Signed projective cubes

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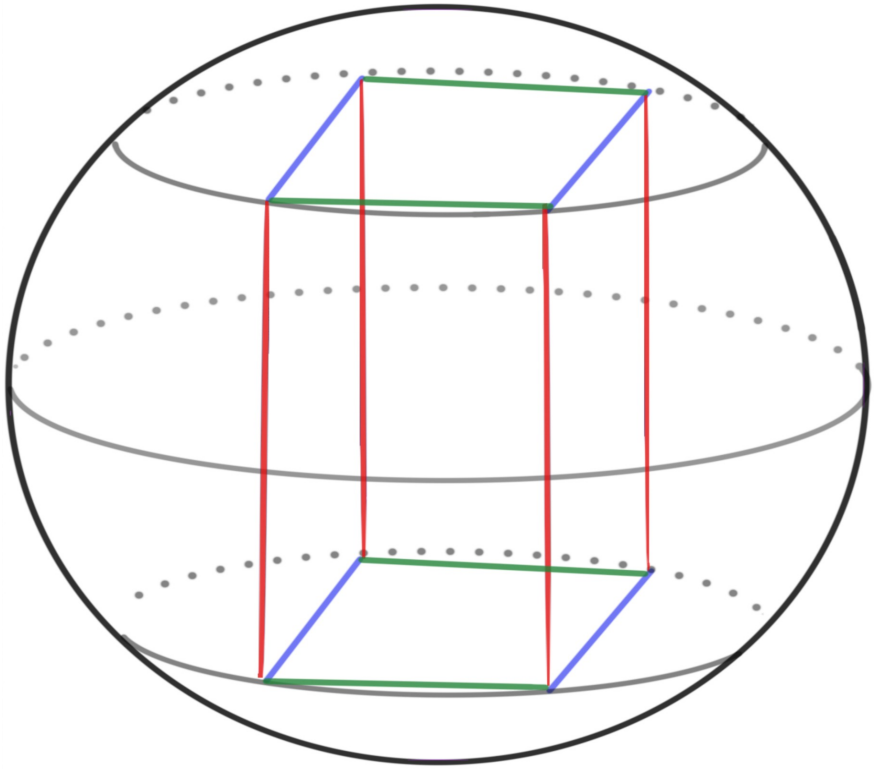
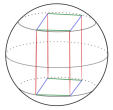
IRIF, CNRS

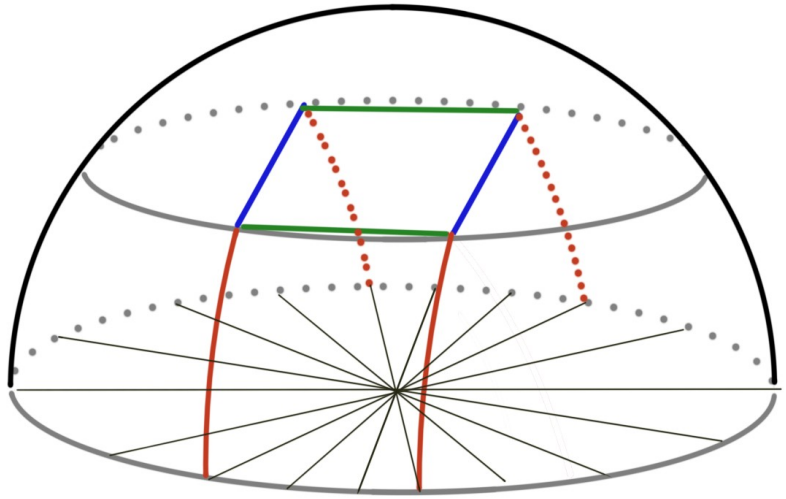
Flajolet Seminar, 25 Nov. 2021

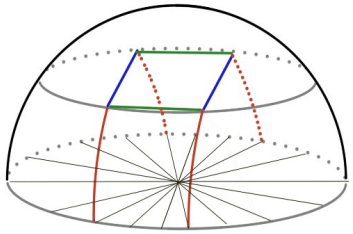
H_d embeded in S^d

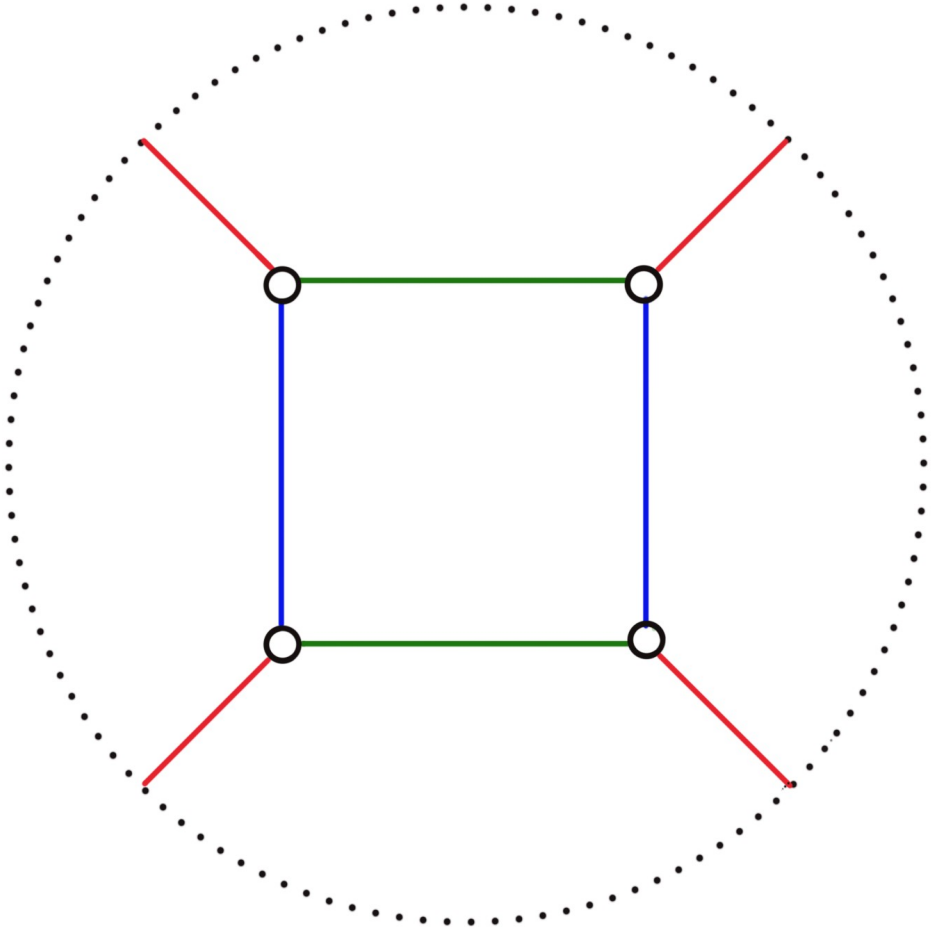


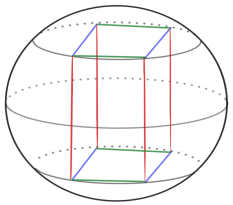


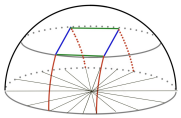
Projection of S^3











Equivalent definition,

As a Cayley signed graph:

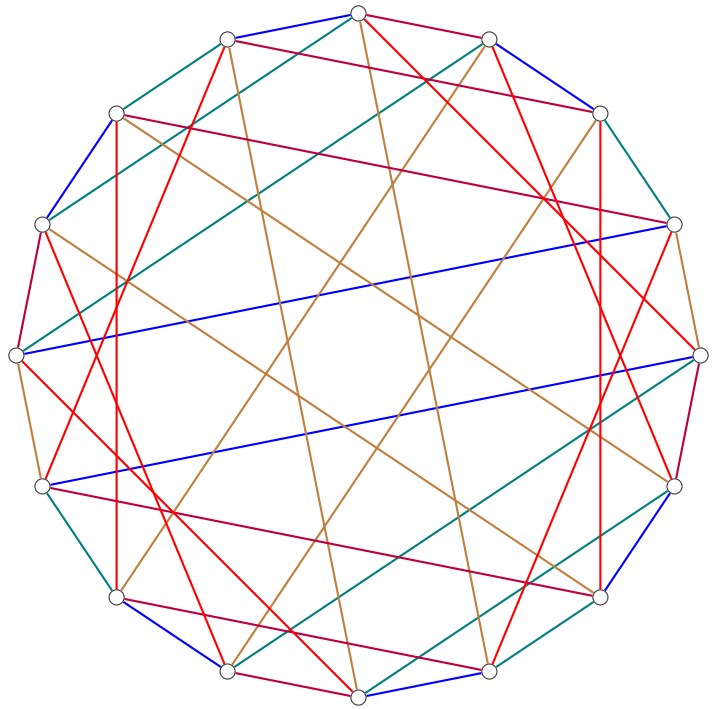
$$SPC(d) := \left(\mathbb{Z}_2^d, \{e_1, e_2, e_3, \cdots e_d, J\}\right)$$

Equivalent definition,

As a Cayley signed graph:

$$\left(\mathbb{Z}_2^k, \{s_1, s_2, \cdots, s_k, s_{k+1}\}\right)$$

minimally $s_1 + s_2 + \cdots + s_k + s_{k+1} = 0$



$PC(4)$: $(\mathbb{Z}_2^4, (e_1, e_2, e_3, e_4, J))$

Questions such as symmetries, inductive definition ...

allready motivates study of signed graphs.

Questions such as symmetries, inductive definition ...

allready motivates study of signed graphs.

Meredith: H_d and $PC(2d)$ are the only
(1974)
non-trivial triple transitive graphs.

