Reminder

signed projective cube of dimension d, $SPC(d)$:

Vertices: $\mathbb{Z}_2^d$

positive edges: xy where $x-y \in \{e_1, e_2 \cdots e_d\}$

negative edges: xy where $x-y = J$.

Homework. $SPC(d) \longrightarrow SPC(d-2)$

Conjecture. Given a signed planar graph (G, σ), if

$$g_{ij}(G, \sigma) \geq g_{ij}(SPC(d)) \text{ for every } ij \in \mathbb{Z}_2^2,$$

then $(G, \sigma) \longrightarrow SPC(d)$.

d even: SPC(d) can be switched to all edges being −.

d odd: SPC(d) is a signed bipartite graph.

d even: PC(d) bounds the class of planar graphs of odd-girth $\geq d+1$.

d odd: SPC(d) bounds the class of signed bipartite graphs of negative-girth $\geq d+1$.

Notion(s) of perfectness:

$PC(2d)$ is (the?) smallest possible graph of odd-girth $2d+1$ which bounds the class of planar graphs of odd-girth at least $2d+1$.

$SPC(2d-1)$ is (the?) smallest possible signed bipartite graph of negative girth $2d$ which bounds the class of signed bipartite planar graphs of negative girth at least $2d$.

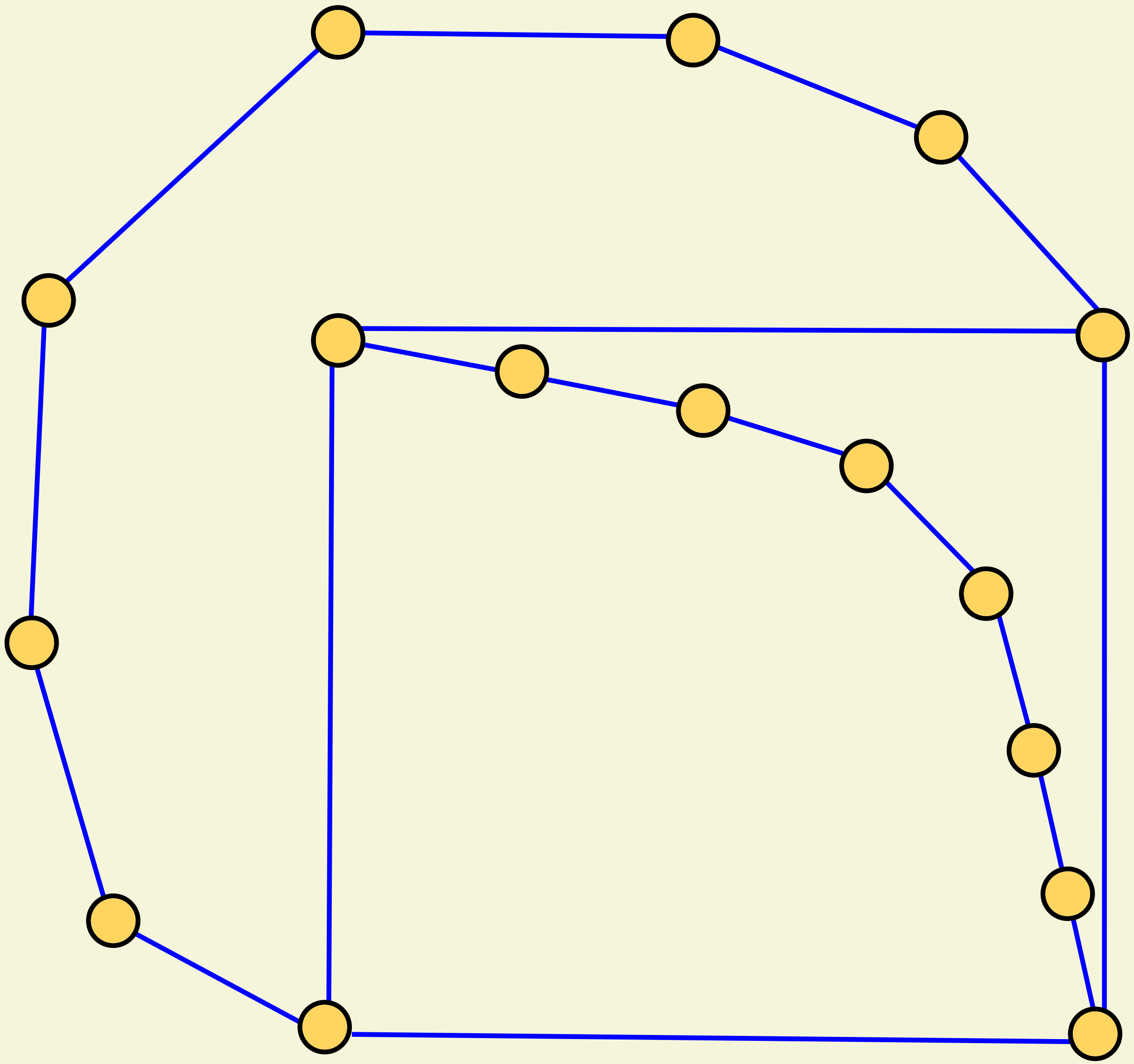
Notion(s) of perfectness:

