Signed projective cubes

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\( H_d \) embeded in \( S^d \)
Projection of $S^3$
Equivalent definition,

As a Cayley signed graph:

$$\text{SPC}(d) := \left( \mathbb{Z}_2^d, \{e_1, e_2, e_3, \ldots e_d, J \} \right)$$
Equivalent definition,

As a Cayley signed graph:

\[
\left( \mathbb{Z}_2^k, \{s_1, s_2, \ldots, s_k, s_{k+1} \} \right)
\]

minimally \( s_1 + s_2 + \cdots + s_k + s_{k+1} = 0 \)
$PC(4): \ (\mathbb{Z}_2^4, (e_1, e_2, e_3, e_4, J))$
Questions such as symmetries, inductive definition ... alreadly motivates study of signed graphs.
Questions such as symmetries, inductive definition ... already motivates study of signed graphs.

Meredith: \(H_d\) and \(PC(2d)\) are the only non-trivial triple transitive graphs.

Extension using signed graphs captures \(PC(2d-1)\).
signed graph: A graph where each edge is assigned a sign
+ positive  - negative

Notation: \((G, \Sigma)\)  \(\xrightarrow{\text{signature}}\) graph

Examples:

\((K_4, -)\)  \((K_5, -)\)
Main terminology: positive cycle (balanced cycle) negative cycle (unbalanced cycle)

reflecting the fact that the rule of "friend of a friend is a friend" applies.

positive/negative closed walk

Balanced signed graph: signed graph with no negative cycle

Antibalanced signed graph: where odd cycles are negative even cycles are positive.
Switching (what makes it different from 2-edge-colored graphs).

To multiply signs of edges incident to a vertex to -

Switching at all vertices of a subset $X$ of vertices:
To multiply signs of all edges in the edge cut $(X, V-X)$ to a -
Geometric presentation of switching

DC\((G, 6)\): Double Cover of \((G, 6)\)

vertices: two copies of \(V(G)\) \(V_L(G) \cup V_R(G)\)

edges: * If \(xy\) is a positive edge, then
\(x_L \sim y_L \quad \& \quad x_R \sim y_R\)

* If \(xy\) is a negative edge, then
\(x_L \sim y_R \quad \& \quad x_R \sim y_L\)

Example

\[\text{Diagram of a square with double cover edges.}\]
EDC (Extended Double Cover)

\((G, \phi)\)

switch at a vertex \(X\)

\(DC(G, \phi)\)

switch the place of \(X_L\) with \(X_R\)

Example

\[
\begin{array}{c}
\text{Example} \\
\begin{array}{c}
\text{Graph 1} \\
\text{Graph 2}
\end{array}
\end{array}
\]
EDC$(G, 6)$: Extended Double Cover of $(G, 6)$

vertices: two copies of $V(G)$ \quad $V_L(G) \cup V_R(G)$

positive edges: * If $xy$ is a positive edge, then 
$X_L \sim Y_L \quad \& \quad X_R \sim Y_R$

* If $xy$ is a negative edge, then 
$X_L \sim Y_R \quad \& \quad X_R \sim Y_L$

negative edges: $X_L \sim X_R$
$SPC(2) = EDC(SPC(1))$

$K_4$ viewed as a Möbius ladder
Stripes and twists in $K_{4,4}$
Intersection graph of lines in an algebraic surface

Clebsch, Segre, Del Pezzo (of degree 4)?

Greenwood-Gleason graph (for application in Ramsey theory)

Clebsch graph viewed as combinations of Möbius ladders
$K_2 \ast C_4$
Two presentations of $K_3 \ast C_4$
Other definitions of $\text{SPC}(K)$:

- Product of signed graphs

- Power graphs

- Poset of partition pairs
Properties of SPC(d):

* Its negative girth is $d+1$.
* Every negative cycle is of same parity as $d+1$.
* Every positive cycle is even.
Homomorphisms of signed graphs.

**Definition.** Given signed graphs \((G, \sigma^-)\) & \((H, \pi)\) a mapping of \(V(G)\) to \(V(H)\) (and \(E(G)\) to \(E(H)\)) is said to be a **homomorphism** of \((G, \sigma^-)\) to \((H, \pi)\) if it preserves **adjacencies**, (incidences) and signs of closed walks.

It is said to be **edge-sign preserving homomorphism** if it furthermore, preserves signs of edges.

**Examples**
Comment. The edge mapping is implied unless \((\mathcal{H}, \pi)\) contains a digon.

Theorem. Signed graph \((G, \sigma')\) admits a homomorphism to signed graph \((\mathcal{H}, \pi)\) if and only if for some switching \((G, \sigma')\) there exists an edge-sign preserving homomorphism of \((G, \sigma')\) to \((\mathcal{H}, \pi)\).

General references for homomorphisms of signed graphs

Naser Asr
Rollova
Sopena
2015

Naser Asr
Sopena
Zaslavsky
2021
Important note:

Notions of "isomorphism" and "automorphism" depend on our view and choice of homomorphism: edge-sign preserving homomorphism or switch homomorphism. Associated definitions like "vertex transitive" and "edge-transitive" change accordingly.

under edge-sign preserving homomorphism

the only automorphism:

\[
\begin{align*}
  a & \leftrightarrow d \\
  b & \leftrightarrow c
\end{align*}
\]

under switch homomorphism

Dihedral group $D_4$ (8 elements)

Thus $C_4$ is:

vertex-transitive & edge-transitive.
Observation. In a mapping of $(G, \sigma')$ to $(H, \pi)$ the image of every closed walk is a closed walk which has a same (parity) of length and a same sign.

