Packing number of a signed graph \((G, \sigma)\)

(signature)

Maximum number of switching equivalent signatures \(\sigma_1, \sigma_2, \ldots, \sigma_K\)
no two of which assign negative sign to a same edge.

Notation: \(P(G, \sigma)\)  \hspace{1em} Note: \(P(G, +) = \infty\)

Examples
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Examples

Lemma. \(P(G, \sigma) \leq g_-(G, \sigma)\).

When equality holds we say \((G, \sigma)\) packs.
Observation 1. \( P(G, -) \) is always an odd number.

Observation 2. If \( G \) is bipartite, then \( P(G, 6) \) is always an even number.
Definition. \( \text{SPC}^0(d) \) is obtained from \( \text{SPC}(d) \) by adding a positive loop to each vertex.

Theorem. \( P(G, \sigma) \geq d+1 \iff (G,\sigma) \to \text{SPC}^0(d) \)
Proof of the theorem.
special case: If \( \hat{G} \) is connected and has no positive odd closed walk, then \( P(\hat{G}) \) is the largest \( d \) such that

\[
\hat{G} \rightarrow SPC(d).
\]

Corollary. A graph \( G \) is 4-colorable if and only if \( P(G, -) \geq 3 \).

(equivalently \( P(G, -) > 1 \))

Corollary. A signed graph \((G, \sigma)\) is \( \{\pm 1, \pm 2\}\)-colorable if and only if \( P(G, -\sigma) \geq 2 \).
Conjecture. Every signed planar graph with no positive odd walk packs.
negative cycle cover v.s. signature

A set of edges that covers every negative cycle.

Observation: Every equivalent signature is a negative cycle cover.
Theorem. Every minimal negative cycle cover is a (minimal) signature.

[Harry 1959]
Theorem. If \((G, \delta)\) has no \(k^2\) minor, then the maximum number of edge-disjoint negative cycle covers is equal to the length of a shortest negative cycle.

\[ k^2 \]

[Gan, Jhonson]
Restating the theorem:

Every signed graph with no $k_3^2$-minor packs.
Theorem. Let $C$ be a minor closed class of 4-colorable graphs. Let $G$ be a bipartite graph in the class and let $\sigma$ be any signature on $G$. Then $P(G, \sigma) \geq 4$.

Note 1. $P(G, \sigma) \geq 3$ for every $G$ in $C$.

Note 2. $C$ is included in the class of $K_5$-minor free graphs.

Note 3. The number of edges of each bipartite graph in $C$ is at most $2n-4$. (average degree strictly less than 4)