Circular Coloring of Signed Graphs

Zhouningxin Wang

Université de Paris

wangzhou4@irif.fr

(A joint work with Reza Naserasr, Xuding Zhu)

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Circular coloring of graphs

Given a real number $r$, a circular $r$-coloring of a graph $G$ is a mapping $f : V(G) \to C^r$ such that for any edge $uv \in E(G)$,

$$d_{(\text{mod } r)}(f(u), f(v)) \geq 1.$$ 

The circular chromatic number of $G$ is defined as

$$\chi_c(G) = \inf\{r : G \text{ admits a circular } r\text{-coloring}\}.$$
Circular coloring of graphs

- A 3-chromatic graph is not 2-colorable, but if its circular chromatic number is near 2, then it is somehow “just barely” not 2-colorable.

- By Grotzsch’s theorem, every triangle-free planar graph is 3-colorable. In generalizing this to circular chromatic number, we may ask what threshold on girth is needed to force the circular chromatic number to be at most $2 + \frac{1}{t}$.

Jaeger-Zhang conjecture [C.-Q. Zhang 2002]

Every planar graph of odd-girth $4k + 1$ admits a circular $(2 + \frac{1}{k})$-coloring.
Homomorphism of signed graphs

- A **signed graph** is a graph $G = (V, E)$ together with an assignment $\{+, -\}$ on its edges, denoted by $(G, \sigma)$.
- A **switching** at vertex $v$ is to switch the signs of all the edges incident to this vertex.
- The **sign** of a closed walk is the product of signs of all the edges of this walk.
- A **homomorphism** of signed graph $(G, \sigma)$ to a signed graph $(H, \pi)$ is a mapping $\varphi$ from $V(G)$ and $E(G)$ correspondingly to $V(H)$ and $E(H)$ such that the adjacency, the incidence and the signs of the closed walks are preserved.
- If there exists one, we write $(G, \sigma) \rightarrow (H, \pi)$. 
Homomorphism of signed graphs

- An **edge-sign preserving homomorphism** of a signed graph $(G, \sigma)$ to $(H, \pi)$ is a mapping $f : V(G) \rightarrow V(H)$ such that for every positive (respectively, negative) edge $uv$ of $(G, \sigma)$, $f(u)f(v)$ is a positive (respectively, negative) edge of $(H, \pi)$.

- If there exists one, we write $(G, \sigma) \xrightarrow{s.p.} (H, \pi)$.

**Proposition**

Given two signed graphs $(G, \sigma)$ and $(H, \pi)$,

$$(G, \sigma) \rightarrow (H, \pi) \iff \exists \sigma' \equiv \sigma, (G, \sigma') \xrightarrow{s.p.} (H, \pi).$$
Double Switching Graphs

Given a signed graph \((G, \sigma)\) on the vertex set \(V = \{x_1, \ldots, x_n\}\), the Double Switching Graph of \((G, \sigma)\), denoted \(\text{DSG}(G, \sigma)\), is a signed graph built as follows:

- We have two disjoint copies of \(V\), \(V^+ = \{x_1^+, x_2^+, \ldots, x_n^+\}\) and \(V^- = \{x_1^-, x_2^-, \ldots, x_n^-\}\) in \(\text{DSG}(G, \sigma)\).
- Each set of vertices \(V^+, V^-\) then induces a copy of \((G, \sigma)\).
- Furthermore, a vertex \(x_i^-\) connects to vertices in \(V^+\) as it is obtained from a switching on \(x_i\).
Double Switching Graphs

Figure: Signed graphs $(C_4, e)$ and $DSG(C_4, e)$

**Theorem [R.C. Brewster and T. Graves 2009]**

Given signed graphs $(G, \sigma)$ and $(H, \pi)$,

$$(G, \sigma) \rightarrow (H, \pi) \iff (G, \sigma) \xrightarrow{s.p.} DSG(H, \pi).$$
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4 Discussion
Circular coloring of signed graphs