[Naserasr, Sen, Sun]

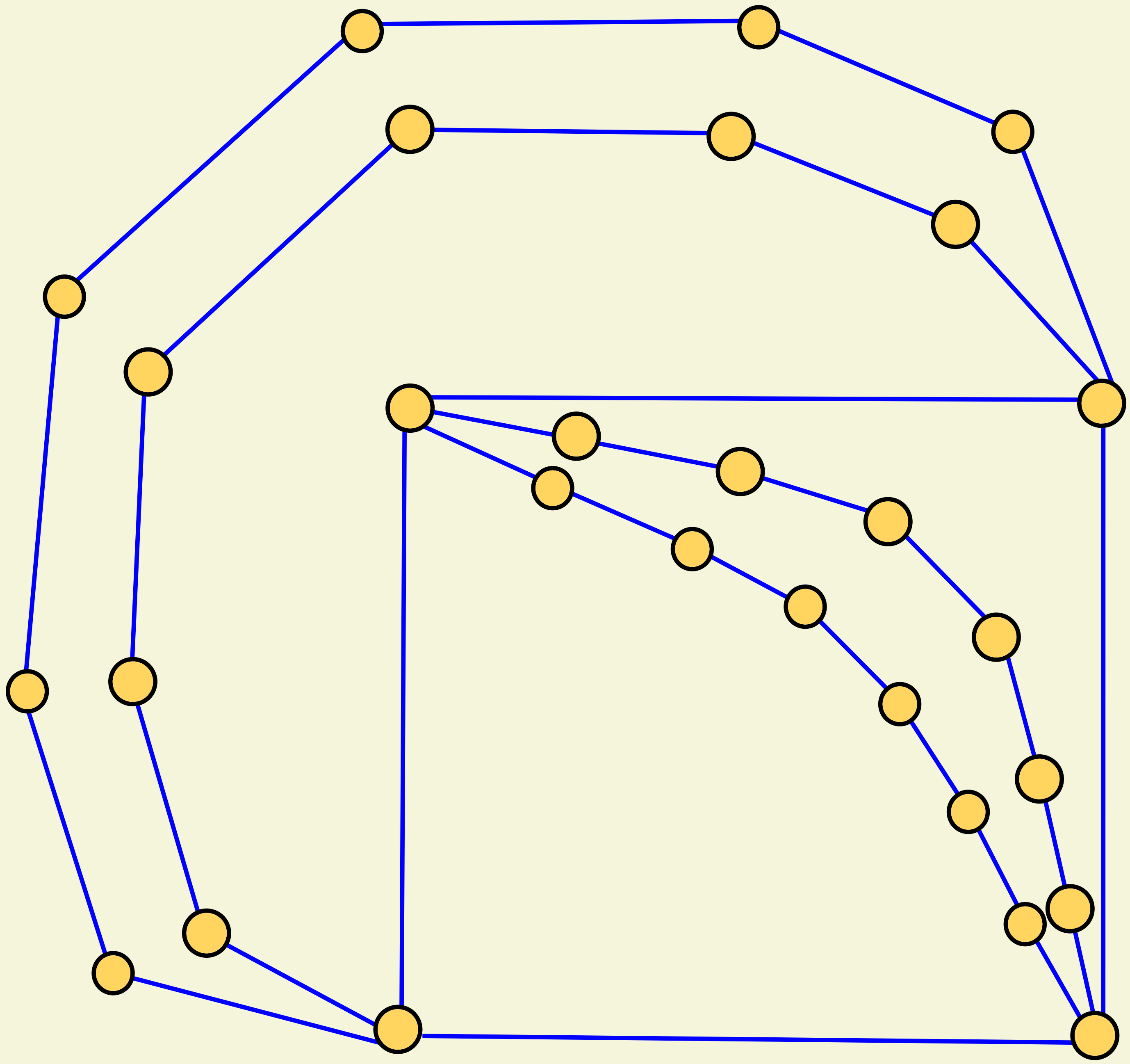
Theorem. There exists a planar graph of odd-girth $2d+1$ with a set W of 2^{2d} vertices such that for any two vertices u and v in W, identifying u and v would create an odd cycle of length at most $2d-1$.

