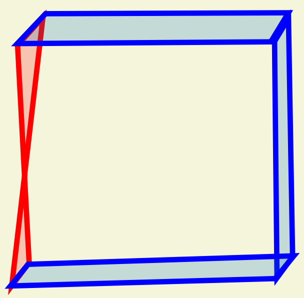
* If xy is a negative edge, then
$x_L \sim y_R$ & $x_R \sim y_L$

negative edges: $x_L \sim x_R$

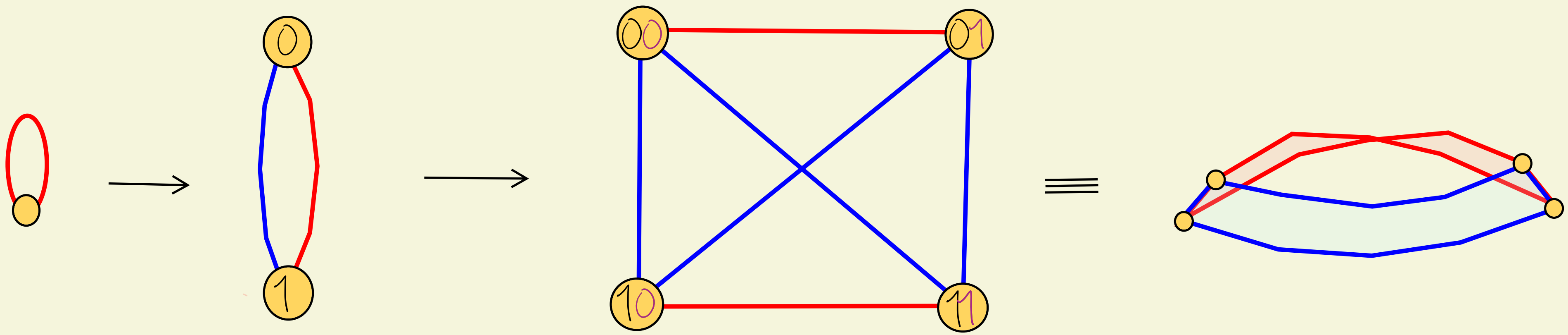
Example

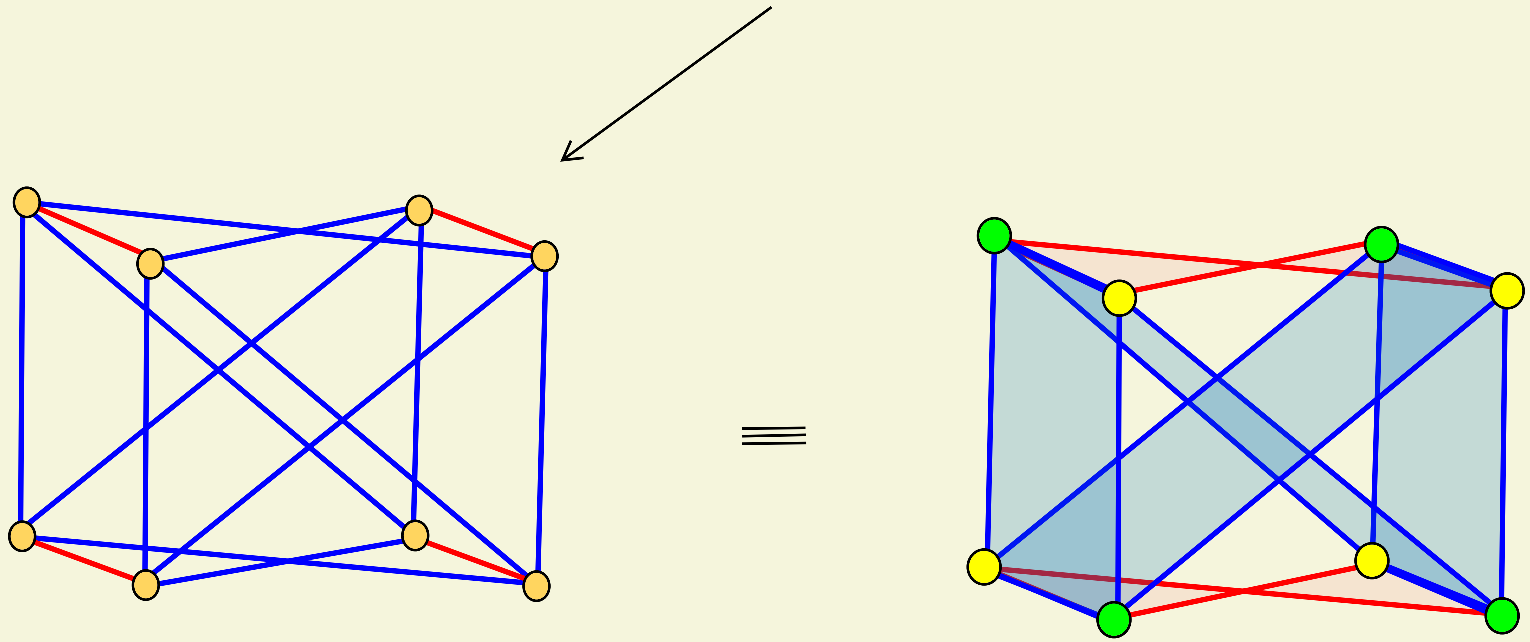


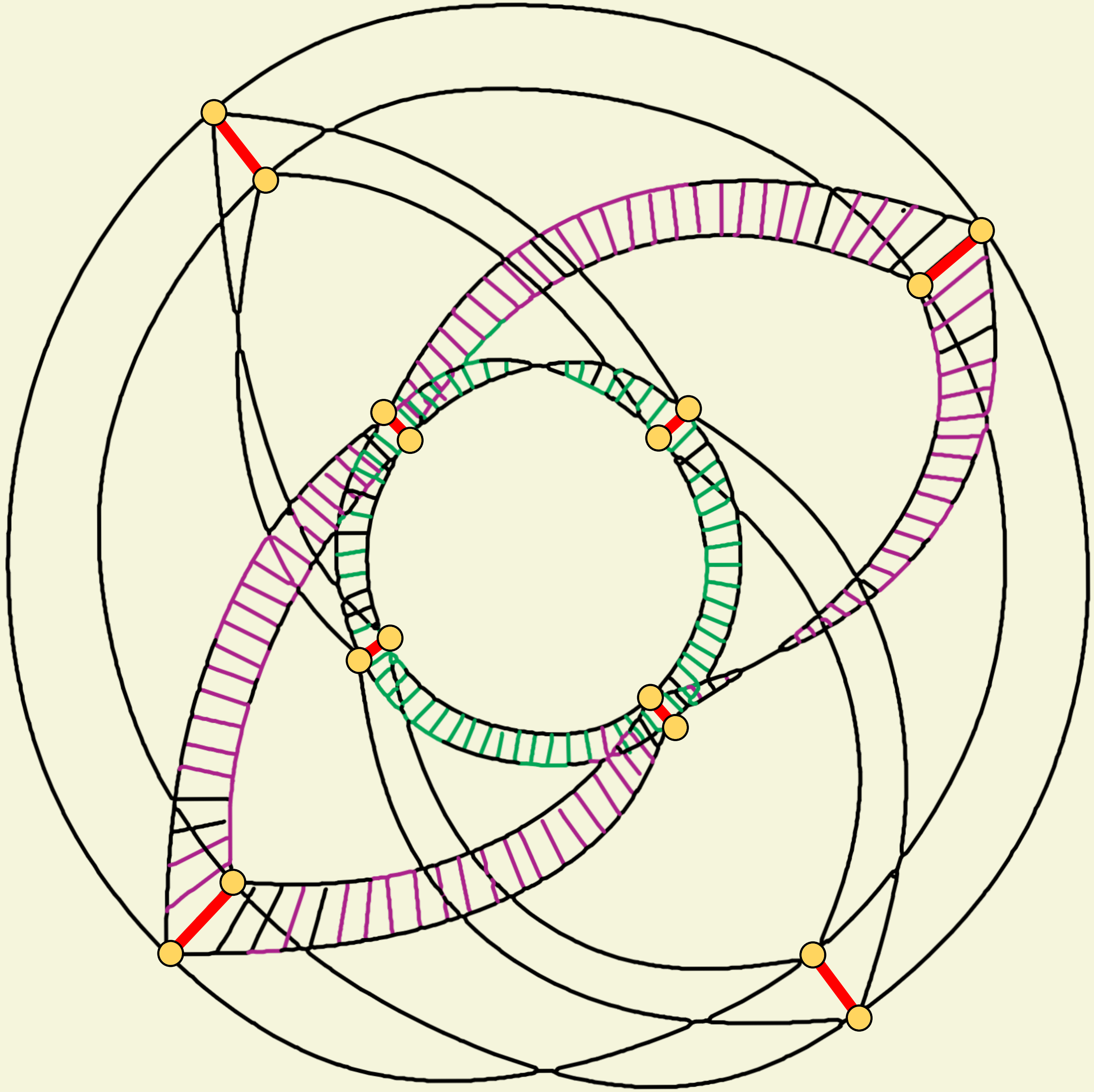




Special examples



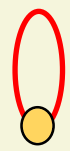




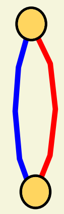
First definition of SPC(d), the

Signed Projective Cube of dimension d

SPC(0): the projective cube of dimension 0.



SPC(1): the projective cube of dimension 1.