Given a signed graph \((G, \sigma)\) with no positive loop and a real number \(r\), a **circular \(r\)-coloring** of \((G, \sigma)\) is a mapping \(f : V(G) \to C^r\) such that for each positive edge \(uv\) of \((G, \sigma)\),

\[
d_{(\text{mod } r)}(f(u), f(v)) \geq 1,
\]

and for each negative edge \(uv\) of \((G, \sigma)\),

\[
d_{(\text{mod } r)}(f(u), \overline{f(v)}) \geq 1.
\]

The **circular chromatic number** of \((G, \sigma)\) is defined as

\[
\chi_c(G, \sigma) = \inf\{r \geq 1 : (G, \sigma) \text{ admits a circular } r\text{-coloring}\}.
\]
Circular coloring of signed graphs

Results on some classes of signed graphs

Discussion

Circular chromatic number

Refinement of 0-free 2k-coloring of signed graphs

Definition [T. Zaslavsky 1982]

Given a signed graph \((G, \sigma)\) and a positive integer \(k\), a 0-free 2k-coloring of \((G, \sigma)\) is a mapping \(f : V(G) \rightarrow \{\pm 1, \pm 2, \ldots, \pm k\}\) such that for any edge \(uv\) of \((G, \sigma)\), \(f(u) \neq \sigma(uv)f(v)\).

Proposition

Assume \((G, \sigma)\) is a signed graph and \(k\) is a positive integer. Then \((G, \sigma)\) is 0-free 2k-colorable if and only if \((G, \sigma)\) is circular 2k-colorable.
Circular coloring of signed graphs

Results on some classes of signed graphs

Discussion

Circular chromatic number

Equivalent definition

Note that for $s, t \in [0, r)$, $d_{(\text{mod } r)}(s, t) = \min\{|s - t|, r - |s - t|\}$.

- A **circular $r$-coloring** of a signed graph $(G, \sigma)$ is a mapping $f : V(G) \rightarrow [0, r)$ such that for each positive edge $uv$,

  $$1 \leq |f(u) - f(v)| \leq r - 1$$

  and for each negative edge $uv$,

  either $|f(u) - f(v)| \leq \frac{r}{2} - 1$ or $|f(u) - f(v)| \geq \frac{r}{2} + 1$. 

Circular coloring of signed graphs

Equivalent definition: \((p, q)\)-coloring of signed graphs

For \(i, j, x \in \{0, 1, \ldots, p - 1\}\), we define
\[
d_{(\text{mod } p)}(i, j) = \min\{|i - j|, p - |i - j|\}
\]
and \(\bar{x} = x + \frac{p}{2} \pmod{p}\).

Assume \(p\) is an even integer and \(q \leq \frac{p}{2}\) is a positive integer. A \((p, q)\)-coloring of a signed graph \((G, \sigma)\) is a mapping \(f : V(G) \to \{0, 1, \ldots, p - 1\}\) such that for any positive edge \(uv\),
\[
d_{(\text{mod } p)}(f(u), f(v)) \geq q,
\]
and for any negative edge \(uv\),
\[
d_{(\text{mod } p)}(f(u), \bar{f}(v)) \geq q.
\]

The circular chromatic number of \((G, \sigma)\) is
\[
\chi_c(G, \sigma) = \inf\{\frac{p}{q} : (G, \sigma) \text{ has a } (p, q)\text{-coloring}\}.
\]
Circular chromatic number

Signed circular clique

Circular chromatic number of signed graphs are also defined through graph homomorphism.

For integers $p \geq 2q > 0$ such that $p$ is even, the signed circular clique $K^s_{p;q}$ has vertex set $[p] = \{0, 1, \ldots, p - 1\}$, in which

- $ij$ is a positive edge if $q \leq |i - j| \leq p - q$;
- $ij$ is a negative edge if $|i - j| \leq \frac{p}{2} - q$ or $|i - j| \geq \frac{p}{2} + q$. 
Circular chromatic number

Signed circular clique

Lemma

Given a signed graph \((G, \sigma)\) and a positive even integer \(p\), a positive integer \(q\) with \(p \geq 2q\), \((G, \sigma)\) has a \((p, q)\)-coloring if and only if \((G, \sigma) \xrightarrow{s.p.} K_{p,q}^s\).