Extension using signed graphs captures $PC(2d-1)$.

signed graph: A graph where each edge is assigned a sign

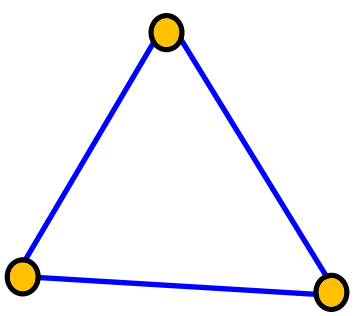
+ positive - negative

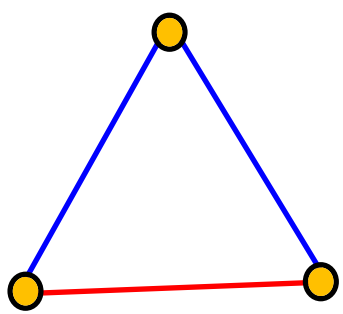
Notation: (G, σ)

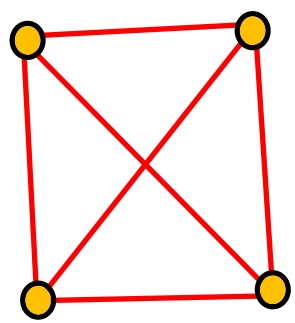
G → graph

σ → signature

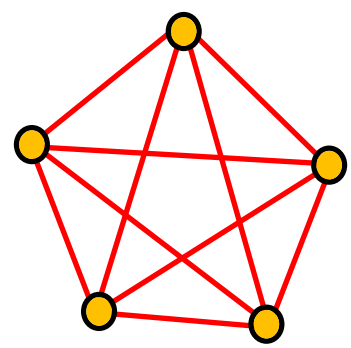
Examples:







$(K_4, -)$



$(K_5, -)$

Main terminology: positive cycle (balanced cycle) negative cycle (unbalanced cycle)

↓

reflecting the fact that the rule of "friend of a friend is a friend" applies.

positive/negative closed walk

Balanced signed graph: signed graph with no negative cycle

Antibalanced signed graph: where odd cycles are negative
even cycles are positive.

Switching (what makes it different from 2-edge-colored graphs).

To multiply signs of edges incident to a vertex to $-$

Switching at all vertices of a subset X of vertices:

To multiply signs of all edges in the edge cut $(X, V-X)$ to a $-$

Geometric presentation of switching

$DC(G,\sigma)$: Double Cover of (G,σ)

vertices: two copies of $V(G)$ $\quad V_L(G) \cup V_R(G)$

edges:
* If xy is a positive edge, then
  $x_L \sim y_L$ & $x_R \sim y_R$
* If xy is a negative edge, then
  $x_L \sim y_R$ & $x_R \sim y_L$

Example

EDC (Extended Double Cover)

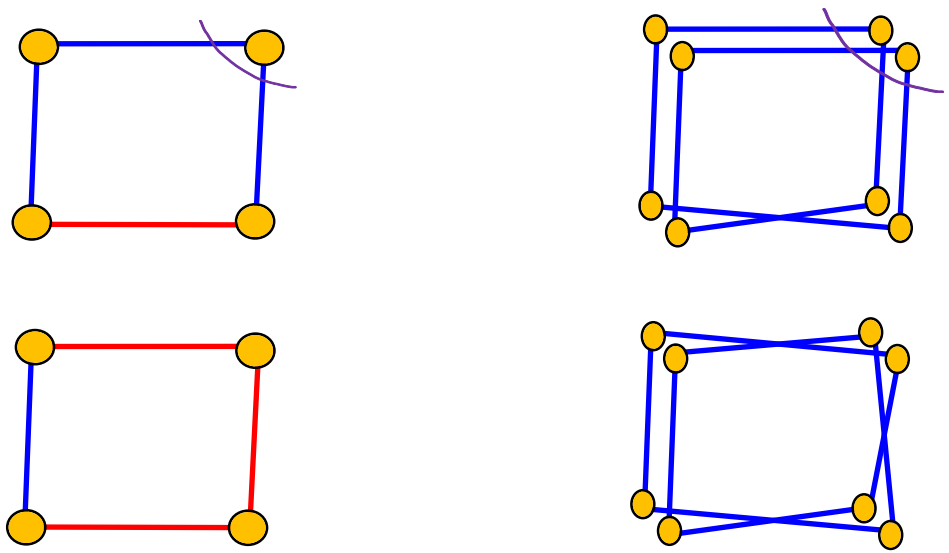
(G, σ)

switch at a vertex x

$DC(G, \sigma)$

switch the place of x_L with x_R

Example

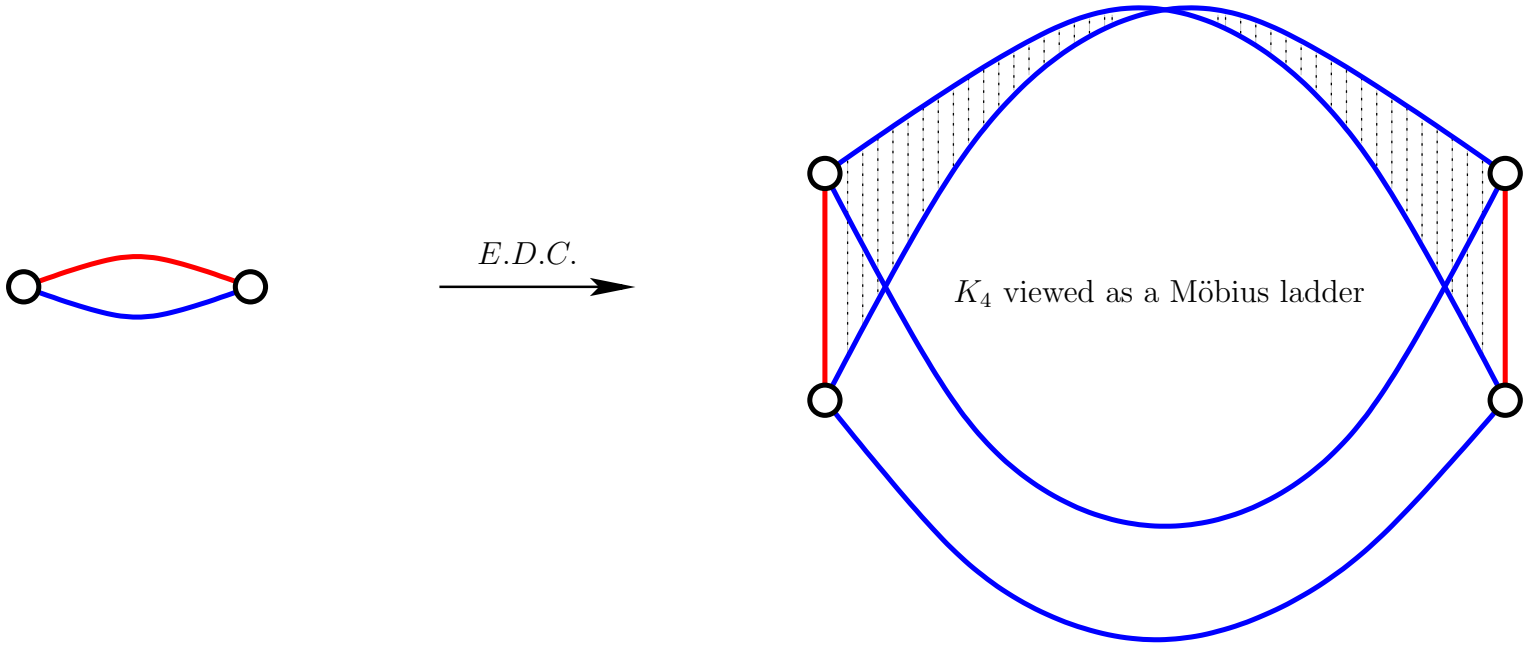


$EDC(G, \sigma)$: Extended Double Cover of (G, σ)

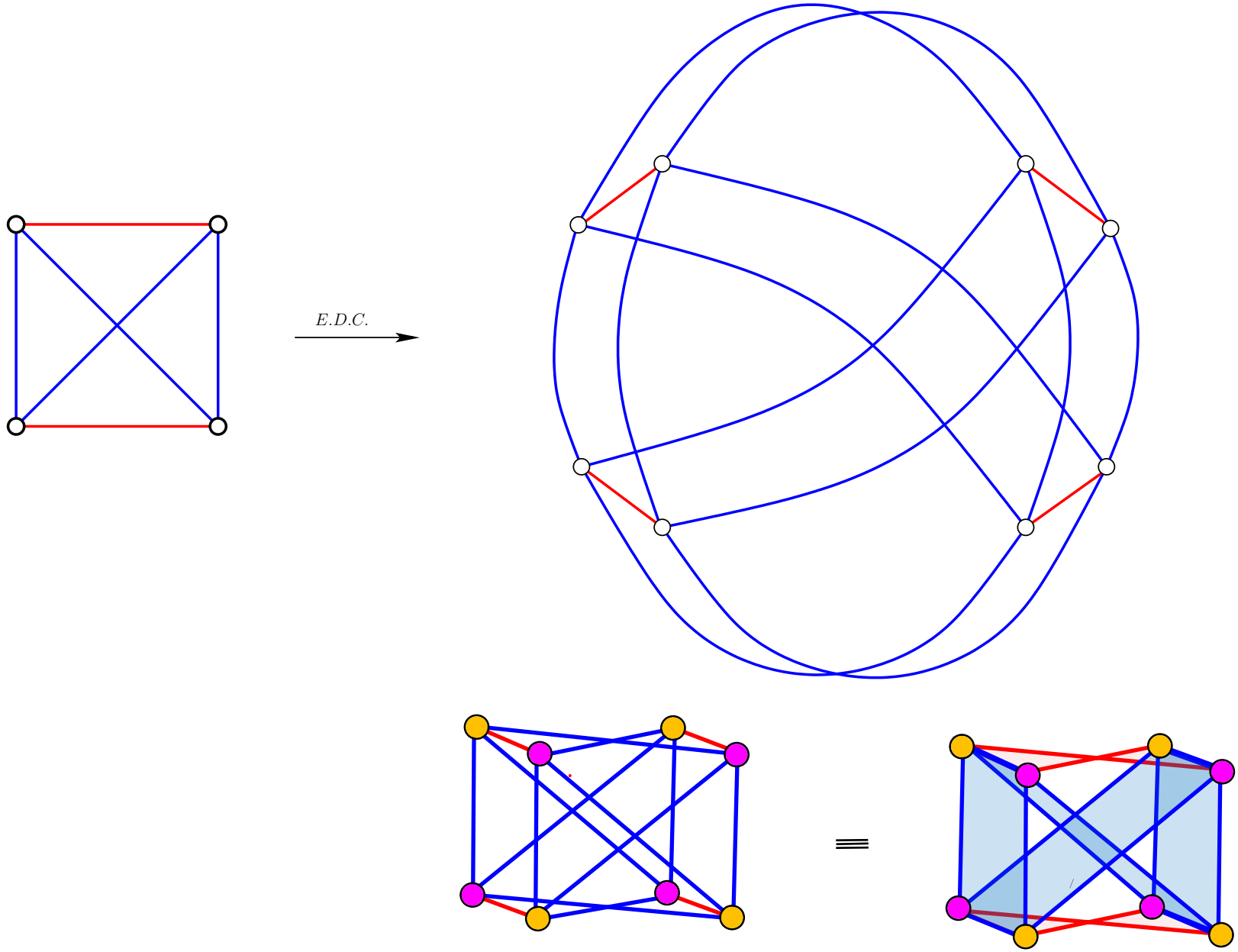
vertices: two copies of $V(G)$ $\quad V_L(G) \cup V_R(G)$

positive edges:
* If xy is a positive edge, then $x_L \sim y_L$ & $x_R \sim y_R$
* If xy is a negative edge, then $x_L \sim y_R$ & $x_R \sim y_L$

