Homomorphism bounds and edge-colourings of K_4-minor-free graphs

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Abstract

We present a necessary and sufficient condition for a graph of odd-girth $2k+1$ to bound the class of K_4-minor-free graphs of odd-girth (at least) $2k+1$, that is, to admit a homomorphism from any such K_4-minor-free graph. This yields a polynomial-time algorithm to recognize such bounds. Using this condition, we first prove that every K_4-minor free graph of odd-girth $2k+1$ admits a homomorphism to the projective cube of dimension $2k$. This supports a conjecture of the third author which generalizes the four-color theorem and relates to several outstanding conjectures such as Seymour's conjecture on edge-colorings of planar graphs. Strengthening this result, we show that the Kneser graph $K(2k+1,k)$ satisfies the conditions, thus implying that every K_4-minor free graph of odd-girth $2k+1$ has fractional chromatic number exactly $2+\frac{1}{k}$. Knowing that a smallest bound of odd-girth $2k+1$ must have at least $\binom{k+2}{2}$ vertices, we build nearly optimal bounds of order $4k^2$. Furthermore, we conjecture that the suprema of the fractional and circular chromatic numbers for K_4-minor-free graphs of odd-girth $2k+1$ are achieved by a same bound of odd-girth $2k+1$. If true, this improves, in the homomorphism order, earlier tight results on the circular chromatic number of K_4-minor-free graphs. We support our conjecture by proving it for the first few cases. Finally, as an application of our work, and after noting that Seymour provided a formula for calculating the edge-chromatic number of K_4-minor-free multigraphs, we show that stronger results can be obtained in the case of K_4-minor-free regular multigraphs.

1. Introduction

In this paper, graphs are simple and loopless. While loops will never be considered, multiple edges will be considered, but graphs containing multiple edges will specifically be referred to as multigraphs. A homomorphism f of a graph G to a graph H is a mapping of $V(G)$ to $V(H)$ that preserves the edges, that is, if x and y are adjacent in G, then $f(x)$ and $f(y)$ are adjacent in H. If such a homomorphism exists, we note $G \to H$. Observe that a homomorphism of a graph G to the complete graph K_k is the same as a proper k-colouring of G. Given a class $\mathcal{C}$ of graphs and a graph H, we say that H *bounds* $\mathcal{C}$ if every graph in $\mathcal{C}$ admits a homomorphism to H. We also say that $\mathcal{C}$ is *bounded* by H, or that H is a *bound* for $\mathcal{C}$. For example, the Four-Colour Theorem states that K_4 bounds the class of planar graphs. It is of interest to also ask for (small) bounds having some additional properties. In this paper we study the problem of determining bound(s) of smallest possible order for the class of K_4-minor-free graphs of given odd-girth, with the restriction that the bound itself also has the same odd-girth (the *odd-girth* of a graph is the smallest length of an odd cycle).

Homomorphism bounds and minors. The *core* of a graph G is the smallest subgraph of G to which G admits a homomorphism (it is known to be unique up to isomorphism). A graph G is a *core* if it is its own core. For additional concepts in graph homomorphisms, we refer to the book by Hell and Nešetřil [18]. Let $\mathcal{I} = \{I_1, I_2, \ldots, I_i\}$ be a class of cores. We define $forb_h(\mathcal{I})$ to be the class of all graphs (or cores) to which no member of $\mathcal{I}$ admits a homomorphism. For example for $\mathcal{I} = \{K_n\}$, $forb_h(\mathcal{I})$ is the class of K_n-free graphs and for $\mathcal{I} = \{C_{2k-1}\}$ it is the class of graphs of odd-girth at least $2k+1$. Similarly, given $\mathcal{J} = \{J_1, J_2, \ldots, J_j\}$, we use the notation $forb_m(\mathcal{J})$ to denote the class of all graphs that have no member of $\mathcal{J}$ as a minor. (Recall that a graph H is a *minor* of a graph G if H can be obtained from G by a sequence of edge-contractions and vertex- and edge-deletions.) The following is a fundamental theorem in the study of the relation between minors and homomorphisms.