Hence the circular chromatic number of \((G, \sigma)\) is

\[
\chi_c(G, \sigma) = \inf \\left\{ \frac{p}{q} : p \text{ is even and } (G, \sigma) \xrightarrow{s.p.} K_{p,q}^s \right\}.
\]

Lemma

If \((G, \sigma) \xrightarrow{s.p.} (H, \pi)\), then \(\chi_c(G, \sigma) \leq \chi_c(H, \pi)\).

Lemma

Given even positive integers \(p, p'\), if \(\frac{p}{q} \leq \frac{p'}{q'}\), then \(K_{p,q}^s \xrightarrow{s.p.} K_{p',q'}^s\).
Signed circular clique

Let \( \hat{K}_p^s \) be the signed subgraph of \( K^s_{p,q} \) induced by vertices \( \{0, 1, \ldots , \frac{p}{2} - 1\} \). Notice that \( K^s_{p,q} = DSG(\hat{K}_p^s) \).

The circular chromatic number of \( (G, \sigma) \) is also

\[
\chi_c(G, \sigma) = \inf \left\{ \frac{p}{q} : p \text{ is even and } (G, \sigma) \rightarrow \hat{K}_p^s \right\}.
\]

Figure: \( K^s_{8,3} \)

Figure: \( \hat{K}_8^s \)
Circular chromatic number

Circular chromatic number of cycles

For a non-zero integer \( \ell \), we denote by \( C_\ell \) the cycle of length \(|\ell|\) whose sign agrees with the sign of \( \ell \).

**Proposition**

\[
\chi_c(C_{2k}) = \chi_c(C_{-(2k+1)}) = 2; \quad \chi_c(C_{2k+1}) = \frac{2k+1}{k}; \\
\chi_c(C_{-2k}) = \frac{4k}{2k-1}.
\]

Observe that the signed graph \( \hat{K}^s_{4k;2k-1} \) is obtained from \( C_{-2k} \) by adding a negative loop at each vertex.
Circular coloring of signed graphs

Introduction

Circular chromatic number

$C_{2k+1}$-coloring and $C_{-2k}$-coloring

Proposition

- Given a graph $G$, $G \rightarrow C_{2k+1}$ if and only if $\chi_c(G) \leq \frac{2k+1}{k}$;
- Given a signed bipartite graph $(G, \sigma)$,

$$(G, \sigma) \rightarrow C_{-2k}$$ if and only if $\chi_c(G, \sigma) \leq \frac{4k}{2k-1}$.
Signed indicators

Let $G$ be a graph and let $\Omega$ be a signed graph.

- A **signed indicator** $\mathcal{I}$ is a triple $\mathcal{I} = (\Gamma, u, v)$ such that $\Gamma$ is a signed graph and $u, v$ are two distinct vertices of $\Gamma$.

- **Replacing e of G with a copy of I** is the following operation: Take the disjoint union of $\Omega$ and $\mathcal{I}$, delete the edge $e$ from $\Omega$, identify $x$ with $u$ and identify $y$ with $v$.

- Given a signed indicator $\mathcal{I}$, we denote by $G(\mathcal{I})$ the signed graph obtained from $G$ by replacing each edge with a copy of $\mathcal{I}$.

- Given two signed indicators $\mathcal{I}_+$ and $\mathcal{I}_-$, we denote by $\Omega(\mathcal{I}_+, \mathcal{I}_-)$ the signed graph obtained from $\Omega$ by replacing each positive edge with a copy of $\mathcal{I}_+$ and replacing each negative edge with a copy of $\mathcal{I}_-$. 
Signed indicator

Assume $\mathcal{I} = (\Gamma, u, v)$ is a signed indicator and $r \geq 2$ is a real number.