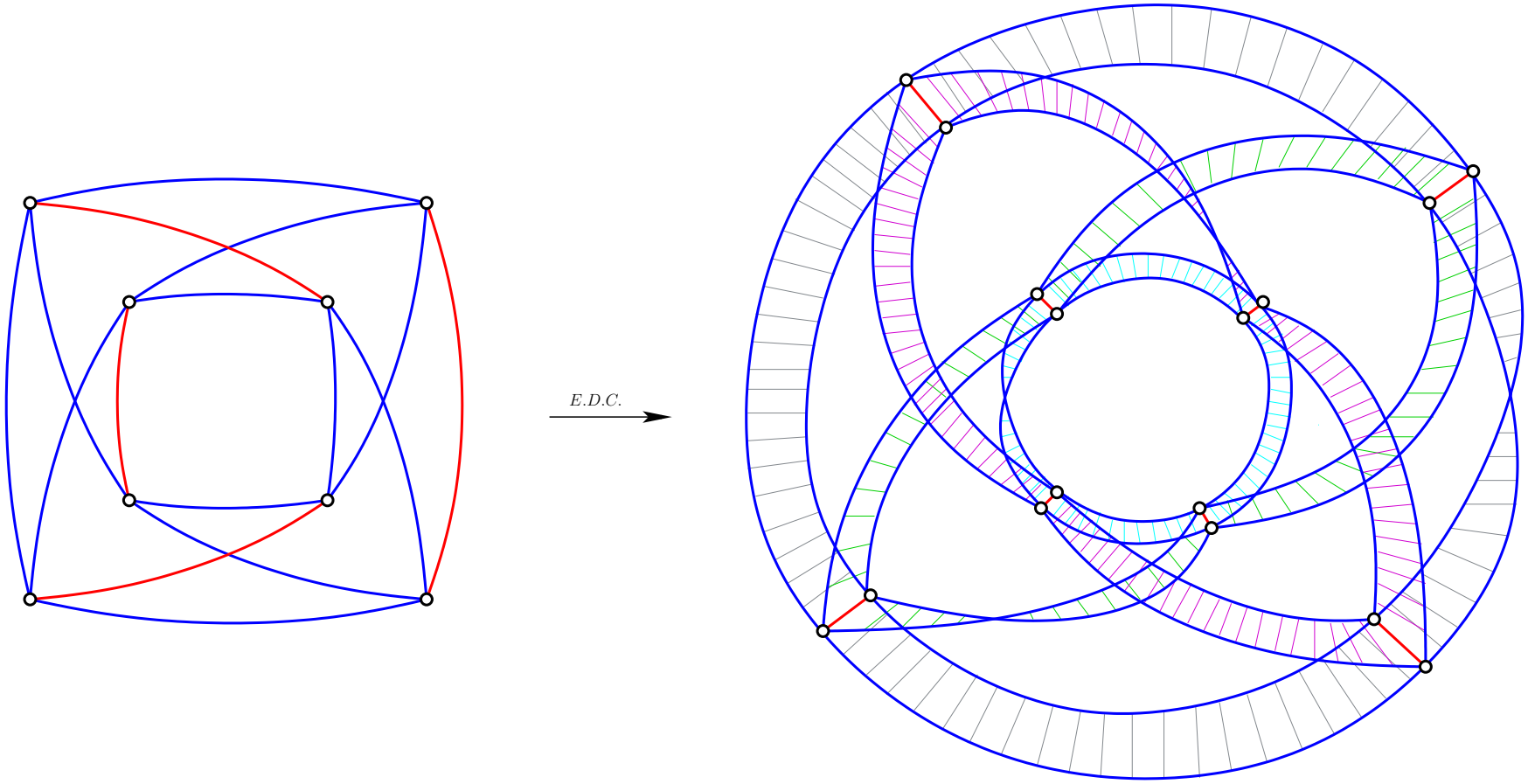
negative edges: $x_L \sim x_R$



$$SPC(2) = EDC(SPC(1)$$



Stripes and twists in $K_{4,4}$

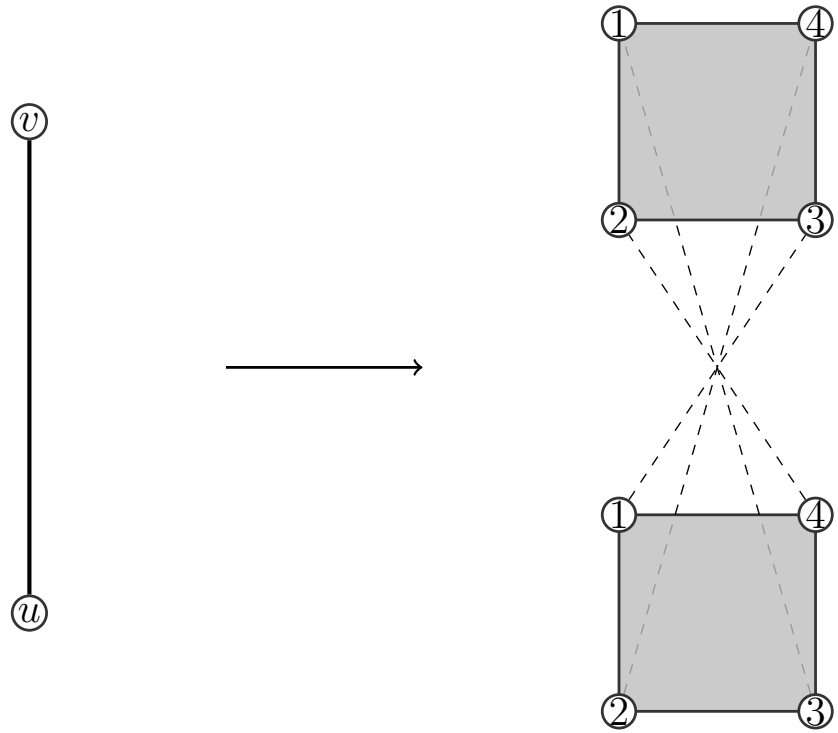


Intersection graph of lines in an algebraic surface

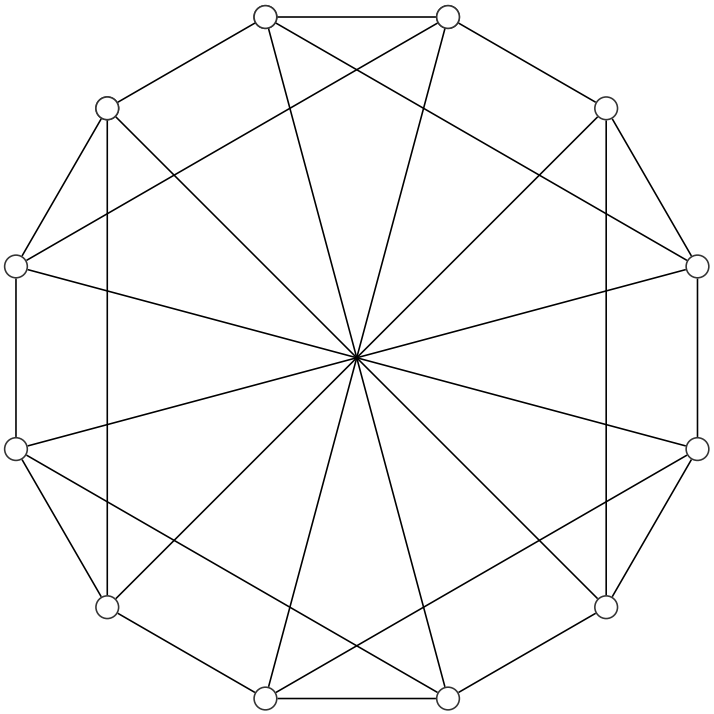
Clebsch, Segre, Del Pezza (of degree 4)?

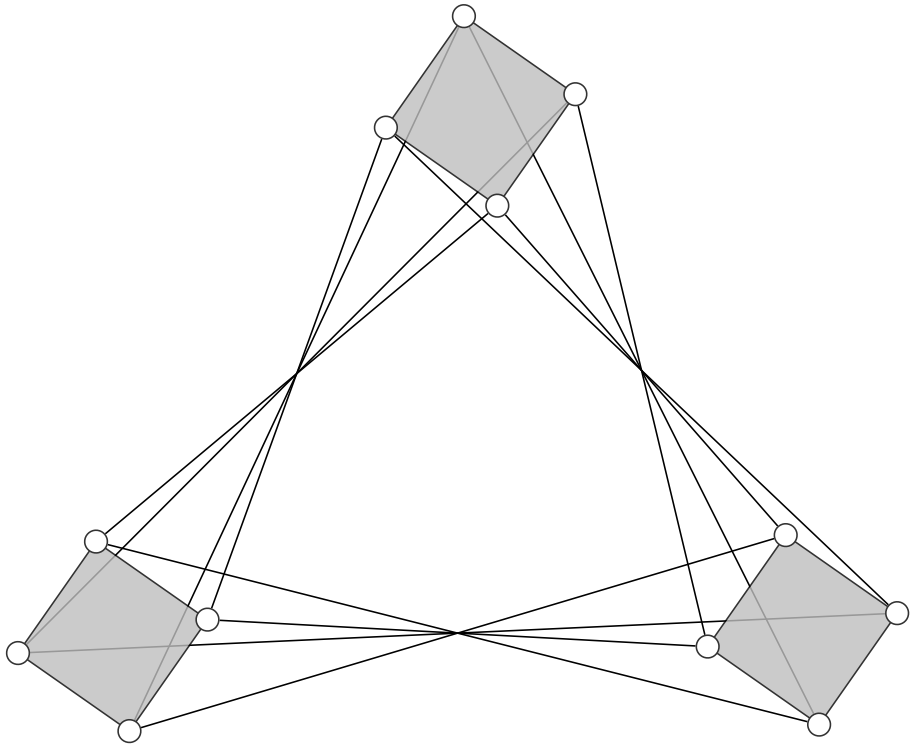
Greenwood-Gleason graph (for application in Ramsey theory)