Theorem 1 (Nešetřil and Ossona de Mendez [34]). *Given a finite set $\mathcal{I}$ of connected cores and a finite set $\mathcal{J}$ of graphs, there is a graph in $forb_h(\mathcal{I})$ which is a bound for the class $forb_h(\mathcal{I}) \cap forb_m(\mathcal{J})$.*

It is worthwhile to note that a bound for $forb_h(\mathcal{I}) \cap forb_m(\mathcal{J})$ belonging itself to $forb_h(\mathcal{I}) \cap forb_m(\mathcal{J})$ may not exist, for example there is no triangle-free planar graph bounding the class of triangle-free planar graphs [27]; see Theorem 3.37 in the third author's PhD thesis [28] for a more general statement for any odd-girth. By a similar argument, it is easily observed that there is no K_4-minor-free graph of odd-girth $2k+1$ ($k \geq 2$) bounding the class of K_4-minor-free graphs of odd-girth $2k+1$.

Even though the proof of Theorem 1 is constructive, the bound obtained by this construction is generally very large. Thus, we pose the following problem.

Problem 2. *What is a graph in $forb_h(\mathcal{I})$ of smallest possible order that bounds $forb_h(\mathcal{I}) \cap forb_m(\mathcal{J})$?*

Finding the answer to Problem 2 can be a very difficult task in general. For example, Hadwiger's conjecture would be answered if we find the optimal solution with $\mathcal{I} = \{K_n\}$ and $\mathcal{J} = \{K_n\}$. Nevertheless, the answer to some special cases is known. As an example, the main result of [11] implies in particular the following claim, which corresponds to $\mathcal{I} = \{C_{2k-1}\}$ and $\mathcal{J} = \{K_4, K_{2,3}\}$.

Theorem 3 (Gerards [11]). *The cycle C_{2k+1} bounds the class of outerplanar graphs of odd-girth at least $2k+1$.*

Problem 2 asks for optimal bounds in terms of the order. As observed in [33, 34], such a bound (unless it is itself in the class $forb_m(\mathcal{J})$) cannot be optimal with respect to the homomorphism order. More precisely, if $B \in forb_h(\mathcal{I})$ is a bound for $forb_h(\mathcal{I}) \cap forb_m(\mathcal{J})$ and $B \notin forb_m(\mathcal{J})$, then there exists another bound $B' \in forb_h(\mathcal{I})$ such that $B' \to B$ and $B \nrightarrow B'$.

One of the interesting cases of Problem 2 is when $\mathcal{I} = \{C_{2k-1}\}$ and $\mathcal{J} = \{K_{3,3}, K_5\}$. In other words, the problem consists of finding a smallest graph of odd-girth $2k+1$ which bounds the class of planar graphs of odd-girth at least $2k+1$. This particular problem was studied by several authors, see for example [14, 27, 29, 30]. A proposed answer was formulated as a conjecture, using the notions that are developed next.

Projective hypercubes and bounds for planar graphs. Hypercubes are among the most celebrated families of graphs. Recall that the *hypercube* of dimension d, denoted $H(d)$, is the graph whose vertices are all binary words of length d and where two such words are adjacent if their Hamming distance is 1. In other words, $H(d)$ is the Cayley graph $(\mathbb{Z}_2^d, \{e_1, e_2, \ldots, e_d\})$ where $\{e_1, e_2, \ldots, e_d\}$ is the standard basis. With a natural embedding of the d-dimensional hypercube $H(d)$ on the d-dimensional sphere S^d, hypercubes can be regarded as the discrete approximation of S^d. Recall that the projective space of dimension d is obtained from identifying antipodal points of S^{d+1}. Then, the image of $H(d+1)$ under such projection is called the *projective hypercube* of dimension d, denoted $PC(d)$ (sometimes also called *projective cube* or *folded cube*). It is thus the graph obtained from the $(d+1)$-dimensional hypercube by identifying all pairs of antipodal vertices. It can be checked that this is the same as the graph obtained from the d-dimensional hypercube by adding an edge between every pair of antipodal vertices. Thus, $PC(d)$ is the Cayley graph $(\mathbb{Z}_2^d, S = \{e_1, e_2, \ldots, e_d, J\})$ where $\{e_1, e_2, \ldots, e_d\}$ is the standard basis and J is the all-1 vector. Therefore, $PC(1)$ is K_2, $PC(2)$ is K_4, $PC(3)$ is $K_{4,4}$ and $PC(4)$ is the celebrated Clebsch graph (which is also referred to as the Greenwood-Gleason graph for its independent appearance in Ramsey theory).