- For $a, b \in [0, r)$, we say the color pair $(a, b)$ is feasible for $\mathcal{I}$ (with respect to $r$) if there is a circular $r$-coloring $\phi$ of $\Gamma$ such that $\phi(u) = a$ and $\phi(v) = b$.
- Define
  \[
  Z(\mathcal{I}, r) = \{ b \in [0, \frac{r}{2}] : (0, b) \text{ is feasible for } \mathcal{I} \text{ with respect to } r \}.
  \]

Lemma

Assume that $\mathcal{I} = (\Gamma, u, v)$ is a signed indicator, $r \geq 2$ is a real number and $Z(\mathcal{I}, r) = [t, \frac{r}{2} - t]$ for some $0 < t < \frac{r}{4}$. Then for any graph $G$,

\[
\chi_c(G) = \frac{\chi_c(G(\mathcal{I}))}{2t}.
\]
Examples

- If $\Gamma$ is a positive 2-path connecting $u$ and $v$, and $\mathcal{I} = (\Gamma, u, v)$, then for any $\epsilon$, $0 < \epsilon < 1$, and $r = 4 - 2\epsilon$,
  \[ Z(\mathcal{I}, r) = [0, 2 - 2\epsilon] = [0, \frac{r}{2} - \epsilon]. \]

- If $\Gamma'$ is a negative 2-path connecting $u$ and $v$, and $\mathcal{I}' = (\Gamma', u, v)$, then for any $\epsilon$, $0 < \epsilon < 1$, and $r = 4 - 2\epsilon$,
  \[ Z(\mathcal{I}', r) = [\epsilon, \frac{r}{2}]. \]

- If $\Gamma''$ consists of a negative 2-path and a positive 2-path connecting $u$ and $v$, and $\mathcal{I}'' = (\Gamma'', u, v)$, then for any $\epsilon$, $0 < \epsilon < 1$, and $r = 4 - 2\epsilon$,
  \[ Z(\mathcal{I}'', r) = [\epsilon, \frac{r}{2} - \epsilon]. \]
**Indicator construction** $S(G)$

Given a graph $G$, a signed graph $S(G)$ is built as follows.

**Corollary**

For any graph $G$,

$$\chi_c(S(G)) = 4 - \frac{4}{\chi_c(G) + 1}.$$
Signed indicators

Signed indicator

Lemma

Assume that $\mathcal{I}_+$ and $\mathcal{I}_-$ are indicators, $r \geq 2$ is a real number and

$$Z(\mathcal{I}_+, r) = [t, \frac{r}{2}], \quad Z(\mathcal{I}_-, r) = [0, \frac{r}{2} - t]$$

for some $0 < t < \frac{r}{2}$. Then for any signed graph $\Omega$,

$$\chi_c(\Omega) = \frac{\chi_c(\Omega(\mathcal{I}_+, \mathcal{I}_-))}{t}.$$
Tight cycle argument

Assume \((G, \sigma)\) is a signed graph and \(\phi : V(G) \to [0, r)\) is a circular \(r\)-coloring of \((G, \sigma)\). The partial orientation \(D = D_\phi(G, \sigma)\) of \(G\) with respect to a circular \(r\)-coloring \(\phi\) is defined as follows: 

\((u, v)\) is an arc of \(D\) if and only if one of the following holds:

- \(uv\) is a positive edge and 
  \((\phi(v) - \phi(u))(\text{mod } r) = 1\).
- \(uv\) is a negative edge and 
  \((\phi(v) - \phi(u))(\text{mod } r) = 1\).

Arcs in \(D_\phi(G, \sigma)\) are called tight arcs of \((G, \sigma)\) with respect to \(\phi\). A directed cycle in \(D_\phi(G, \sigma)\) is called a tight cycle with respect to \(\phi\).
Tight cycle argument

Lemma

Let \((G, \sigma)\) be a signed graph and let \(\phi\) be a circular \(r\)-coloring of \((G, \sigma)\). If \(D_\phi(G, \sigma)\) is acyclic, then there exists an \(r_0 \leq r\) such that \((G, \sigma)\) admits an \(r_0\)-circular coloring.