Clebsch graph viewed as combinations of Möbius ladders

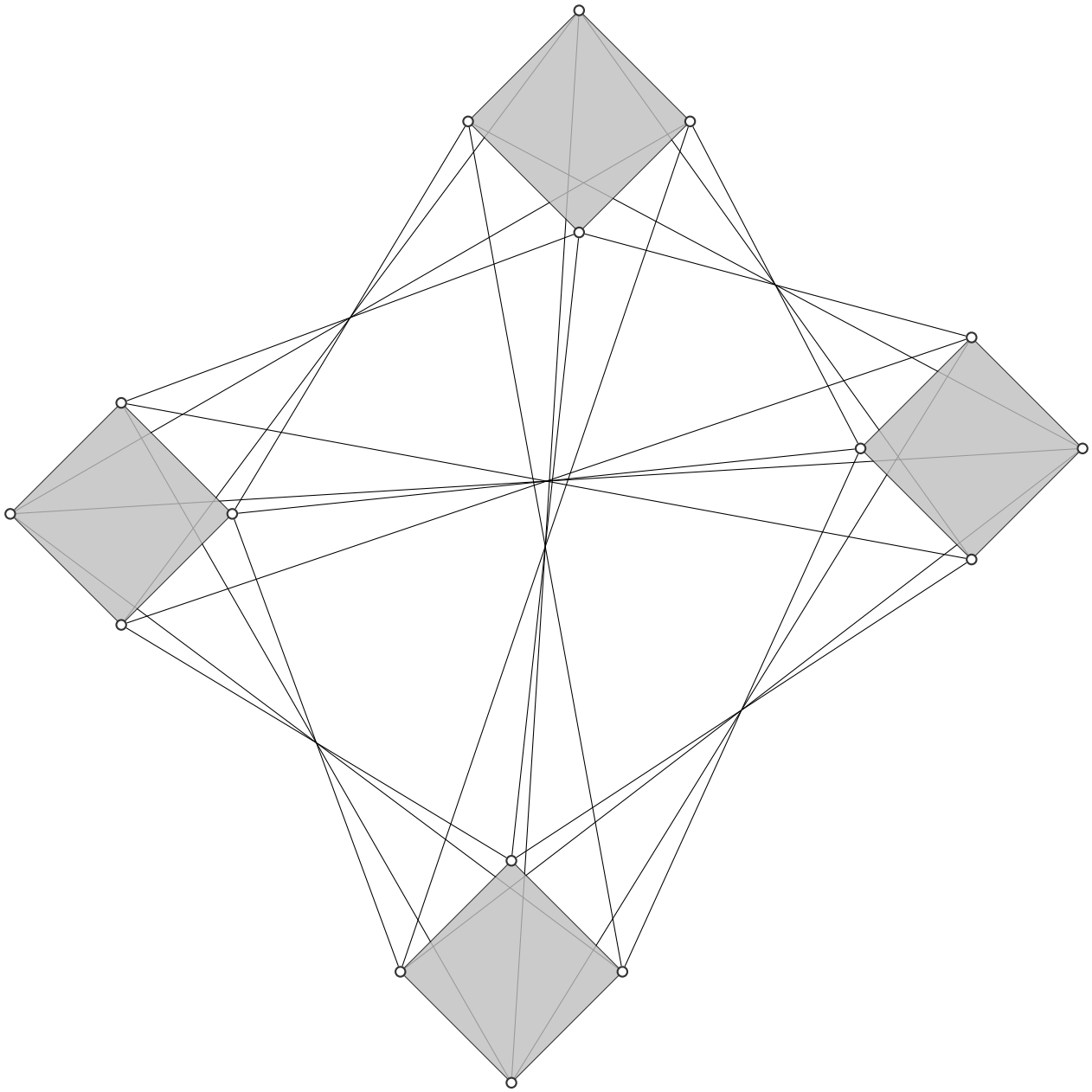


$K_2 * C_4$





Two presentations of $K_3 * C_4$



Other definitions of SPC(K):

- Product of signed graphs

- Power graphs

- Poset of partition pairs

Properties of SPC(d):

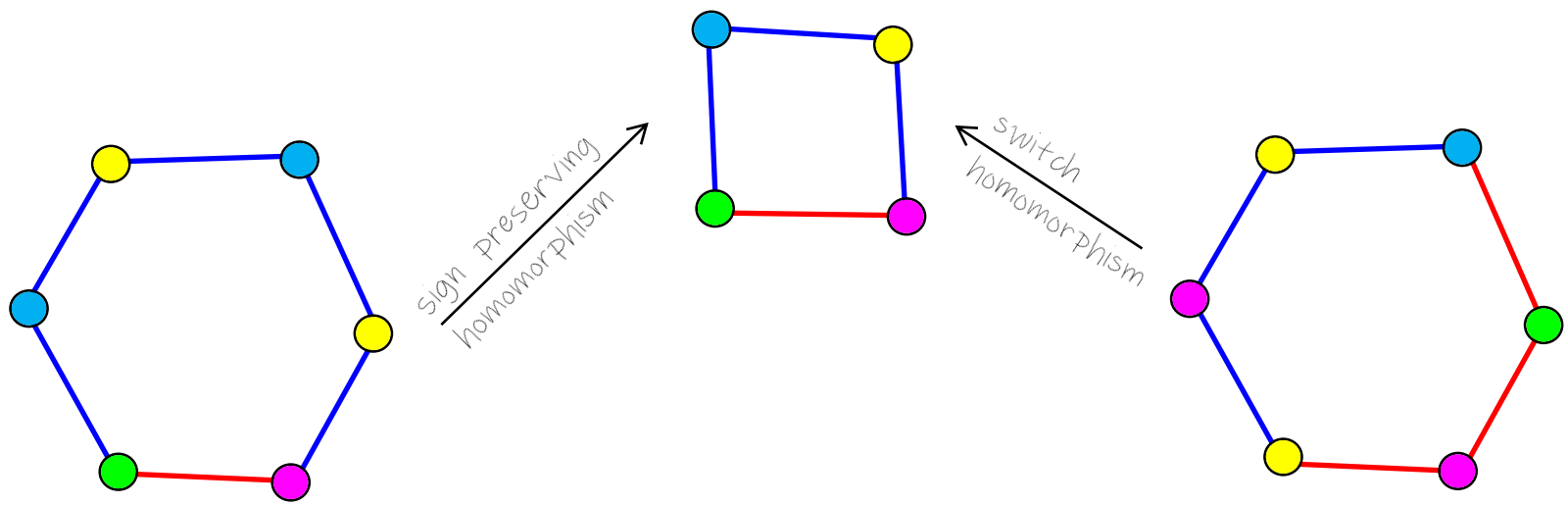
* Its negative girth is $d+1$.
* Every negative cycle is of same parity as $d+1$.
* Every positive cycle is even.

Homomorphisms of signed graphs.

Definition. Given signed graphs (G,σ) & (H,π) a mapping of $V(G)$ to $V(H)$ (and $E(G)$ to $E(H)$) is said to be a <u>homomorphism</u> of (G,σ) to (H,π) if it preserves djacencies, (incidences) and signs of closed walks.

It is said to be <u>edge-sign prserving homomorphism</u> if it furthermore, preserves signs of edges.

Examples



Comment. The edge mapping is implied unless (H, π) contains a digon.

Theorem. Signed graph (G, σ) admits a homomorphism to signed graph (H, π) if and only if for some switching (G, σ') there exists an edge-sign preserving homomorphism of (G, σ') to (H, π).

General references for homomorphisms of signed graphs

Naserasr
Rollova
Sopena
2015

Naserasr
Sopena
Zaslavsky
2021

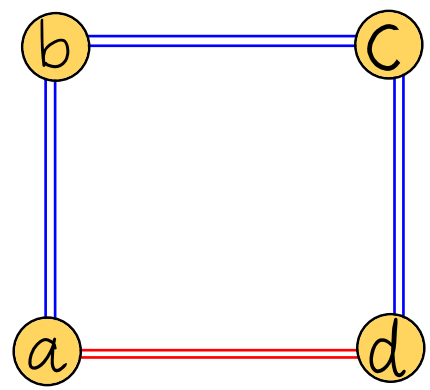
Important note:

Notions of "isomorphism" and "automorphism" depend on our view and choice of homomorphism: edge-sign preserving homomorphism or switch homomorphism. Associated definitions like "vertex transitive" and "edge-transitive" change accordingly.

under edge-sign preserving homomorphism

the only automorphism:

$$a \rightleftarrows d$$
$$b \rightleftarrows c$$



under switch homomorphism

Dihedral group D_4 (8 elements)

Thus C_{-4} is:

vertex-transitive &

edge-transitive.

Observation. In a mapping of (G,σ) to (H,π) the image of every closed walk is a closed walk which has a same (parity) of length and a same sign.

This leads to four notions of girth:

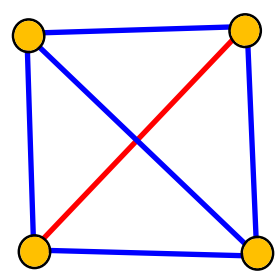
$g_{00}(G,\sigma)$: length of shortest positive even closed walk,

$g_{10}(G,\sigma)$: length of shortest negative even closed walk,

$g_{01}(G,\sigma)$: length of shortest positive odd closed walk,

$g_{11}(G,\sigma)$: length of shortest negative odd closed walk.

Example.



$g_{00}(G,\sigma)=2$, $g_{10}(G,\sigma)=4$, $g_{01}(G,\sigma)=3$, $g_{11}(G,\sigma)=3$

Theorem. A signed graph $(G\ \sigma)$ maps to $SPC(d)$ if and only if $E(G)$ can be partitioned into $E_1, E_2, \ldots E_{d+1}$ such that (G, E_i), for each i, is switch equivalent to (G, σ).

Naserasr
Rollova
Sopena
2013

Strengthen to a general packing problem is part of Weiqiang Yu's Ph.D. project.

Conjecture. Given a signed planar graph (G, σ), if

[Naserasr, Guenin]

$$g_{ij}(G, \sigma) \geq g_{ij}(SPC(d)) \text{ for every } ij \in \mathbb{Z}_2^2,$$

then $(G, \sigma) \longrightarrow SPC(d)$.