$$SPC(d) = EDC(SPC(d-1))$$

Second definition. (where the name comes from)

H_d : Hypercube of dimension d

$\left(\mathbb{Z}_2^d, \{e_1, e_2, e_3, \dots e_d\}\right)$

σ_i: - sign to the edges corresponding to e_i

+ sign to all other edges

SPC(d) is obtained from (H_{d+1}, σ_i) by identifying antipodal pairs of vertices.

Third definition (as augmented cube)

SPC_d is obtained from H_d by considering its edges as positive edges and adding a negative edge between each pair of antipodal vertices.

Forth definition (as Cayley graph)

$$SPC(d) := \left(\mathbb{Z}_2^d, \{e_1, e_2, e_3, \dots e_d, J\}\right)$$

$\{e_1, e_2, e_3, \dots e_d, J\}$ can be replaced with any basis as positive elements and their sum as the negative element.

Homework. Prove that all these definitions result in a same signed graph.

Fifth definition (as power graph of C_{-d})

Vertices: Subsets of vertices of C_{-d-1}.

(those of even order)

Edges: $A \sim B$ if their symmetric difference is an edge of C_{-d-1}.

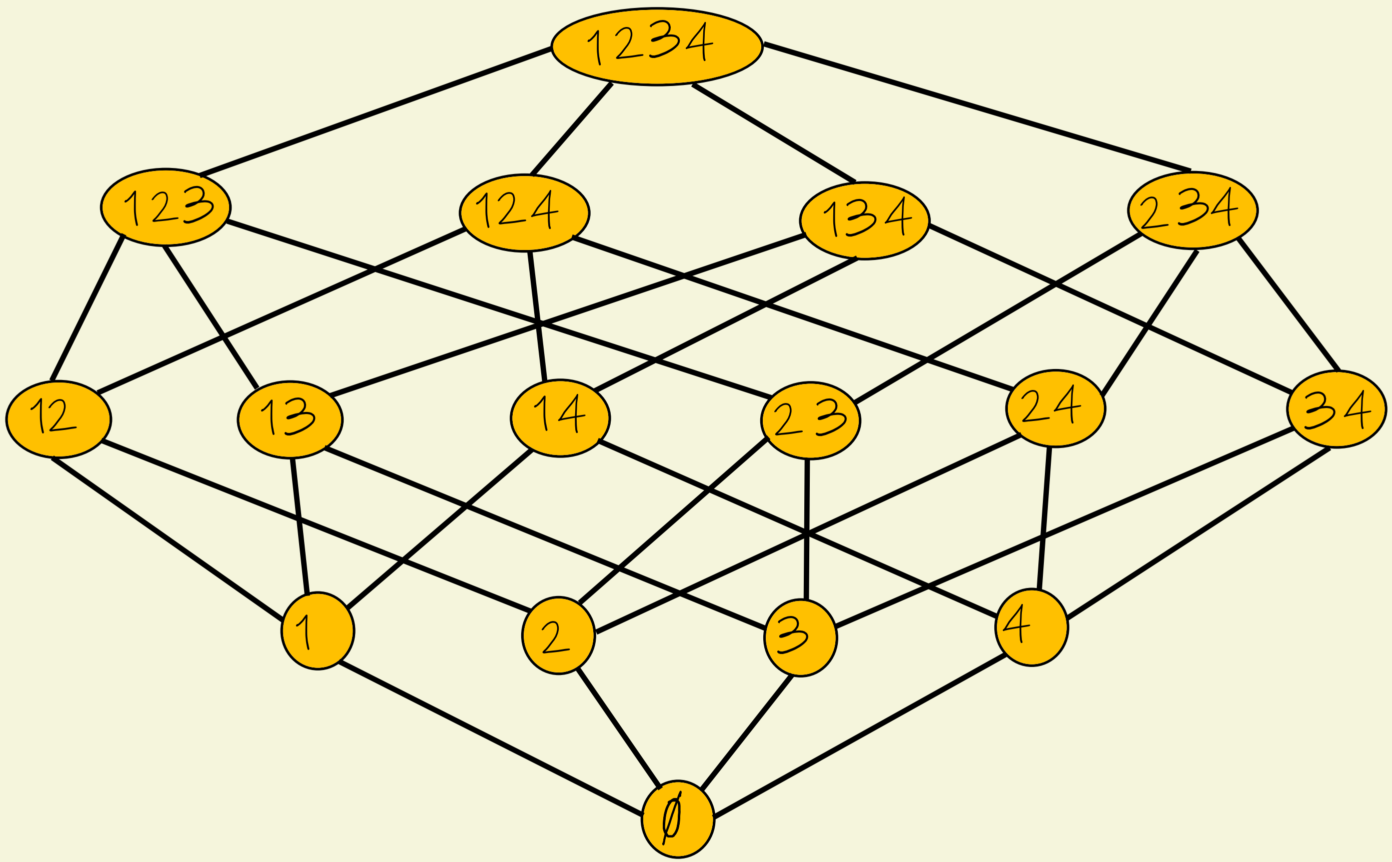
Signature: sign of AB is the same as sign of the edge it corresponds to, i.e. $A \Delta B$.