Notice that assume \(D_\phi(G, \sigma)\) is acyclic and among all such \(\phi\), \(D_\phi(G, \sigma)\) has minimum number of arcs, then \(D_\phi(G, \sigma)\) has no arc.

Lemma

Given a signed graph \((G, \sigma)\), \(\chi_c(G, \sigma) = r\) if and only if \((G, \sigma)\) is circular \(r\)-colorable and every circular \(r\)-coloring \(\phi\) of \((G, \sigma)\) has a tight cycle.
Tight cycle argument

Proposition

Any signed graph \((G, \sigma)\), which is not a forest, has a cycle with \(s\) positive edges and \(t\) negative edges such that

\[
\chi_c(G, \sigma) = \frac{2(s + t)}{2a + t}
\]

for some non-negative integer \(a\).

Corollary

Given a signed graph \((G, \sigma)\) on \(n\) vertices, \(\chi_c(G, \sigma) = \frac{p}{q}\) for some \(p \leq 2n\) and \(q\).
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4 Discussion
Classes of signed graphs

Given a class $\mathcal{C}$ of signed graphs,

$$\chi_c(\mathcal{C}) = \sup\{\chi_c(G, \sigma) | (G, \sigma) \in \mathcal{C}\}.$$ 

- $\mathcal{SBP}$ the class of signed bipartite planar simple graphs,
- $\mathcal{SD}_d$ the class of signed $d$-degenerate simple graphs,
- $\mathcal{SP}$ the class of signed planar simple graphs.
Signed bipartite planar graphs

Proposition

\[ \chi_c(SBP) = 4. \]

Let \( \Gamma_1 \) be a positive 2-path connecting \( u_1 \) and \( v_1 \). For \( i \geq 2 \),

Even step

Odd step
Signed bipartite planar graphs

Lemma

$$\chi_c(\Gamma_n) = \frac{4n}{n + 1}.$$
Signed bipartite planar graphs

Results on signed bipartite planar graphs with girth condition

- $\chi_c(SBP_6) \leq 3$. (Corollary of a result that every signed bipartite planar graph of negative girth 6 admits a homomorphism to $(K_{3,3}, M)$ [R. Naserasr and Z. Wang 2021+])

- $\chi_c(SBP_8) \leq \frac{8}{3}$. (Corollary of a result that $C_{-4}$-critical signed graph has density $|E(G)| \geq \frac{3|V(G)|-2}{4}$ [R. Naserasr, L-A. Pham and Z. Wang 2020+])
Proposition

For any positive integer $d$, $\chi_{c}(SD_d) = 2\lfloor \frac{d}{2} \rfloor + 2$.

Sketch of the proof:

- First we show that every $(G, \sigma) \in SD_d$ admits a circular $(2\lfloor \frac{d}{2} \rfloor + 2)$-coloring.

For the tightness,

- For odd integer $d$, we consider the signed complete graphs $(K_{d+1}, +)$.
- For $d = 2$, we consider the signed graph $\Gamma_n$ built before.
- For even integer $d \geq 4$, we construct a signed $d$-degenerate graph $(G, \sigma)$ such that $\chi_{c}(G, \sigma) = d + 2$.
Signed \( d \)-degenerate graphs

Proof for even \( d \geq 4 \)

- Define a signed graph \( \Omega_d \) as follows.

\[
(\mathcal{K}_d, +)
\]

For any \( i \neq j \in [d] \)

- Let \( \varphi \) be a circular \( r \)-coloring of \( \Omega_d \) where \( r < d + 2 \).