$PC(2k) \longrightarrow PC(2k-2) \quad \cdots \quad PC(6) \longrightarrow PC(4) \longrightarrow PC(2)$

$\uparrow$?

planar graphs of odd-grth at least $2k+1$

$\uparrow$ ✓

planar graphs of odd-girth at least 7

$\uparrow$ ✓

triangle-free planargraphs

$\uparrow$ ✓

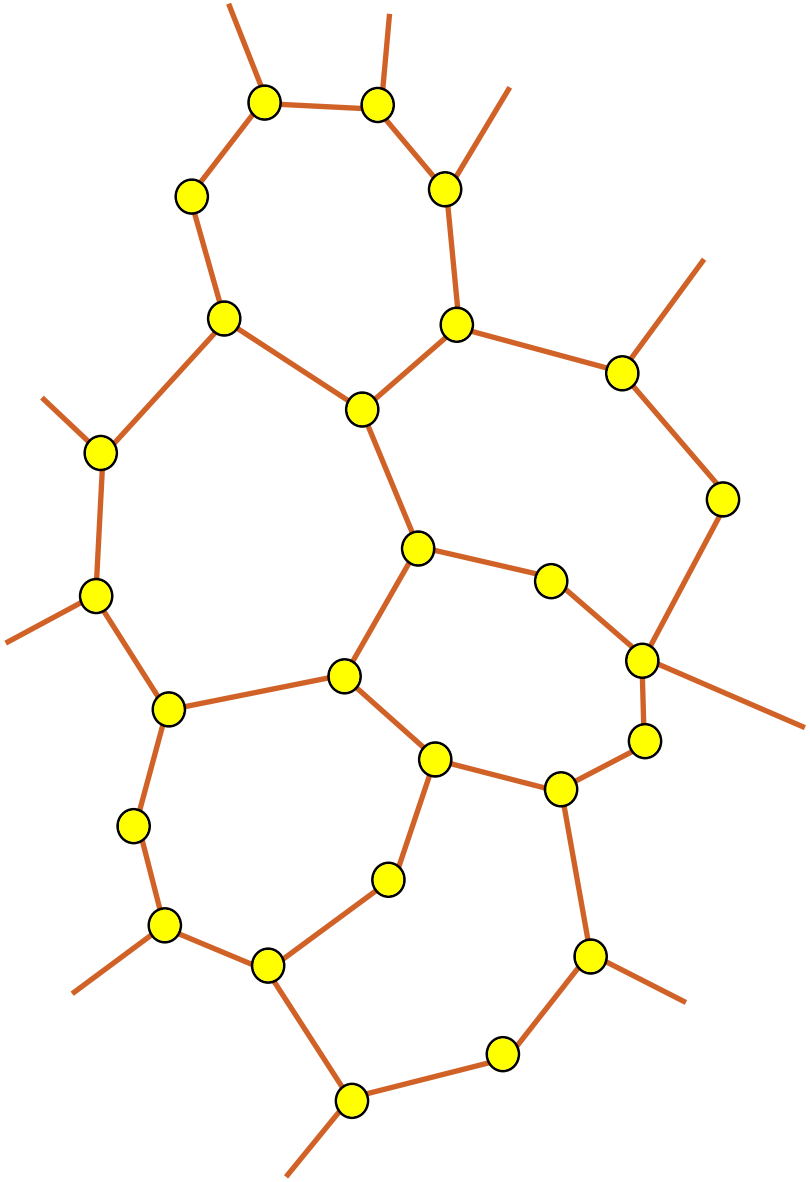
planar graphs

$PC(2k) \longrightarrow PC(2k-2) \quad \cdots \quad PC(6) \longrightarrow PC(4) \longrightarrow PC(2)$

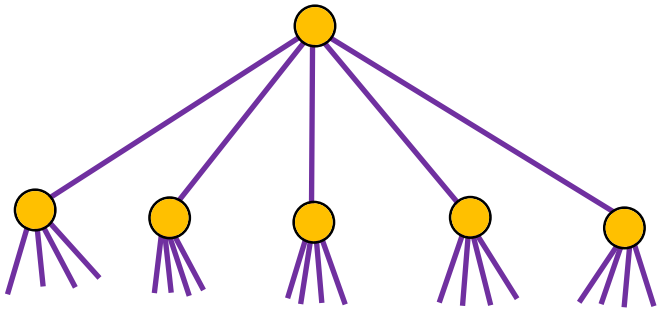
↑ ?		↑ ✓	↑ ✓	↑ ✓
planar graphs of odd-girth at least $2k+1$		planar graphs of odd-girth at least 7	triangle-free planargraphs	planar graphs

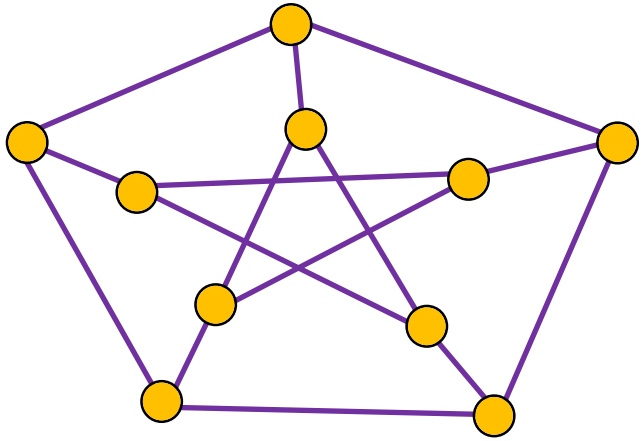
Analogous questions are asked for mapping signed bipartite planar graphs to $SPC(2d-1)$.

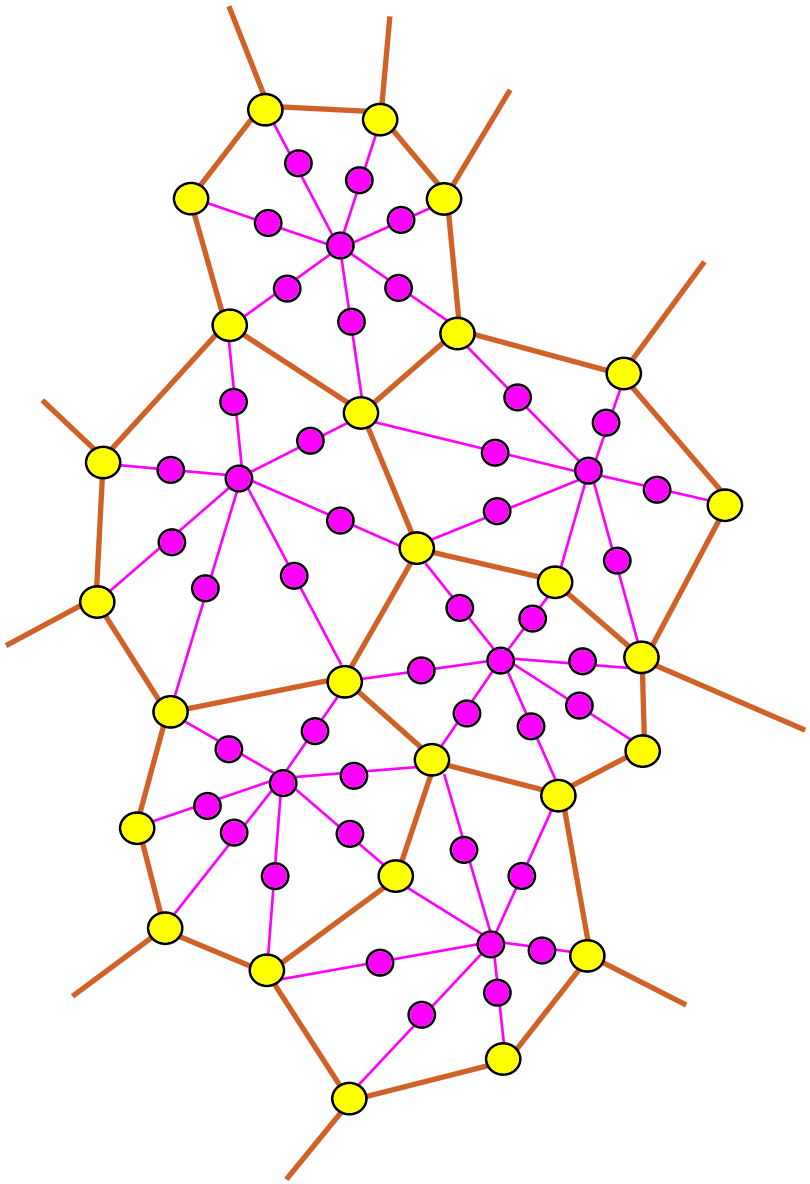
$SPC(2k+1)$	$SPC(2k-1)$	$\cdots$ $SPC(7)$	$SPC(5)$	$SPC(3)$
↑ ?		↑ ✓	↑ ✓	↑ ✓
signed bipartite planar graphs of negative girth $2k+2$		signed bipartite planar graphs of negative girth 8	signed bipartite planar graphs of negative girth 6	signed bipartite planar simple graphs



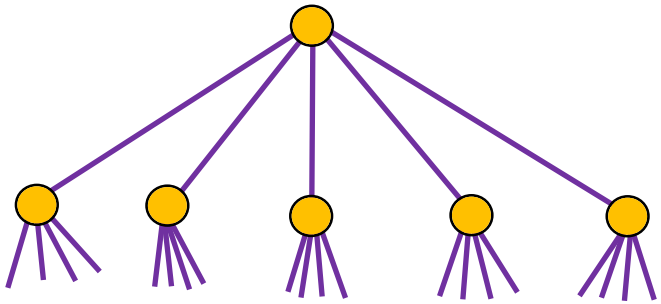
PC(4)

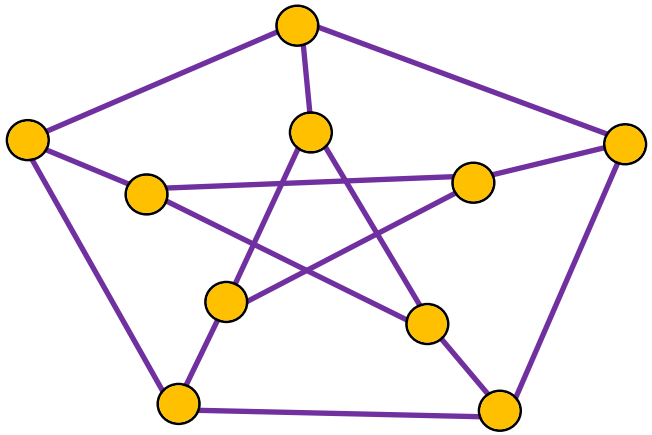






PC(4)





Conjecture. Every planar graph of odd-girth at least $2k+3$ maps to $K(2k+1, k)$.

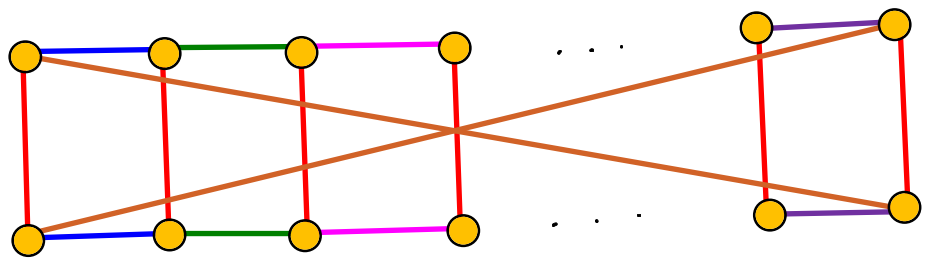
Would imply the best possible bound of $2 + \frac{1}{k}$ for the fractional chromatic number of this class.