Sixth definition

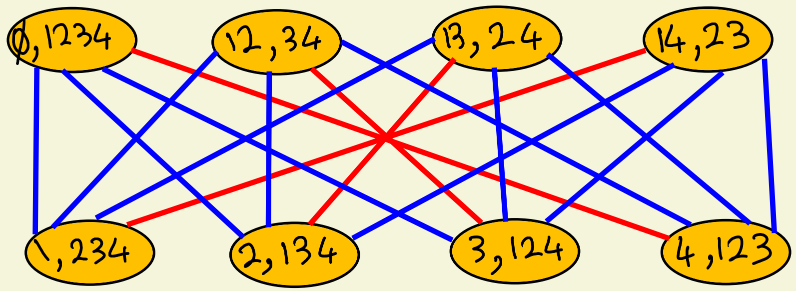
Definition of Hypercube as a poset (a reminder)

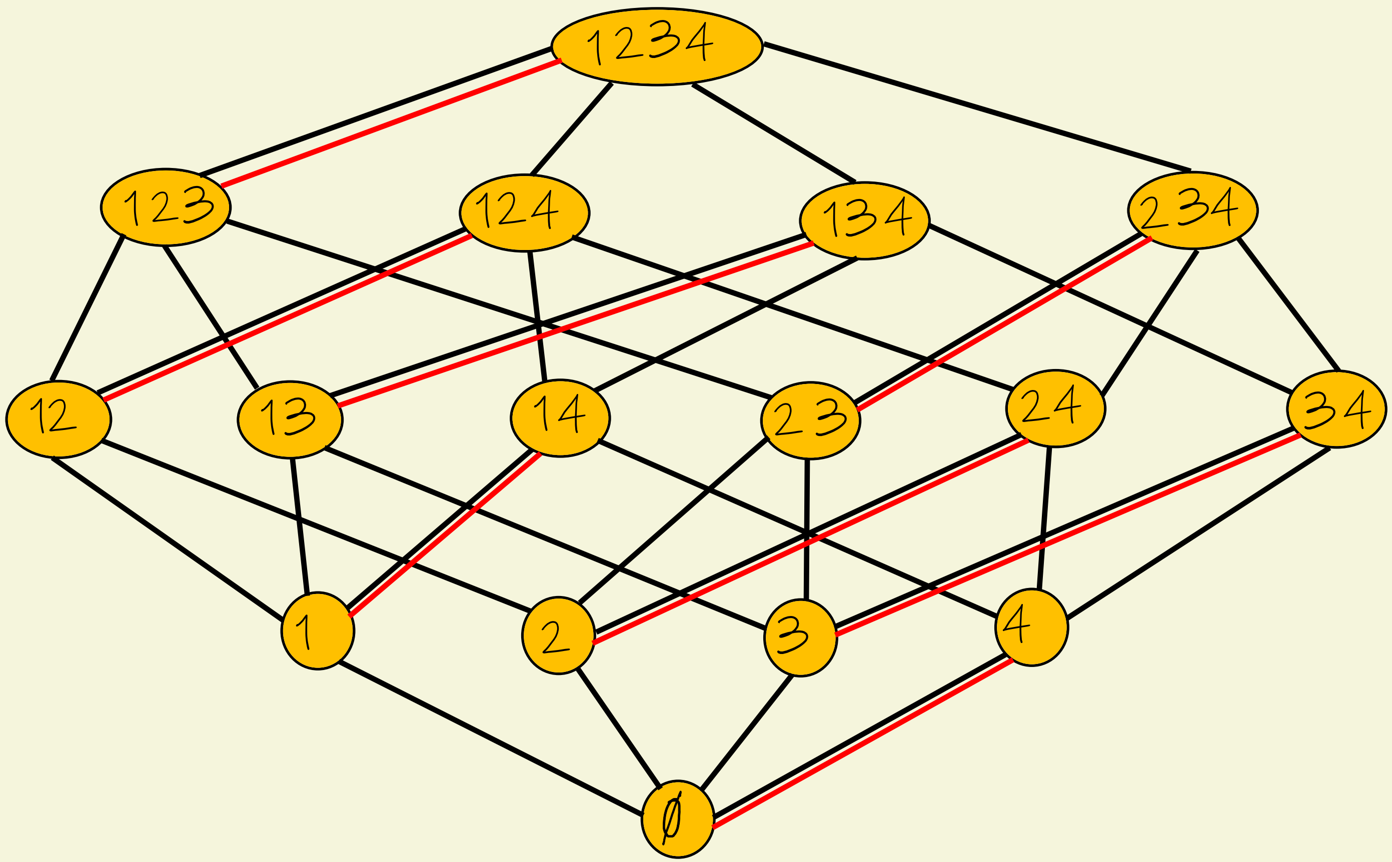
$$V(H_d) = 2^{[d]}$$

$$E(H_d): \quad A \sim B \quad \text{if} \quad A \subset B \;\; \& \;\; |B| = |A| + 1$$