Similar result holds for signed bipartite case.

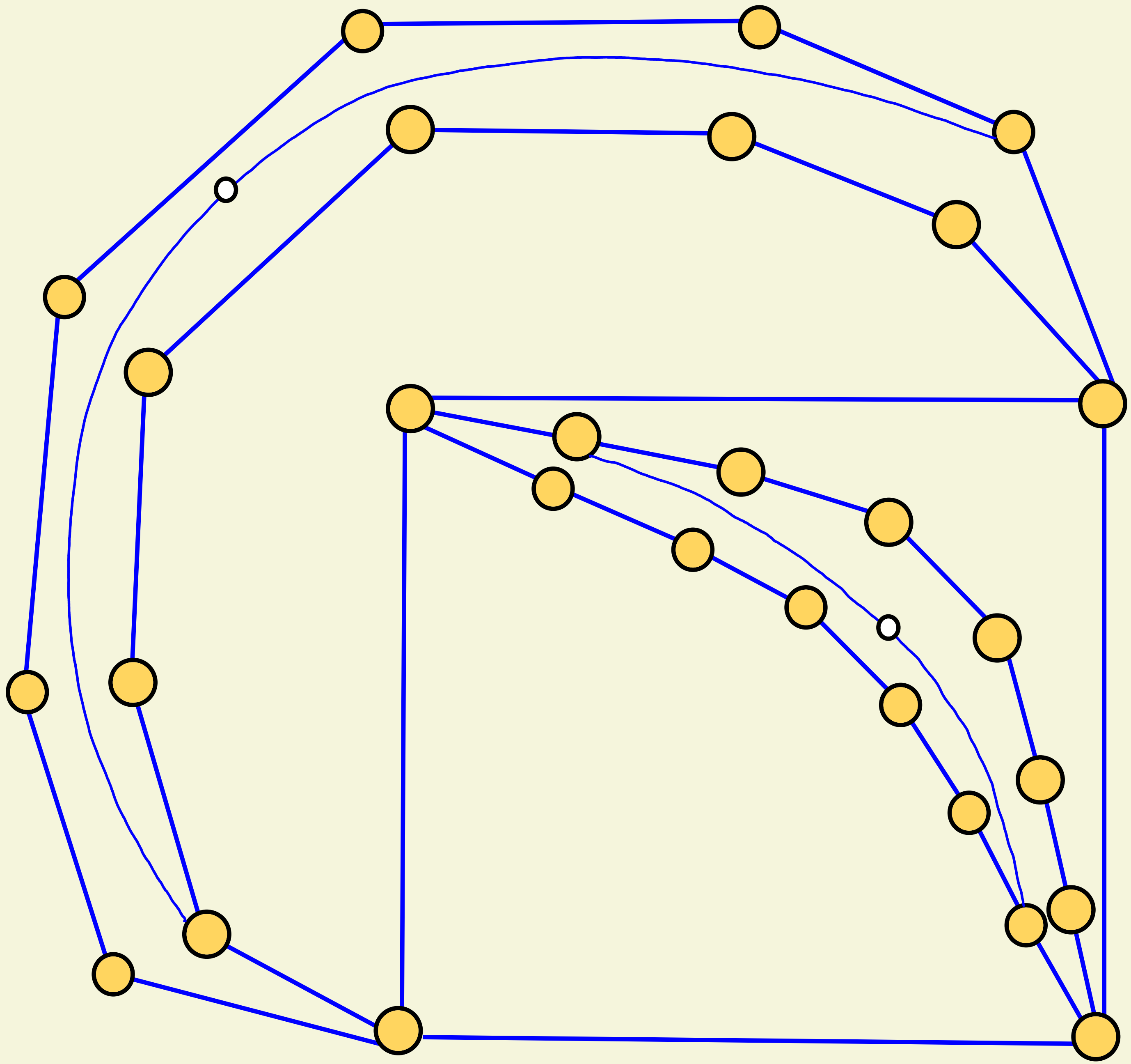
Construction:



Construction:



Construction:



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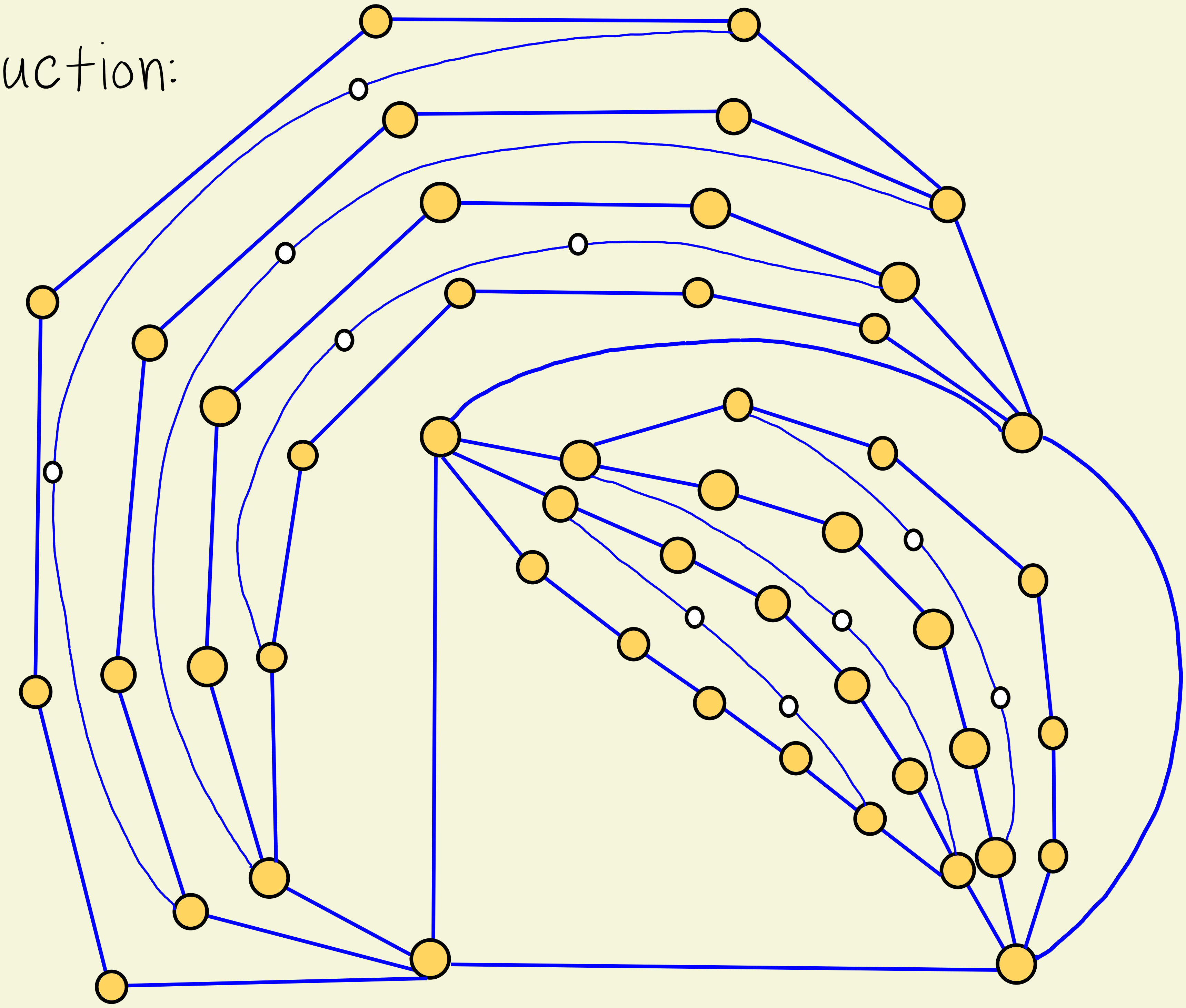