Conjecture. Every planar graph of odd-girth at least $4k+1$ maps to C_{2k+1}.

[Jaeger–Zhang]

Would imply the best possible bound of $2+\frac{1}{k}$ for the circular chromatic number of this class.

Conjecture. Every planar graph of odd-girth at least $4k-1$ maps to M_{2k+1}.



Would imply the best possible bound of $2+\frac{2}{2k-1}$ for the circular chromatic number of this class.

$PC(2k) \longrightarrow PC(2k-2) \quad \cdots \quad PC(6) \longrightarrow PC(4) \longrightarrow PC(2)$

↑ ?		↑ ✓	↑ ✓	↑ ✓
planar graphs of odd-girth at least $2k+1$		planar graphs of odd-girth at least 7	triangle-free planargraphs	planar graphs

Analogous questions are asked for mapping signed bipartite planar graphs to $SPC(2d-1)$.

$SPC(2k+1) \quad SPC(2k-1) \quad \cdots \quad SPC(7) \quad SPC(5) \quad SPC(3)$

↑ ?		↑ ✓	↑ ✓	↑ ✓
signed bipartite planar graphs of negative girth $2k+2$		signed bipartite planar graphs of negative girth 8	signed bipartite planar graphs of negative girth 6	signed bipartite planar simple graphs

$$\begin{tikzcd}
PC(2k) \arrow[r] \arrow[rrrr, bend right] \arrow[rrrrr, bend right] & PC(2k-2) & \cdots & PC(6) \arrow[rr] & & PC(4) \arrow[r] & PC(2)
\end{tikzcd}$$

$$\begin{tikzcd}
SPC(2k+1) \arrow[r] \arrow[rrrr, bend right] \arrow[rrrrr, bend right] & SPC(2k-1) & \cdots & SPC(7) \arrow[rr] & & SPC(5) \arrow[r] & SPC(3)
\end{tikzcd}$$

Question. Can any of these homomorphisms be not surjective?

$$\begin{tikzcd}
PC(2k) \arrow[r] \arrow[rrrr, bend right] \arrow[rrrrr, bend right] & PC(2k-2) & \cdots & PC(6) \arrow[r, "\text{Onto } \checkmark"] & PC(4) & PC(2)
\end{tikzcd}$$

Onto ✓ Payan 1992

$$\begin{tikzcd}
SPC(2k+1) \arrow[r] \arrow[rrrr, bend right] \arrow[rrrrr, bend right] & SPC(2k-1) & \cdots & SPC(7) & SPC(5) & SPC(3)
\end{tikzcd}$$

Onto ✓ Meirun Chen Naserasr

Question. Can any of these homomorphisms be not surjective?

Conjecture. No

Beaudou Naserasr Tardif 2015

Circular coloring (of signed graphs)

For graphs: survey papers by X. Zhu

For signed graphs: Ph.D. project of Zhouningxin Wang
(in joint work with Zhu)

Natural setting: to seat people around a round table
so that 1. Friends are not too far,
2. enemies are not too close.

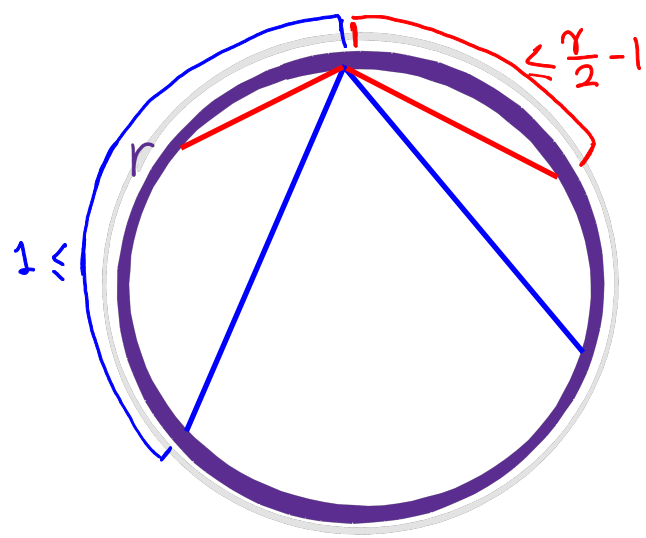
Circular coloring (of signed graphs)

Formal definition (exchanging the roles +&-)

Mapping vertices of (G, σ) to the points of circle (of circomference r) such that:

$$x \overset{+}{\sim} y \Rightarrow d(c(x), c(y)) \geq 1.$$

$$x \overset{-}{\sim} y \Rightarrow d(c(x), c(y)) \leq r/2 - 1$$



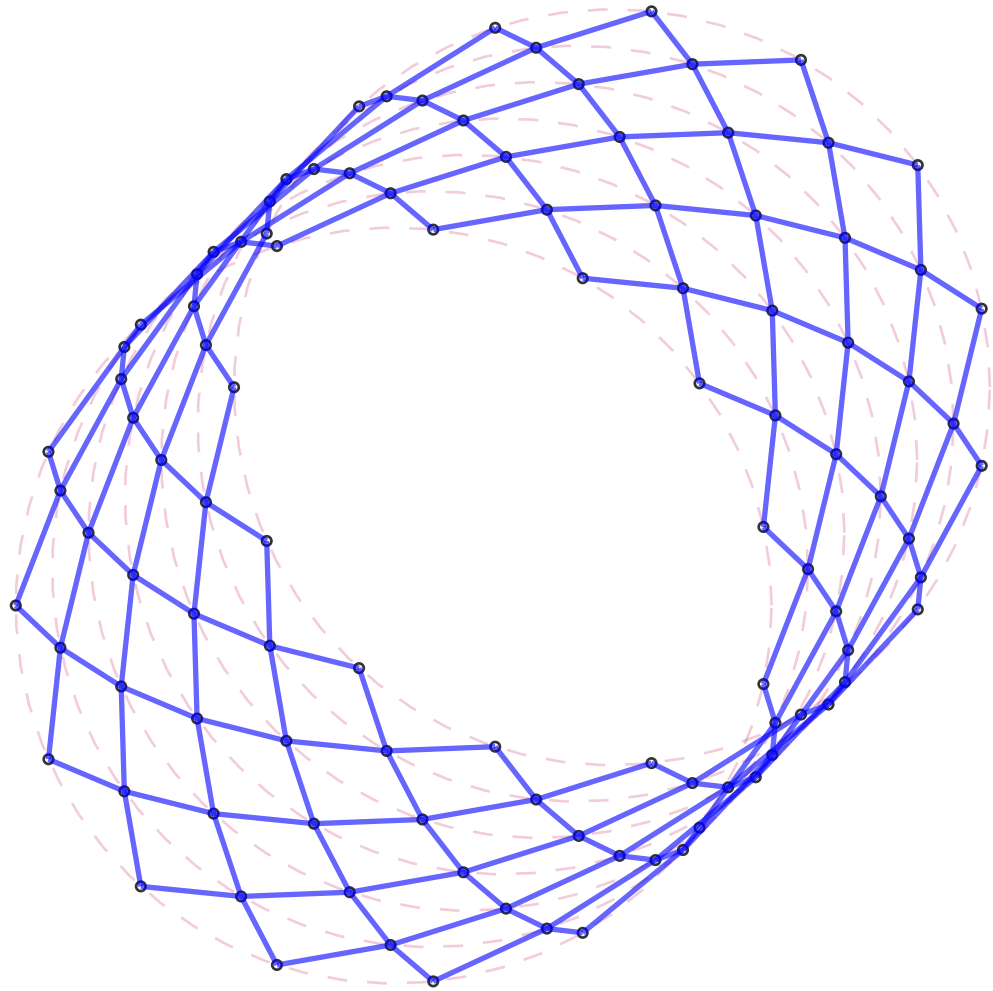
Our goal: to introduce subgraph of signed projective cube whose circular chromatic number is 4.

Generalized Mycielski

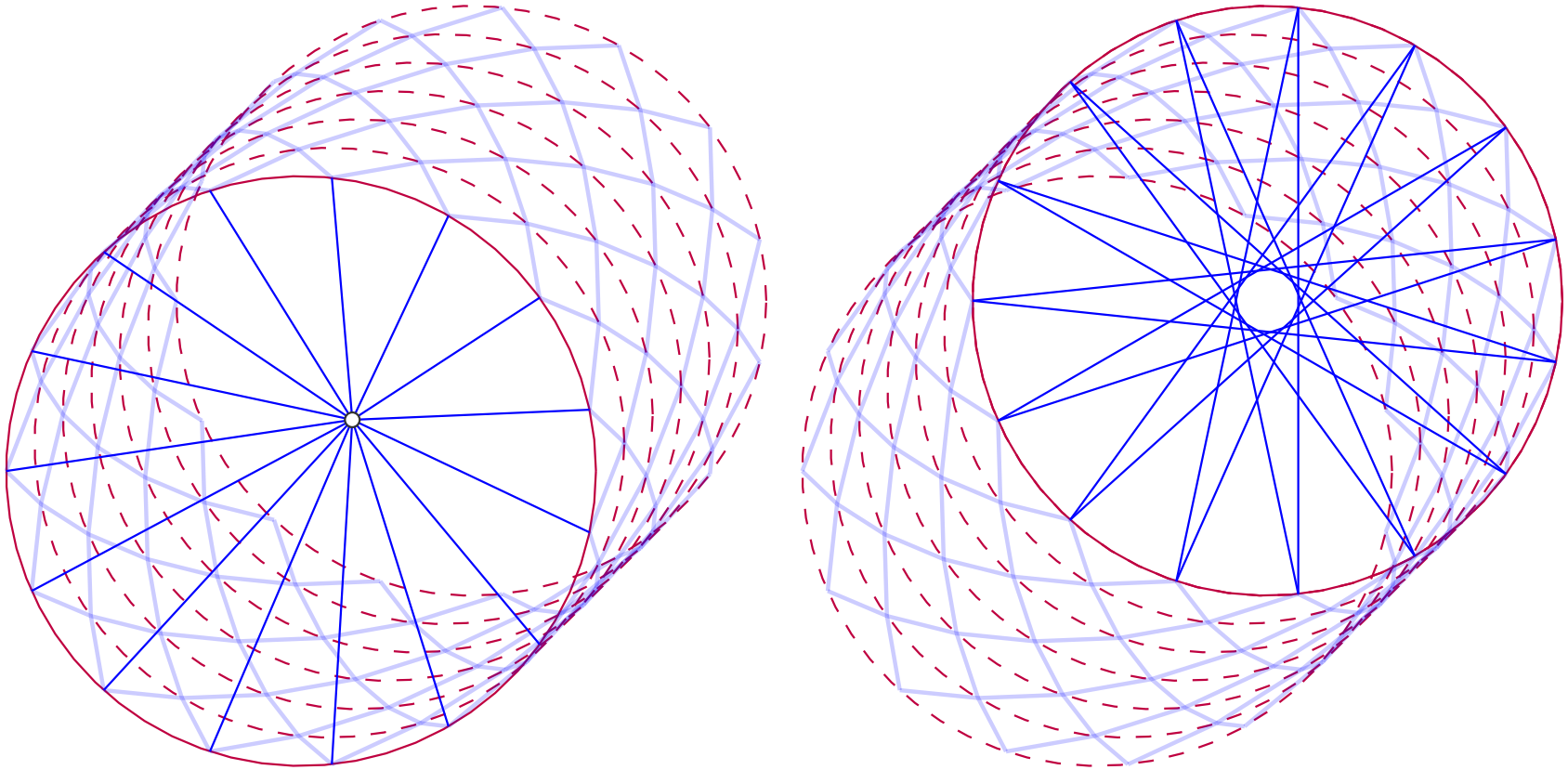
Defined and studied by:

Stiebitz [85], Van Ngoc [87],

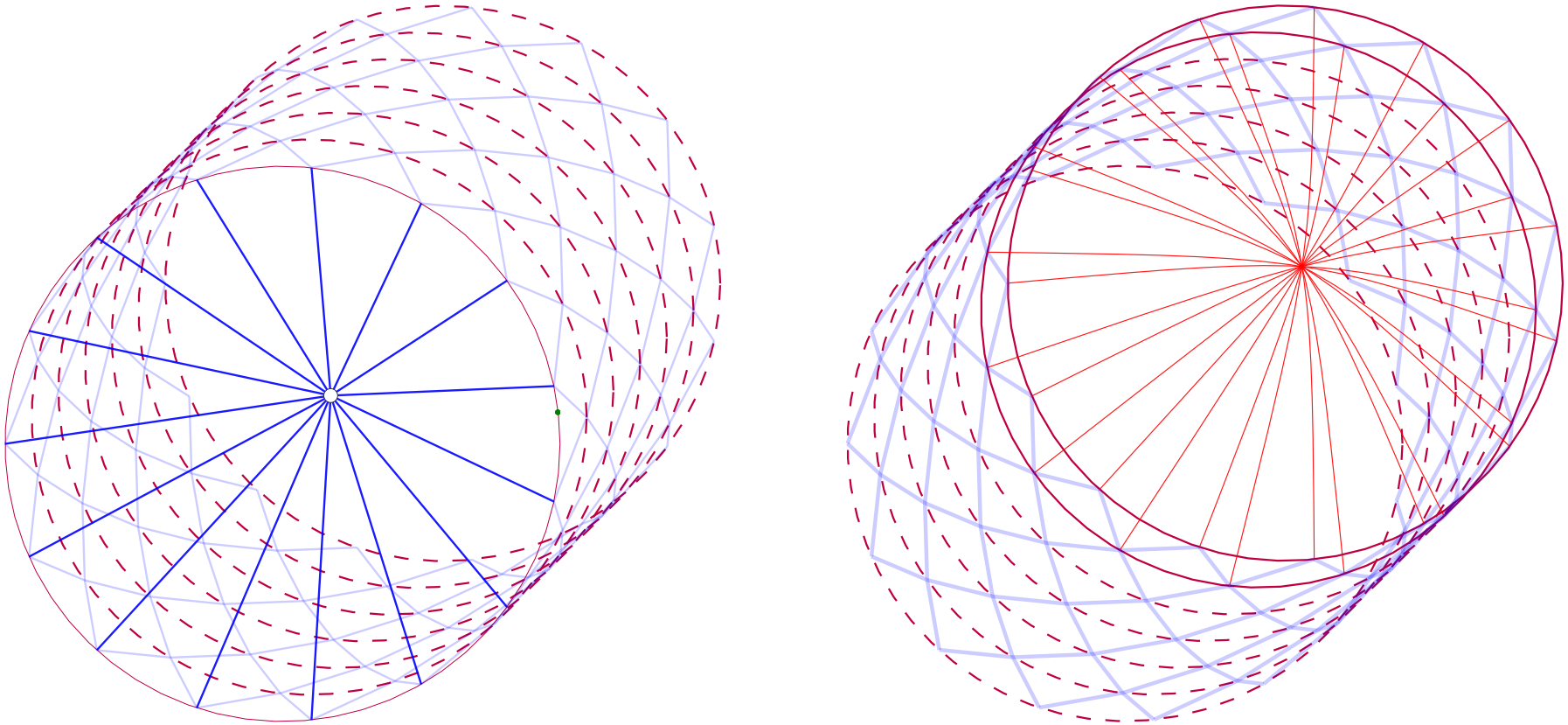
Payan [92], Van Ngoc + Tuza [95], Young [96]



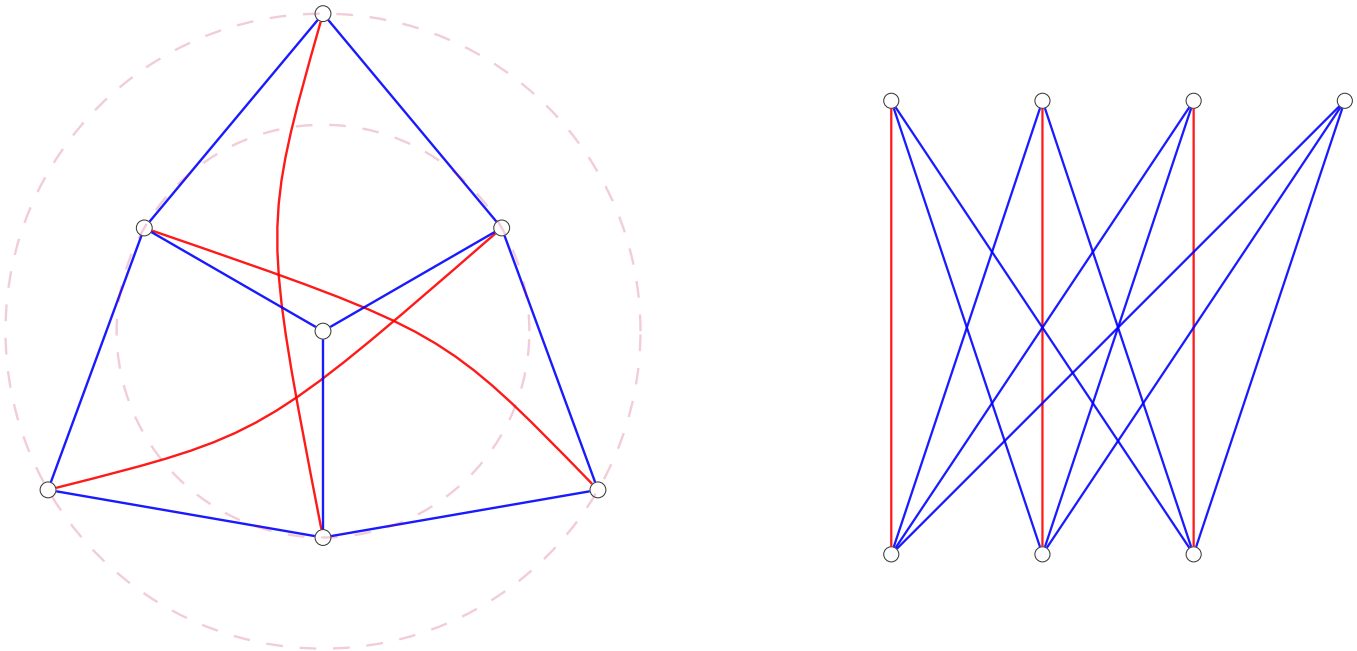
$C_{\ell \times (2k+1)}$ with layers highlighted