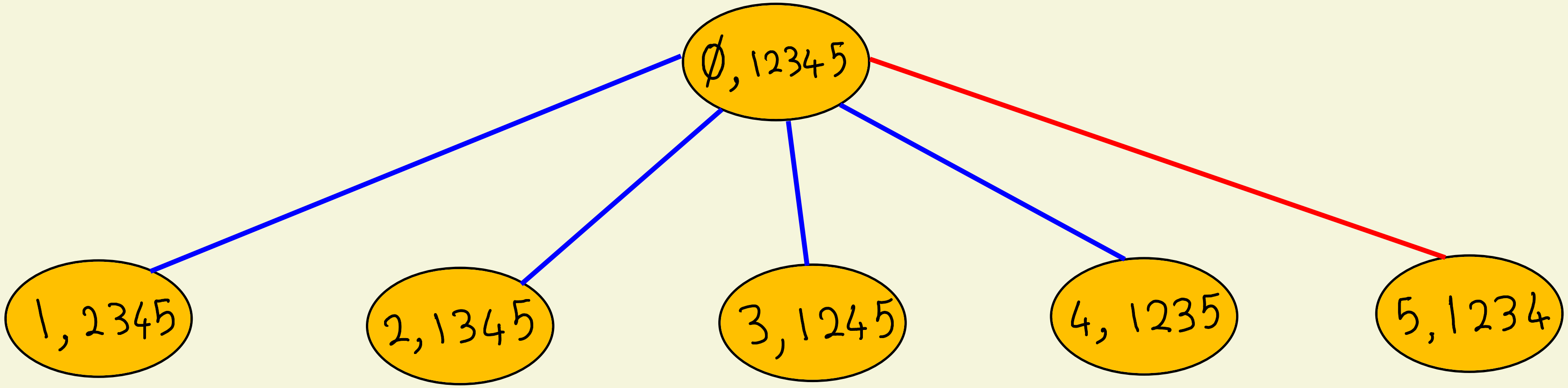


SPC_d as a projected poset

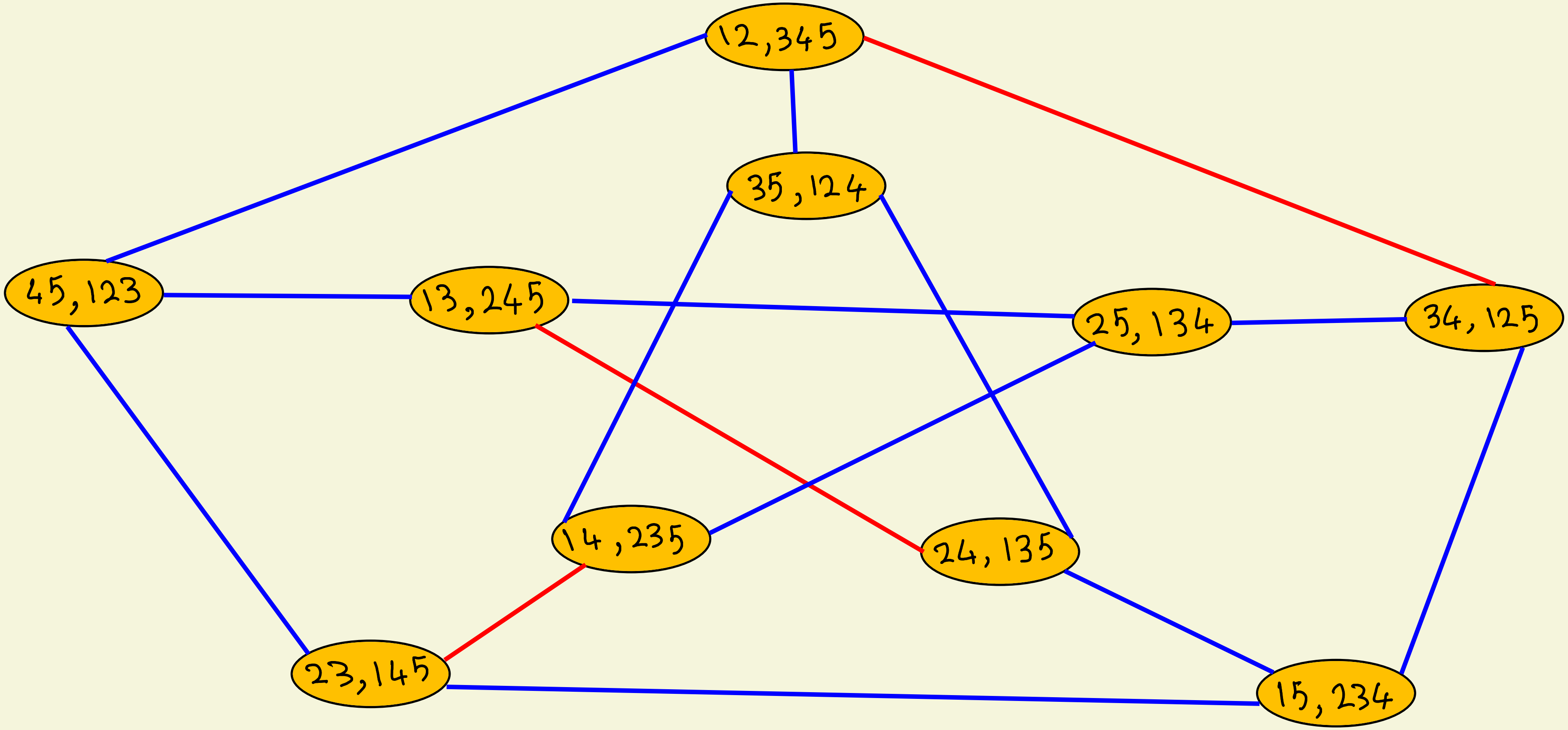




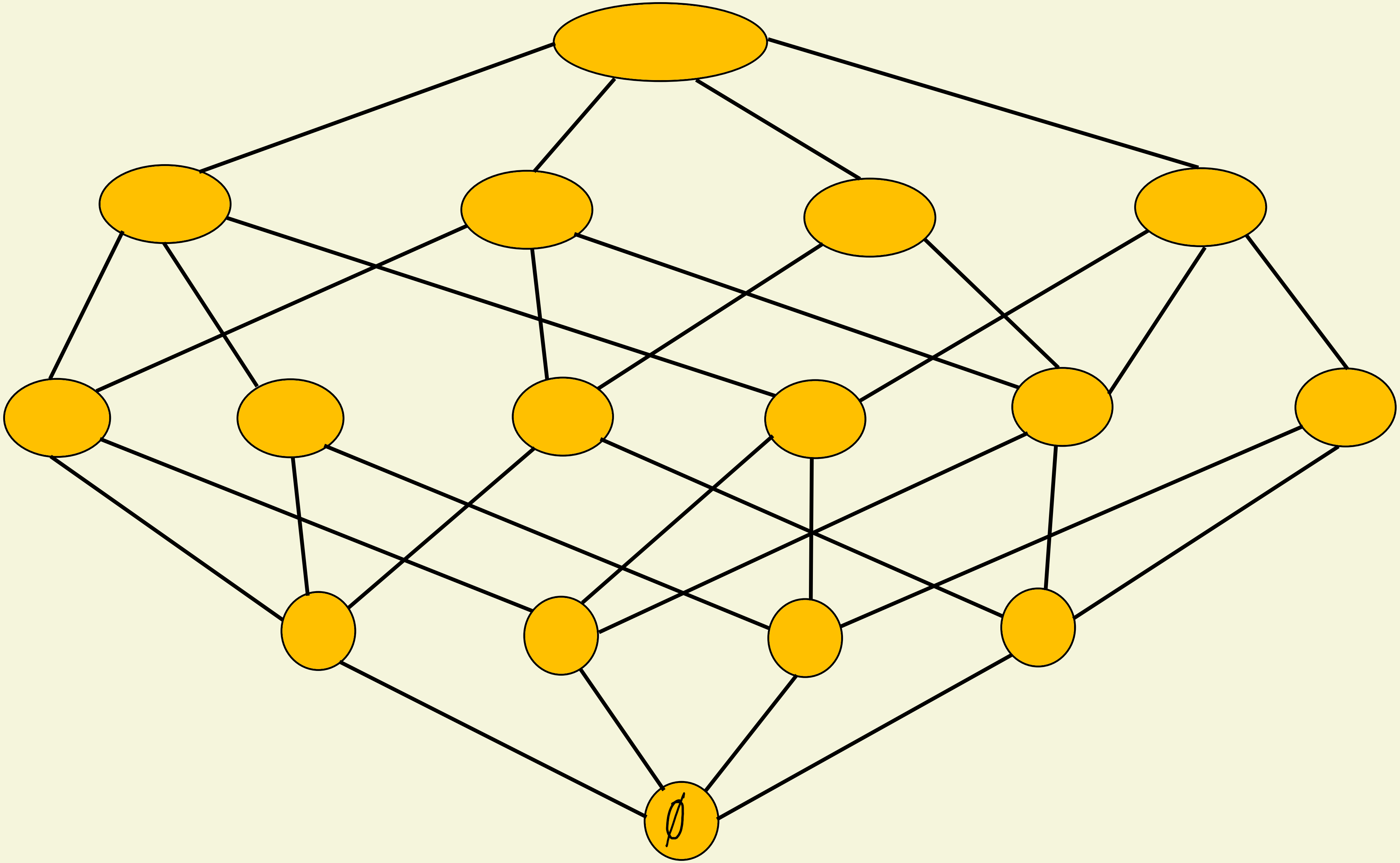
SPC_5:



∅, 12345
1, 2345
2, 1345
3, 1245
4, 1235
5, 1234



12, 345
35, 124
45, 123
13, 245
25, 134
34, 125
14, 235
24, 135
23, 145
15, 234



Properties of SPC(d):

* Its negative girth is $d+1$.
* Every negative cycle is of same parity as $d+1$.
* Every positive cycle is even.

Proofs are left as homework.

Theorem. A signed graph $(G\ \sigma)$ maps to $SPC(d)$ if and only if $E(G)$ can be partitioned into $E_1, E_2, \ldots E_{d+1}$ such that (G, E_i), for each i, is switch equivalent to (G, σ).

Proof

Theorem. A signed graph $(G\ \sigma)$ maps to $SPC(d)$ if and only if $E(G)$ can be partitioned into $E_1, E_2, \ldots E_{d+1}$ such that (G, E_i), for each i, is switch equivalent to (G, σ).

Corollary. A graph G is 4-colorable if and only if $E(G)$ can be partitioned into 3 sets E_1, E_2, E_3 such that:

"each E_i contains an odd number of edges of each odd cycle and an even number of edges of each even cycle."