As a generalization of the Four-Colour Theorem, the following conjecture was proposed by the third author.

Conjecture 4 (Naserasr [29]). *The projective hypercube $PC(2k)$ bounds the class of planar graphs of odd-girth at least $2k+1$.*

Since $PC(2)$ is isomorphic to K_4, the first case of this conjecture is the Four-Colour Theorem. The conjecture is related to determining the edge-chromatic number of a class of planar multigraphs, as we will explain later. Moreover, it is shown by Naserasr, Sen and Sun [31] that no bound of odd-girth $2k+1$ of smaller order can exist. Thus this conjecture proposes a solution to Problem 2 for the case $\mathcal{I} = \{C_{2k-1}\}$ and $\mathcal{J} = \{K_{3,3}, K_5\}$.

Edge-colouring and fractional colouring. Recall that colouring a graph G corresponds to partitioning its vertices into independent sets, and the chromatic number of G, denoted $\chi(G)$, is the smallest size of such a partition. Analogously, edge-colouring a multigraph G corresponds to partitioning its edge set into matchings, and the edge-chromatic number of G, denoted $\chi'(G)$, is the smallest size of such a partition. Each of these parameters can be regarded as an optimal solution to an integer program where independent sets (or matchings) are given value 0 or 1 together with inequalities indicating that each vertex (or edge) receives a total weight of at least 1, that is, it is coloured (or covered).

After relaxing these possible values from $\{0,1\}$ to all non-negative real numbers, we obtain the notion of fractional chromatic number, denoted $\chi_f(G)$, and fractional edge-chromatic number of multigraphs, denoted $\chi'_f(G)$. The fractional chromatic number of a graph G can equivalently be defined as the smallest value of $\frac{p}{q}$ over all positive integers p, q, $2q \leq p$, such that $G \to K(p,q)$, where $K(p,q)$ is the *Kneser graph* whose vertices are all q-subsets of a p-set and where two vertices are adjacent if they have no element in common.

While computing each of $\chi(G)$, $\chi_f(G)$ and $\chi'(G)$ are known to be NP-complete problems (see [22], [24] and [20] respectively), it is shown by Edmonds [7] (see also [41]) that $\chi'_f(G)$ can be computed in polynomial time. We refer to the book of Scheinermann and Ullman [38] for details. For every multigraph G, the inequality $\chi'_f(G) \leq \chi'(G)$ clearly holds. Seymour conjectured that, up to an integer approximation, equality holds for every planar multigraph.

Conjecture 5 (Seymour [39, 40]). *For each planar multigraph G, $\chi'(G) = \lceil \chi'_f(G) \rceil$.*

The restriction of Conjecture 5 to planar 3-regular multigraphs corresponds to a claim of Tait (every bridgeless cubic planar graphs is 3-edge-colourable) from the late 19th century [43]. As Tait has shown, this is equivalent to the Four-Colour Theorem.

Conjecture 5 has been studied extensively for the special case of planar *r-graphs*, for which the fractional edge-chromatic number is known to be exactly r [39, 41]. An r-regular multigraph is an r-graph if for each set X of an odd number of vertices, the number of edges leaving X is at least r. Hence, Conjecture 5 restricted to this class is stated as follows.

Conjecture 6 (Seymour [39, 40]). *Every planar r-graph is r-edge-colourable.*

For any odd integer $r = 2k+1$ ($k \geq 1$), the claim of Conjecture 6 is proved to be equivalent to the claim of Conjecture 4 for k by Naserasr [29]. Conjecture 6 has been proved for $r \leq 8$ in [15] ($r = 4, 5$), [6] ($r = 6$), [3, 8] ($r = 7$) and [4] ($r = 8$). We note that these proofs use induction on r and thus use the Four-Colour Theorem as a base step.

The claim of Conjecture 5 when restricted to the class of K_4-minor-free graphs (a subclass of planar graphs) was proved by Seymour [42]. A simpler (unpublished) proof of this result is given more recently by Fernandes and Thomas [10].