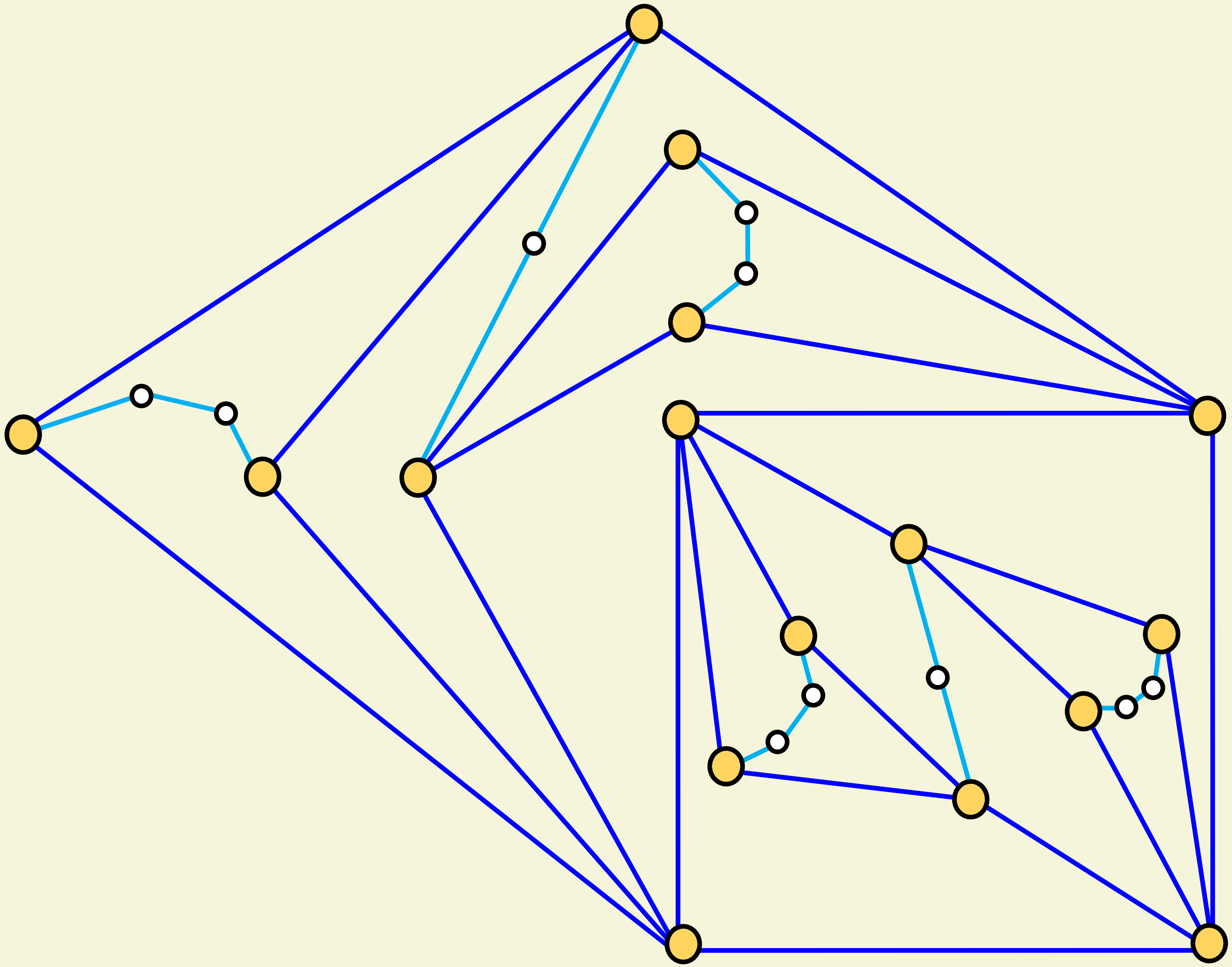


Final result for d=4 (odd girth 5)



Remarks:

The constructions are of treewidth 3.