Conjecture. Given a signed planar graph (G, σ), if

[Naserasr, Guenin] $g_{ij}(G, \sigma) \geq g_{ij}(SPC(d))$ for every $ij \in \mathbb{Z}_2^2$,

then $(G, \sigma) \longrightarrow SPC(d)$.

Conjecture. Given a signed planar graph (G, σ), if

[Naserasr, Guenin]

$$g_{ij}(G, \sigma) \geqslant g_{ij}(SPC(d)) \text{ for every } ij \in \mathbb{Z}_2^2,$$

then $(G, \sigma) \longrightarrow SPC(d)$.

Comments.

1. To formulate the conjecture for odd values of d, B. Guenin introduced the notion of "switch" homomorphisms of signed graphs.

2. The conjecture is expected to hold on larger classes of signed graphs:

   planar $\subseteq$ K_5-minor-free $\subseteq$ no$(K_5,-)$-minor.

3. It generalizes the four-color theorem and it is strongly related to some related conjectures.

Comments

Folding Lemma. [Klostermeyer-zhang, Naserasr-Rollova-Sopena] If (G, σ) is a signed planar connected graph where $g_{01}(G, \sigma) = \infty$, then it has a planar image where every face is of length $g_{-}(G, \sigma)$.

Corollary. In a minimum counterexample to the conjecture for $SPC(d)$ every facial cycle is a negative cycle of length $d+1$.

Fraction chromotic number.

(n, k)-coloring of G: assignment of colors from $[n]$ to the vertices of G such that 1. each vertex gets k colors, 2. adjacent vertices have no common color.

$$\chi_k(G) := \min \{ n \mid G \text{ admits an } (n, k)\text{-coloring} \}.$$

$$\chi_f(G) = \inf \frac{\chi_k(G)}{k}$$

Note: Given $\frac{p}{q} > 2$ to decide if an input graph G satisfies

$\chi_f(G) \le \frac{p}{q}$ is NP- complete.

$\chi'_k(G)$: minimum number of colors used to assign k-colors to each edge such that each color class is a matching.

$$\chi'_f(G) = \inf \frac{\chi'_k(G)}{k}$$

Follows from the definition:

Given a color i and a subset Z of vertices,

at most $\frac{|Z|}{2}$ edges induced by Z are of color i.

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at most $\frac{|Z|}{2}$ edges induced by Z are of color i.

Thus, the total number of colors is at least: $\dfrac{|E(Z)|}{\left\lceil \frac{|Z|}{2} \right\rceil}$

Focusing on odd subsets we have:

$$\chi'_f(G) \geq \frac{2|E(Z)|}{|Z|-1}$$

for every odd subset Z of vertices.

Definition

$$\Lambda(G) = \max_{Z \text{ odd}} \left\{ \frac{2|E(Z)|}{|Z|-1} \right\}.$$

Theorem. For any multigraph G we have

$$\chi_f'(G) = \max\{\Delta(G), \Lambda(G)\}.$$

Corollary. If G is a k-regular multigraph, then $\chi_f'(G) = k$ if and only for every subset Z of $V(G)$ with $|Z|$ being odd the edge cut $(Z, V \setminus Z)$ has at least k edges.

Proof of the corollary is left as a homework.

Conjecture. For every planar multigraph we have

[Seymour]

$$\chi'(G) = \lceil \chi'_f(G) \rceil.$$

Note: Restriction to planar cubic graph is the Tait's statement which is equivalent to the 4CT.

Conjecture. If G s a planar k-regular multigraph where

(restrictio to special case)

for every odd subset Z of vertices, the edg cut $(Z, V \setminus Z)$ is of size at least k, then $\chi'(G) = k$.

Conjecture. Given a signed planar graph (G, σ), if

[A, case d] $g_{ij}(G, \sigma) \geq g_{ij}(SPC(d))$ for every $ij \in \mathbb{Z}_2^2$,

then $(G, \sigma) \longrightarrow SPC(d)$.

Conjecture. If G s a planar k-regular multigraph where

[B, case k] for every odd subset Z of vertices, the edg cut

$(Z, V \setminus Z)$ is of size at least k, then $\chi'(G) = k$.

Theorem. For $d=k$, $A(d)$ holds if and only if $B(k)$ holds.

Proof of the theorem.