Without loss of generality, \( \varphi(x_1), \ldots, \varphi(x_d) \) are cyclically ordered on \( C^r \) and assume that \( d_{\text{mod } r}(\varphi(x_1), \varphi(x_2)) \) is maximized. We prove that there is no place for \( y_{1, 1+\frac{d}{2}} \).
Proposition

\[ 4 + \frac{2}{3} \leq \chi_c(SP) \leq 6. \]
Signed planar graphs

$\ell_{\phi;u,v}$: the minimum length of an interval which contains $\phi(u) \cup \phi(v)$.

**Lemma**

Let $r = \frac{14}{3} - \epsilon$ with $0 < \epsilon \leq \frac{2}{3}$. For any circular $r$-coloring $\phi$ of $\tilde{W}$, $\ell_{\phi;u,v} \geq \frac{4}{9}$.

Let $\Gamma$ be obtained from $\tilde{W}$ by adding a negative edge $uv$. Let $\mathcal{I} = (\Gamma, u, v)$.

**Theorem**

Let $\Omega = K_4(\mathcal{I})$. Then $\Omega$ is a signed planar simple graph with $\chi_c(\Omega) = \frac{14}{3}$. 
Signed planar graphs

Sketch of the proof of the theorem

• First we show that $\Omega$ admits a circular $\frac{14}{3}$-coloring. We find a circular $\frac{14}{3}$-coloring $\phi$ of $\Gamma$ such that $\phi(u) = \phi(v) = 0$ and then extend it to each of inner triangles.

• Let $\phi$ be a circular $r$-coloring of $\Omega$ for $r < \frac{14}{3}$. For any $1 \leq i < j \leq 4$, $\frac{4}{9} \leq d(\text{mod } r)(\phi(v_i), \phi(v_j)) \leq \frac{r}{2} - 1$.

Assume that $\phi(x_1), \phi(x_2), \phi(x_3), \phi(x_4)$ are on $C^r$ in this cyclic order.

- $\ell([\phi(v_1), \phi(v_4)]) = \ell([\phi(v_1), \phi(v_2)]) + \ell([\phi(v_2), \phi(v_3)]) + \ell([\phi(v_3), \phi(v_4)]) \geq 3 \times \frac{4}{9} = \frac{4}{3} > \frac{r}{2} - 1$,
- $\ell([\phi(v_4), \phi(v_1)]) \geq r - (\ell([\phi(v_1), \phi(v_3)]) + \ell([\phi(v_2), \phi(v_4)])) \geq 2 > \frac{r}{2} - 1$.

Contradiction.
Signed planar graphs

Results on signed planar graphs with girth condition

- $\chi_c(\mathcal{SP}_4) \leq 4$. (By the 3-degeneracy of triangle-free planar graph)
- $\chi_c(\mathcal{SP}_7) \leq 3$. (Corollary of a result that every signed graph of $\text{mad} < \frac{14}{5}$ admits a homomorphism to $(K_6, M)$ [R. Naserasr, R. Škrekovski, Z. Wang and R. Xu 2020+])
Signed circular chromatic number

For a simple graph $G$, the signed circular chromatic number $\chi^s_c(G)$ of $G$ is defined as

$$\chi^s_c(G) = \max\{\chi_c(G, \sigma) : \sigma \text{ is a signature of } G\}.$$ 

**Proposition**

For every graph $G$, $\chi^s_c(G) \leq 2\chi_c(G)$. 
Signed planar graphs

Signed chromatic number of k-chromatic graph

Theorem

For any integers $k, g \geq 2$ and any $\epsilon > 0$, there is a graph $G$ of girth at least $g$ satisfying that $\chi(G) = k$ and $\chi_c^s(G) > 2k - \epsilon$.

Assume $k, g \geq 2$ are integers. We will prove that for any integer $p$, there is a graph $G$ for which the followings hold:

- $G$ is of girth at least $g$ and has chromatic number at most $k$.
- There is a signature $\sigma$ such that $(G, \sigma)$ is not $(2kp, (p + 1))$-colorable.
Signed planar graphs

Augmented tree

- A complete $k$-ary tree is a rooted tree in which each non-leaf vertex has $k$ children and all the leaves are of the same level.
- An $q$-augmented $k$-ary tree is obtained from a complete $k$-ary tree by adding, for each leaf $v$, $q$ edges connecting $v$ to $q$ of its ancestors. These $q$ edges are called the augmenting edges from $v$.
- For positive integers $k, q, g$, a $(k, q, g)$-graph is a $q$-augmented $k$-ary tree which is bipartite and has girth at least $g$.