Theorem. If G is a graph of treewidth at most 3

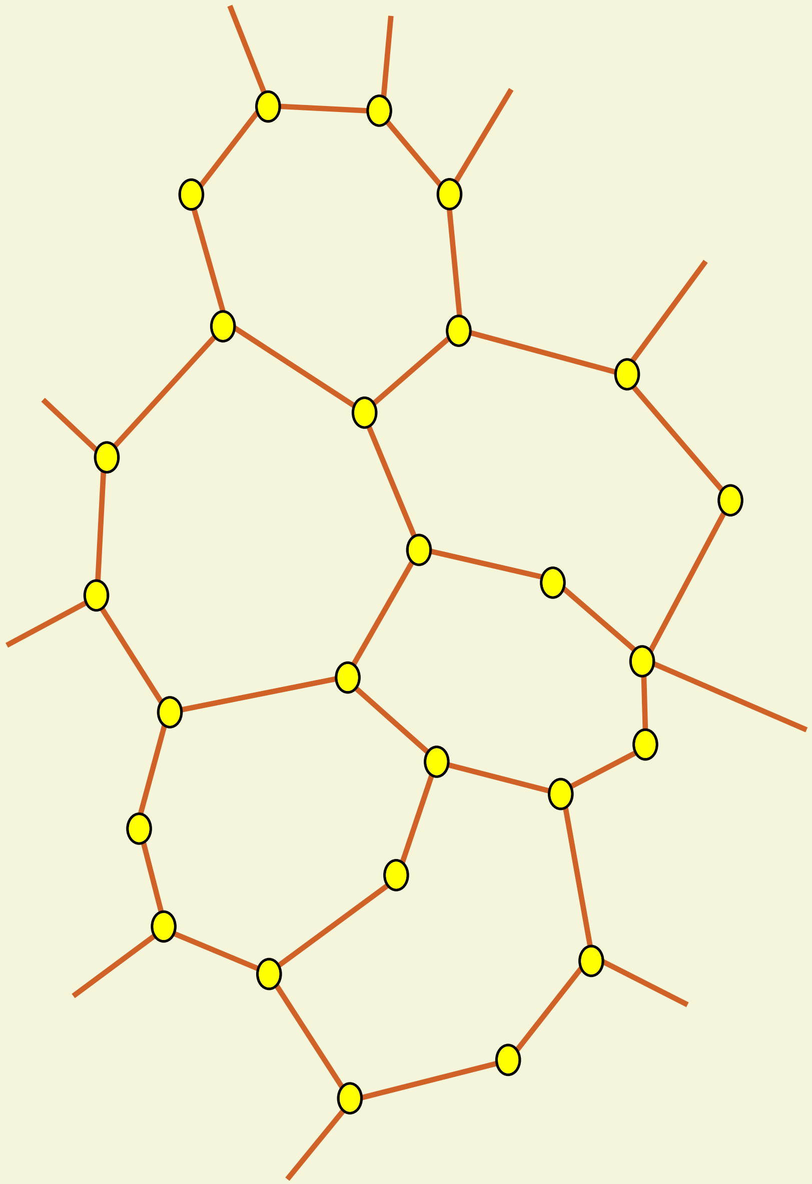
[Meirun Chen, Naserasr]

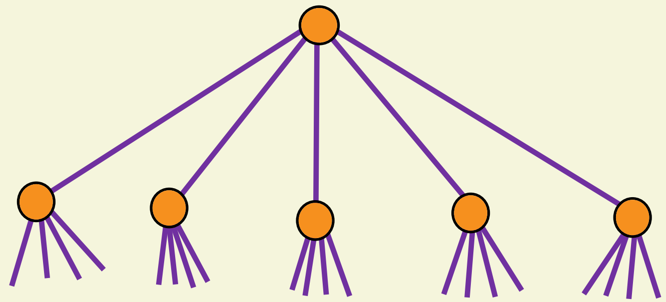
and odd-girth at least $2d+1$, then $G \longrightarrow PC(2d)$