This leads to four notions of girth:

$g_{00}(G,\sigma')$: length of shortest positive even closed walk,
$g_{10}(G,\sigma')$: length of shortest negative even closed walk,
$g_{01}(G,\sigma')$: length of shortest positive odd closed walk,
$g_{11}(G,\sigma')$: length of shortest negative odd closed walk.

Example.

$$g_{00}(G,\sigma')=2, \ g_{10}(G,\sigma')=4, \ g_{01}(G,\sigma')=3, \ g_{11}(G,\sigma')=3$$
Theorem. A signed graph \((G, \delta)\) maps to \(\text{SPC}(d)\) if and only if \(E(G)\) can be partitioned into \(E_1, E_2, \ldots, E_{d+1}\) such that \((G, E_i)\), for each \(i\), is switch equivalent to \((G, \delta)\).

Strengthen to a general packing problem is part of Weigiang Yu's Ph.D. project.
Conjecture. Given a signed planar graph \((G, \sigma)\), if
\[
g_{ij}(G, \sigma) \geq g_{ij}(SPC(d)) \quad \text{for every } ij \in \mathbb{Z}_2^2,
\]
then \((G, \sigma) \rightarrow SPC(d)\).
PC(2k) → PC(2k-2)  ....  PC(6) → PC(4) → PC(2)

↑ ↕
planar graphs
of odd-girth
at least 2k+1

↑ ✓
planar graphs
of odd-girth
at least 7

↑ ✓
triangle-free
planar graphs

↑ ✓
planar graphs
Analogous questions are asked for mapping signed bipartite planar graphs to $SPC(2d-1)$.
Conjecture. Every planar graph of odd-girth at least 

$$2k+3$$ maps to $$K(2k+1, k).$$

Would imply the best possible bound of $$2 + \frac{1}{k}$$
for the fractional chromatic number of this class.
Conjecture. Every planar graph of odd-girth at least

\[4k+1\] maps to \(C_{2k+1}\).

Would imply the best possible bound of \(2 + \frac{1}{k}\) for the circular chromatic number of this class.
Conjecture. Every planar graph of odd-girth at least $4k-1$ maps to $M_{2k+1}$.

Would imply the best possible bound of $2 + \frac{2}{2k-1}$ for the circular chromatic number of this class.
PC(2k) → PC(2k-2) .... PC(6) → PC(4) → PC(2)

planar graphs
of odd-girth
at least 2k+1

planar graphs
of odd-girth
at least 7

triangle-free
planar graphs

planar
graphs

Analogous questions are asked for mapping signed bipartite planar graphs to SPC(2d-1).

SPC(2k+1) SPC(2k-1) .... SPC(7) SPC(5) SPC(3)

↑↑
signed bipartite
planar graphs of
negative girth 2k+2

↑✓
signed bipartite
planar graphs of
negative girth 8

↑✓
signed bipartite
planar graphs of
negative girth 6

↑✓
signed bipartite
planar simple
graphs
Question. Can any of these homomorphisms be not surjective?
Question. Can any of these homomorphisms be not surjective?

Conjecture. No

Beaudou
Naserasr
Tardif
2015
Circular coloring (of signed graphs)

For graphs: survey papers by X. Zhu

For signed graphs: Ph.D. project of Zhouningxin Wang
   (in joint work with Zhu)

Natural setting: to seat people around a round table
   So that 1. Friends are not too far,
   2. enemies are not too close.
Circular coloring (of signed graphs)

Formal definition (exchanging the roles +\&-)

Mapping vertices of \((G, \sigma)\) to the points of circle (of circumference \(r\)) such that:

\[ x \sim_+ y \implies d(c(x), c(y)) \geq 1. \]

\[ x \sim_- y \implies d(c(x), c(y)) \leq \frac{r}{2} - 1. \]
Our goal: to introduce subgraph of signed projective cube whose circular chromatic number is 4.

**Generalized Mycielski**

Defined and studied by:
Stiebitz [85], Van Ngoc [87], Payan [92], Van Ngoc + Tuza [95], Young [96]
$C_{\ell \times (2k+1)}$ with layers highlighted
Constructions on bottom and top layers
Construction of $BM_{i,2k-1}$
$BM_{1,3}$, presented two different ways
Theorem. \( X_c(M_{l, 2K+1}) = X_c(BM_{l, 2K+1}) = 4. \)
Winding Number.
Winding Number

Combinatorial interpretation
\[ \varphi : \mathcal{V}(G) \rightarrow \mathbb{C}^r \]

\[ 2^{\left| \mathcal{E}(G) \right|} \]

possible extensions

\[ \varphi^{sh} \]

extension on the shorter side

(for every edge)
Lemma. \( \varphi \) a circular r-coloring of \( C_{2k} \).

\( (\gamma < 4) \)

\[ \Rightarrow \quad \text{winding} \; \varphi^{sh}(C^0_{2k}) \equiv \text{winding} \; \varphi^{sh}(C^e_{2k}) \mod 2 \]
Problems of interest:

1. Beside a universal vertex how else to force an even winding number?
Problems of interest:

2. considering the conjecture $P_K \leq_{\text{hom}} \text{SPC}(K-1)$, can we replace $P_K$ with a larger set $Q_K$ such that

1. $(G,\sigma) \in P_K \Rightarrow EDC(G,\sigma) \in Q_K$

2. testing membership in $Q_K$ can be done in polynomial time.

Potential suggestions satisfying 2.

i. $K_5$-minor-free instead of planar

ii. $(G,\sigma)$ has no $(K_5,-)$-minor

iii. $(G,\sigma)$ has no $(Petersen,-)$-minor
Problems of interest:

Proposition. $\chi(G) \leq k \iff T_{k-2}(G) \to C_{-k}$.

3. For which graphs $G$, winding number can be used on $T_{k-2}(G)$?
Merci