Using Seymour's result and a characterization theorem, Marcotte extended the result to the class of graphs with no $K_{3,3}$ or $(K_5 - e)$-minor [26], where $K_5 - e$ is obtained from K_5 by removing an edge.

A key tool in proving the equivalence between Conjecture 4 and Conjecture 5 (for a fixed k) in [29] is the Folding Lemma of Klostermeyer and Zhang [25]. In the absence of such a lemma for the subclass of K_4-minor-free graphs, there is no known direct equivalence between the restrictions of Conjecture 6 and Conjecture 4 to K_4-minor-free graphs. However, since K_4-minor-free graphs are planar, the notion of dual is well-defined. Using such a notion, Conjecture 6 restricted to the class of K_4-minor-free graphs is implied by Conjecture 4 restricted to the same class. Thus, in this paper, by proving Conjecture 4 for the class of K_4-minor-free graphs, we obtain as a corollary a new proof of Conjecture 6 for K_4-minor-free $(2k+1)$-graphs. Then, by improving on the homomorphism bound, we also deduce stronger results on the edge-colouring counterpart.

Our results. In this paper, we study the case $\mathcal{I} = \{C_{2k-1}\}$ and $\mathcal{J} = \{K_4\}$ of Problem 2, that is, the case of K_4-minor-free graphs (also known as series-parallel graphs) of odd-girth at least $2k+1$, that we denote by $\mathcal{SP}_{2k+1}$. Our main tool is to prove necessary and sufficient conditions for a graph B of odd-girth $2k+1$ to be a bound for $\mathcal{SP}_{2k+1}$. These conditions are given in terms of the existence of a certain weighted graph (that we call a k-partial distance graph of B) containing B as a subgraph, and that satisfies certain properties. The main idea of the proof is based on homomorphisms of weighted graphs and the characterization of K_4-minor-free graphs as partial 2-trees. This result is presented in Section 3. From this, we are able to deduce a polynomial-time algorithm to decide whether a graph of odd-girth $2k+1$ is a bound for $\mathcal{SP}_{2k+1}$. This algorithm is presented in Section 4. We will then use our main theorem, in Section 5, to prove that the projective hypercube $PC(2k)$ bounds $\mathcal{SP}_{2k+1}$, showing that Conjecture 4 holds when restricted to K_4-minor-free graphs. In fact, we also show that this is far from being optimal (with respect to the order), by exhibiting two families of subgraphs of the projective hypercubes that are an answer: the Kneser graphs $K(2k+1,k)$, and a family of order $4k^2$ (which we call augmented square toroidal grids). Note that the order $O(k^2)$ is optimal, as shown by He, Sun and Naserasr [17], while for planar graphs it is known that any answer must have order at least 2^{2k} (see Sen, Sun and Naserasr [31]). In Section 6, for $k \leq 3$, we determine optimal answers to the problem. For $k=1$, it is well-known that K_3 is a bound; K_3 being a K_4-minor-free graph, it is the optimal bound in many senses (in terms of order, size, and homomorphism order). We prove that the smallest triangle-free graph bounding $\mathcal{SP}_5$ has order 8, and the smallest graph of odd-girth 7 bounding $\mathcal{SP}_7$ has order 15 (and we determine concrete bounds of these orders). These graphs are not K_4-minor-free, and, therefore, these two bounds are not optimal in the sense of the homomorphism order. All our bounds are subgraphs of the corresponding projective hypercubes. These optimal bounds for $k \leq 3$ yield a strengthening of other results about both the fractional and the circular chromatic numbers of graphs in $\mathcal{SP}_5$ and $\mathcal{SP}_7$. In Section 7, we discuss applications of our work to edge-colourings of K_4-minor-free multigraphs. We finally conclude with some remarks and open questions in Section 8.

2. Preliminaries

In this section, we gather some definitions and useful results from the literature.

2.1. General definitions and observations

Given three positive real numbers p, q, r we say the triple $\{p, q, r\}$ satisfies the *triangular inequalities* if we have $p \leq r+q$, $q \leq p+r$ and $r \leq p+q$. Assuming p is the largest of the three, it is enough to check that $p \leq q+r$.