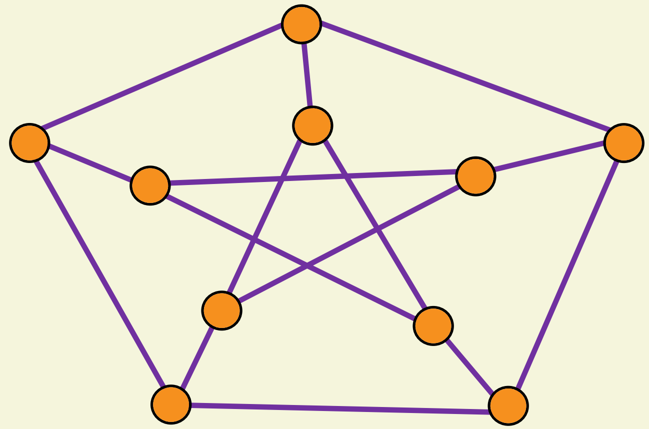
Note: for treewidth at most 2, the best homomorphism bound and the best clique bound do not match.

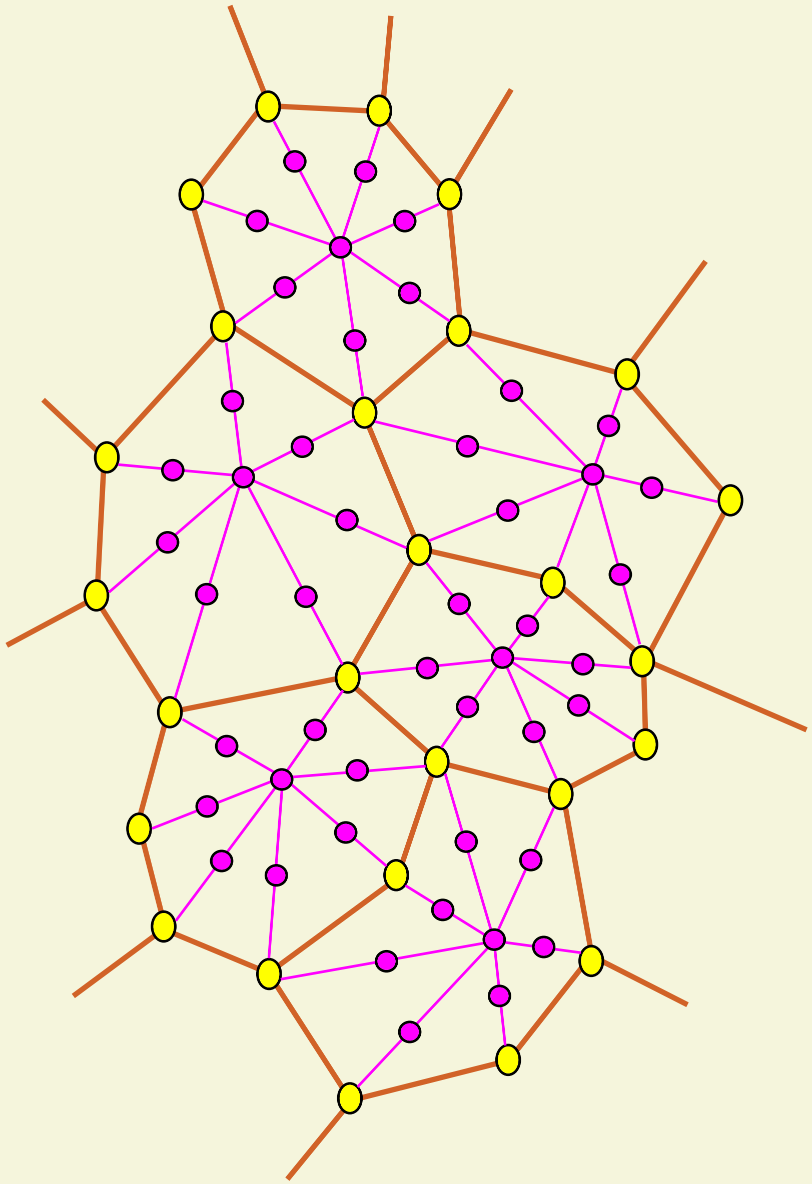
$$
\begin{tikzcd}
PC(2k+1) \arrow[r] & PC(2k-1) & \cdots & PC(6) \arrow[r] & PC(4) \arrow[r] & PC(2) \\
\begin{array}{c}\text{planar graphs}\\ \text{of odd-girth}\\ \text{at least } 2k+1\end{array} \arrow[u, "?"'] & & & \begin{array}{c}\text{planar graphs}\\ \text{of odd-girth}\\ \text{at least } 7\end{array} \arrow[u, "\checkmark"'] & \begin{array}{c}\text{triangle-free}\\ \text{planargraphs}\end{array} \arrow[u, "\checkmark"'] & \begin{array}{c}\text{planar}\\ \text{graphs}\end{array} \arrow[u, "\checkmark"']
\end{tikzcd}
$$