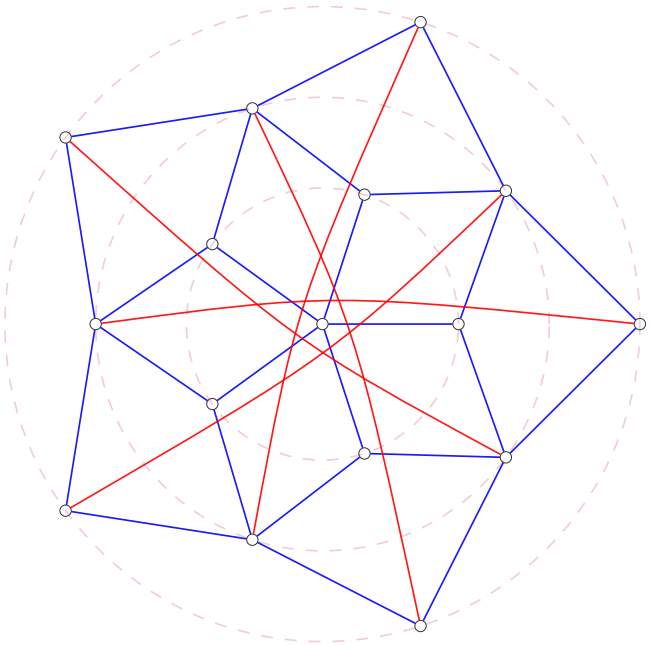
Constructions on bottom and top layers



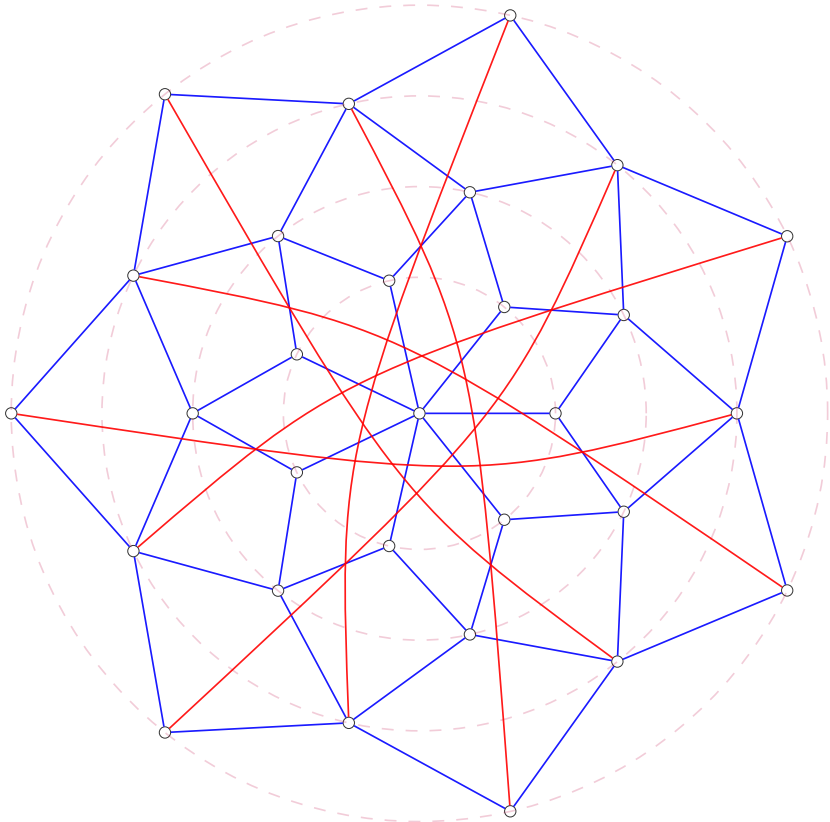
Construction of $BM_{l,2k-1}$



$BM_{1,3}$, presented two different ways



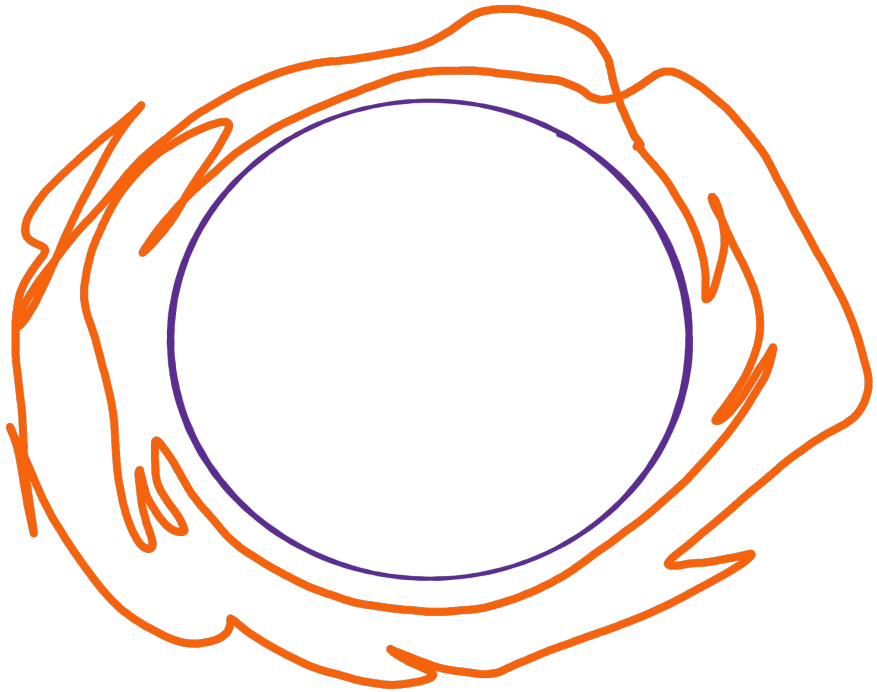
$BM_{2,5}$



$BM_{3,7}$

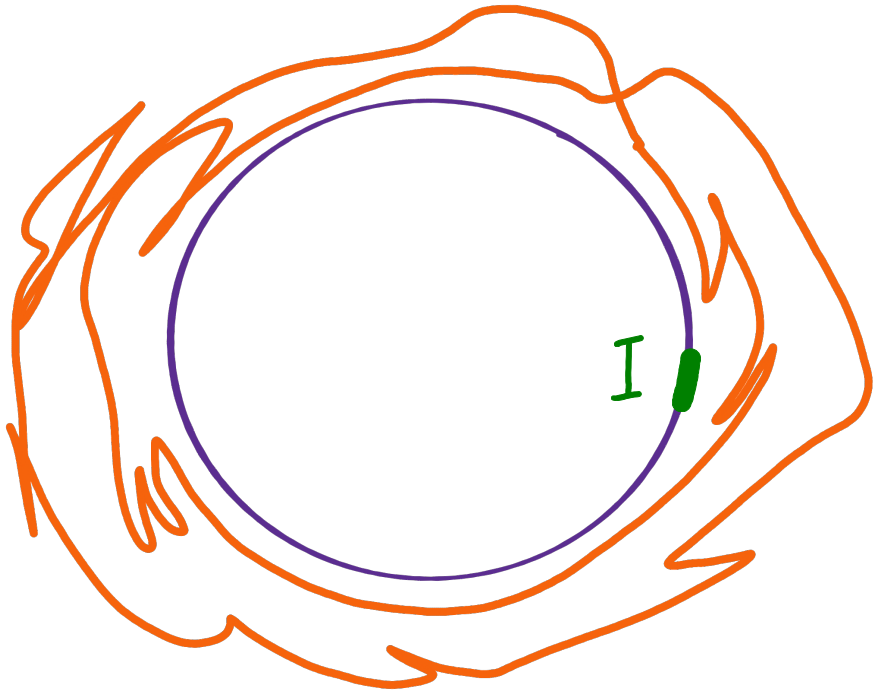
Theorem. $\chi_c(M_{1,2K+1}) = \chi_c(BM_{1,2K+1}) = 4.$

Winding Number.



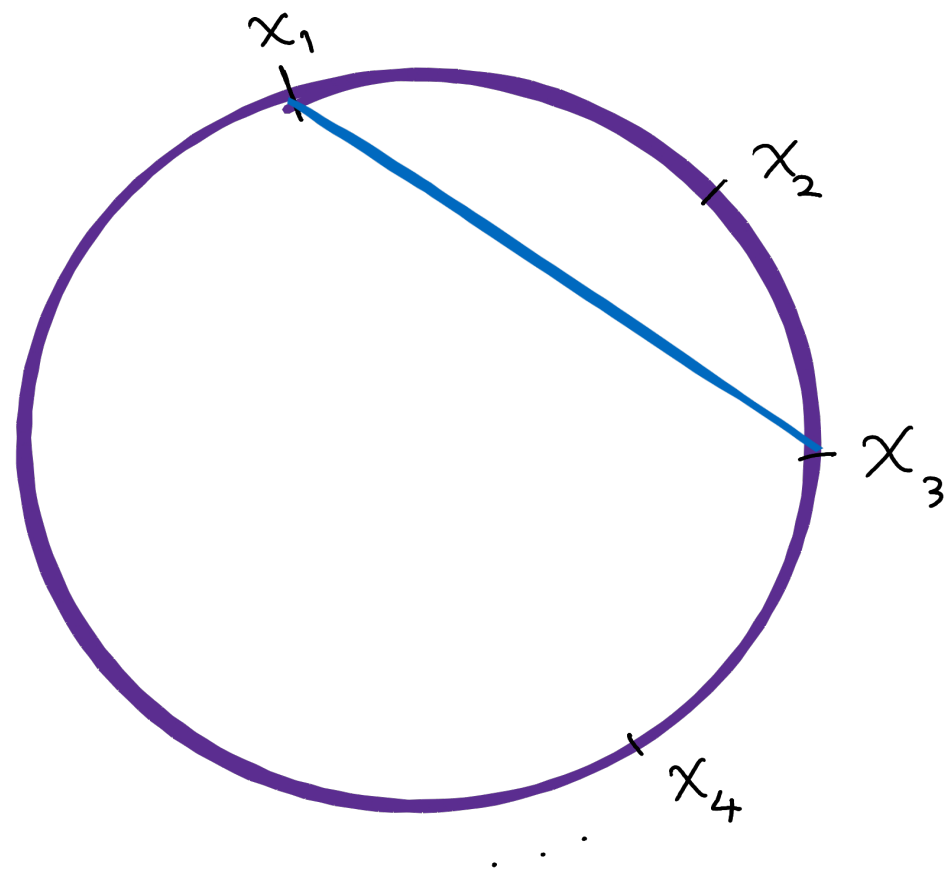
Winding Number.

Combinatorial interpretation



$$\varphi : V(G) \to \mathbb{C}^r$$

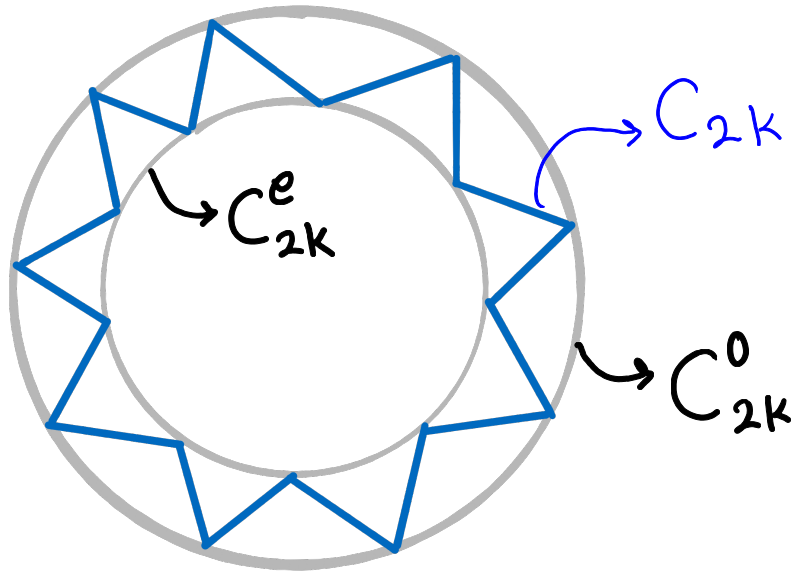
$2^{|E(G)|}$ possible extensions



φ^{sh} extension on the shorter side
(for every edge)

Lemma. φ a circular r-coloring of C_{2K}.
$(r < 4)$

$$\Longrightarrow \quad \text{winding } \varphi^{sh}(C^{o}_{2k}) \equiv \text{winding } \varphi^{sh}(C^{e}_{2k}) \mod 2$$



Problems of interest:

1. Beside a universal vertex how else to force an even winding number?

Problems of interest:

2. considering the conjecture $P_K \leq_{hom} SPC(K-1)$, can we replace P_K with a larger set Q_K such that

   1. $(G,\sigma) \in P_K \Rightarrow EDC(G,\sigma) \in Q_K$,

   2. testing membership in Q_K can be done in polynomial time.

Potential suggestions satisfying 2.

i. K_5-minor-free instead of planar

ii. (G,σ) has no $(K_5,-)$-minor

iii. (G,σ) has no (Petersen,-)-minor

Problems of interest:

Proposition. $\chi(G) \leq k \iff T_{k-2}(G) \to C_{-k}$.

3. For which graphs G, winding number can be used on $T_{k-2}(G)$?

Merci