Lemma [Alon, N., Kostochka, A., Reiniger, B., West, D., and Zhu, X 2016]

For any positive integers $k, q, g \geq 2$, there exists a $(k, q, g)$-graph.
Augmented tree

- A **standard labeling** of a complete $k$-ary tree $T$;
- A $f$-path $P_f$ of $T$ with respect to a given $k$-coloring $f$. 

Signed planar graphs
Construction of k-chromatic graph $G$

- $H$: $(2kp, k, 2kg)$-graph with underline tree $T$.
- $\phi$: a standard $2kp$-labeling of the edges of $T$.
- $\ell(v)$: the level of $v$, i.e., the distance from $v$ to the root vertex in $T$. Let $\theta(v) = \ell(v)(\text{mod } k)$.

For each leaf $v$ of $T$, let $u_{v,1}, u_{v,2}, \ldots, u_{v,k}$ be the vertices on $P_v$ that are connected to $v$ by augmenting edges. Let $u'_{v,i} \in P_v$ be the closest descendant of $u_{v,i}$ with $\theta(u'_{v,i}) = i$ and let $e_{v,i}$ be the edge connecting $u'_{v,i}$ to its child on $P_v$.

Let $s_{v,i} = \phi(e_{v,i})$ and let

- $A_{v,i} = \{s_{v,i}, s_{v,i} + 1, \ldots, s_{v,i} + p\}$,
- $B_{v,i} = \{a + kp : a \in A_{v,i}\}$,
- $C_{v,i} = A_{v,i} \cup B_{v,i}$.
Construction of the signature $\sigma$ on $G$

Note that $B_{v,i}$ is a $kp$-shift of $A_{v,i}$. Two possibilities:

- $A_{v,i} \cap A_{v,j} \neq \emptyset$ (then $B_{v,i} \cap B_{v,j} \neq \emptyset$)

  \[
  d_{(\text{mod } 2kp)}(\phi(e_{v,i}), \phi(e_{v,j})) \leq p.
  \]

- $A_{v,i} \cap B_{v,j} \neq \emptyset$ (then $B_{v,i} \cap A_{v,j} \neq \emptyset$)

  \[
  d_{(\text{mod } 2kp)}(\phi(e_{v,i}), \overline{\phi(e_{v,j})}) \leq p.
  \]

Let $L$ be the set of leaves of $T$. For each $v \in L$, we define one edge $e_v$ on $V(T)$ as follows:

- If $d_{(\text{mod } 2kp)}(\phi(e_{v,i}), \phi(e_{v,j})) \leq p$, then let $e_v$ be a positive edge connecting $u'_{v,i}$ and $u'_{v,j}$.

- If $d_{(\text{mod } 2kp)}(\phi(e_{v,i}), \overline{\phi(e_{v,j})}) \leq p$, then let $e_v$ be a negative edge connecting $u'_{v,i}$ and $u'_{v,j}$.
Proof for "\((G, \sigma)\) is not circular \(\frac{2kp}{p+1}\)-colorable"

Let \((G, \sigma)\) be the signed graph with vertex set \(V(T)\) and with edge set \(\{e_v : v \in L\}\), where the signs of the edges are defined as above.