PC(4)

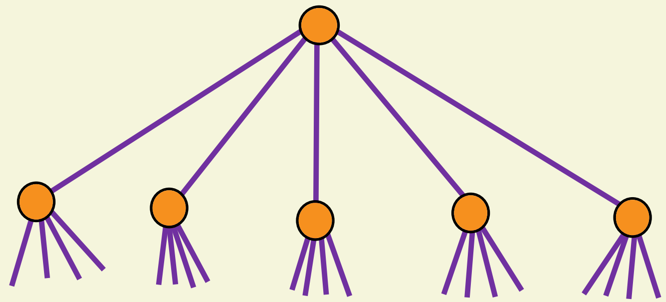


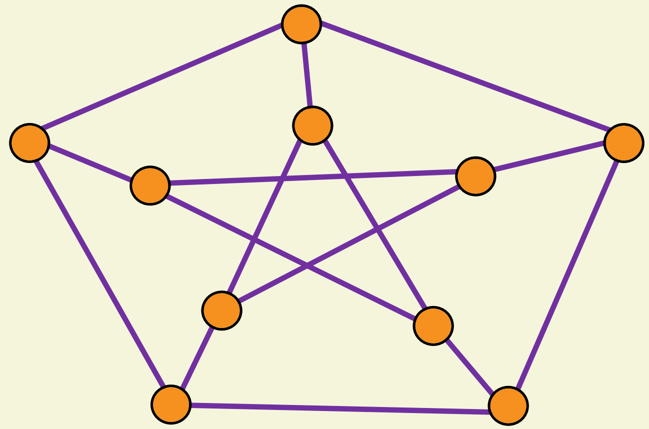






PC(4)





Conjecture. Every planar graph of odd-girth at least $2k+3$ maps to $K(2k+1, k)$.

Would imply the best possible bound of $2+\frac{1}{k}$ for the fractional chromatic number of this class.

Conjecture. Every planar graph of odd-girth at least

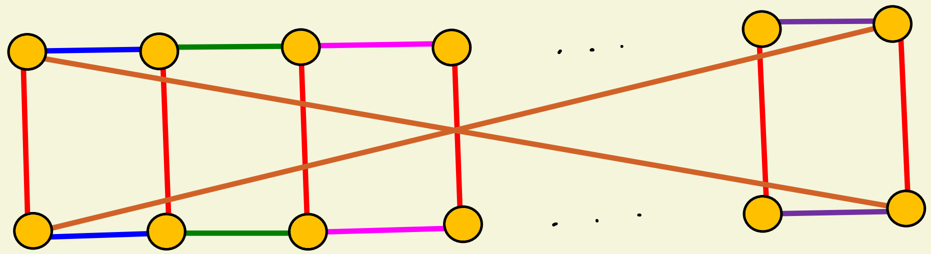
[Jaeger–Zhang]

$4k+1$ maps to C_{2k+1}.

Would imply the best possible? bound of $2+\frac{1}{k}$

for the circular chromatic number of this class.

Conjecture. Every planar graph of odd-girth at least $4k-1$ maps to $\mathcal{M}_{2k+1}$.



Would imply the best possible ? bound of $2+\frac{2}{2k-1}$ for the circular chromatic number of this class.

Analogous questions are asked for mapping signed bipartite planar graphs to $SPC(2d-1)$.