- Assume \(f\) is a \((2kp, p + 1)\)-colorable of \((G, \sigma)\).
- As \(f\) is also a \(2kp\)-coloring of the vertices of \(T\), there is a unique \(f\)-path \(P_v\). Assume that \(e_v = u'_{v,i}u'_{v,j}\). By definition,

\[ f(u'_{v,i}) = \phi(e_v, i) \text{ and } f(u'_{v,j}) = \phi(e_v, j). \]

- If \(e_v\) is a positive edge, then \(d_{(\mod 2kp)}(\phi(e_v, i), \phi(e_v, j)) \leq p\).
- If \(e_v\) is a negative edge, then \(d_{(\mod 2kp)}(\phi(e_v, i), \phi(e_v, j)) \leq p\).

Contradiction.
1 Introduction
   - Circular coloring of graphs
   - Homomorphism of signed graphs

2 Circular coloring of signed graphs
   - Circular chromatic number
   - Signed indicators
   - Tight cycle argument

3 Results on some classes of signed graphs
   - Signed bipartite planar graphs
   - Signed d-degenerate graphs
   - Signed planar graphs

4 Discussion
Mapping signed graphs to signed cycles

Let $C_{\ell}^{o+}$ be signed cycle of length $\ell$ where the number of positive edges is odd. Then $\chi_c(C_{\ell}^{o+}) = \frac{2\ell}{\ell - 1}$.

**Theorem**

Given a positive integer $\ell$ and a signed graph $(G, \sigma)$ satisfying $g_{ij}(G, \sigma) \geq g_{ij}(C_{\ell}^{o+})$ for $ij \in \mathbb{Z}_2^2$, we have $\chi_c(G, \sigma) \leq \frac{2\ell}{\ell - 1}$ if and only if $(G, \sigma) \rightarrow C_{\ell}^{o+}$. 
Circular chromatic number of signed planar graphs

Question

Given a positive integer $\ell$, what is the smallest value $f(\ell)$ (with $f(\infty) = \infty$) such that for every signed planar graph $(G, \sigma)$ satisfying $g_{ij}(G, \sigma) \geq g_{ij}(C_{\ell}^+)/2$ and $g_{ij}(G, \sigma) \geq f(\ell)$ for all $ij \in \mathbb{Z}_2^2$, we have $\chi_c(G, \sigma) \leq \frac{2\ell}{\ell - 1}$. 
Jaeger-Zhang conjecture

When $\ell = 2k + 1$,

Jaeger-Zhang conjecture [C.-Q. Zhang 2002]

Every planar graph of odd-girth $f(2k + 1) = 4k + 1$ admits a circular $\frac{2k+1}{k}$-coloring, i.e., $C_{2k+1}$-coloring.

- $f(3) = 5$ [Grötzsch’s theorem];
- $f(5) \leq 11$ [Z. Dvořák and L. Postle 2017][D. W. Cranston and J. Li 2020];
- $4k + 1 \leq f(2k + 1) \leq 6k + 1$ [C. Q. Zhang 2002; L. M. Lovász, C. Thomassen, Y. Wu and C. Q. Zhang 2013];
Bipartite analogue of Jaeger-Zhang conjecture

When $\ell = 2k$,

Every signed bipartite planar graph of negative-girth $f(2k)$ admits a circular $\frac{4k}{2k-1}$-coloring, i.e., $C_{-2k}$-coloring.

- $f(4) = 8$ [R. Naserasr, L. A. Pham and Z. Wang 2020+];
  ($f(2k) > 4k - 2$ when $k = 2$.)
- $f(2k) \leq 8k - 2$ [C. Charpentier, R. Naserasr and E. Sopena 2020].
Odd-Hadwiger Conjecture

Theorem [P.A. Catlin 1979]
If \((G, -)\) has no \((K_4, -)\)-minor, then \(\chi_c(G, +) \leq 3\).

The Odd-Hadwiger conjecture was proposed independently by B. Gerard and P. Seymour.

Odd-Hadwiger conjecture
If a signed graph \((G, -)\) has no \((K_{k+1}, -)\)-minor, then \(\chi_c(G, +) \leq k\).

Question
Assuming \((G, \sigma)\) has no \((K_{k+1}, -)\)-minor, what is the best upper bound on \(\chi_c(G, -\sigma)\)?
The end. Thank you!