In a graph G, we denote by $N_G^d(v)$ the *distance d-neighbourhood of v*, that is, the set of vertices at distance exactly d from vertex v; $N_G^1(v)$ is simply denoted by $N_G(v)$. In G, the distance between two vertices u and v is denoted $d_G(u,v)$. In these notations, if there is no ambiguity about the graph G, the subscript may be omitted.

An *independent set* in a graph is a set of vertices no two of which are adjacent.

A *walk* in G is a sequence $v_0, \ldots, v_k$ of vertices where two consecutive vertices are adjacent in G. A walk with k edges is a *k-walk*. If the first and last vertices are the same, the walk is a *closed walk* (*closed k-walk*). If no vertex of a walk is repeated, then it is a *path* (k-path). A path whose internal vertices all have degree 2 is a *thread*. If in a closed walk, no inner-vertex is repeated, then it is a *cycle* (k-cycle).

Given a set X of vertices of a graph G, we denote by $G[X]$ the subgraph of G induced by X.

Given two graphs G and H, the cartesian product of G and H, denoted $G \square H$, is the graph on vertex set $V(G) \times V(H)$ where (x,y) is adjacent to (z,t) if either $x=z$ and y is adjacent to t in H, or $y=t$ and x is adjacent to z in G.

An *edge-weighted graph* (G,w) is a graph G together with an edge-weight function $w: E(G) \to \mathbb{N}$. Given two edge-weighted graphs (G,w_1) and (H,w_2), a homomorphism of (G,w_1) to (H,w_2) is a homomorphism of G to H which also preserves the edge-weights. Given a connected graph G of order n, the *complete distance graph* (K_n, d_G) of G is the weighted complete graph on vertex set $V(G)$, and where for each edge u,v of K_n, its weight $\omega(uv)$ is the distance $d_G(u,v)$ between u and v in G. Given any subgraph H of K_n, the *partial distance graph* (H, d_G) of G is the spanning subgraph of (K_n, d_G) whose edges are the edges of H. Furthermore, if for every edge xy of H we have $d_G(x,y) \leq k$, we say that (H, d_G) is a *k-partial distance graph of G*.

Easy but important observations, which we will use frequently, are the following.

Observation 7. *Let G be a graph of odd-girth $2k+1$ and C a cycle of length $2k+1$ in G. Then, for any pair (u,v) of vertices of C, the distance in G between u and v is determined by their distance in C.*

The following fact follows from the previous observation.

Observation 8. *Suppose G and H are two graphs of odd-girth $2k+1$ and that ϕ is a homomorphism of G to H. If u and v are two vertices of G on a common $(2k+1)$-cycle of G, then $d_H(\phi(u), \phi(v)) = d_G(u,v)$.*

2.2. K_4-minor-free graphs

The class of K_4-minor-free graphs has a classic characterization as the set of *partial* 2*-trees*. A 2*-tree* is a graph that can be built from a 2-vertex complete graph K_2 in a sequence $G_0 = K_2, G_1, \ldots, G_t$ where G_i is obtained from G_{i-1} by adding a new vertex and making it adjacent to two adjacent vertices of G_{i-1} (thus forming a new triangle). A partial 2-tree is a graph that is a subgraph of a 2-tree. Since 2-trees are 2-degenerate (that is, each subgraph has a vertex of degree at most 2), any 2-tree of order n has exactly $2n - 3$ edges.

The following fact is well-known and will be useful (for a reference, see for example Diestel's book [5]).

Theorem 9 ([5, Proposition 8.3.1])**.** *A graph is K_4-minor-free if and only if it is a partial 2-tree.*

Note that the set of edge-maximal K_4-minor-free graphs coincides with the set of 2-trees.

Another alternative definition of K_4-minor-free (multi)graphs is via the classic notion of *series-parallel graphs*, indeed a graph is K_4-minor-free if and only if each biconnected component is a series-parallel graph [5]. A (multi)graph G is series-parallel if it contains two vertices s and t such that G can be built using the following inductive definition: (i) an edge whose endpoints are labelled s and t is series-parallel; (ii) the graph obtained from two series-parallel graphs by identifying their s-vertices and their t-vertices, and labeling the new vertices s and t correspondingly is series-parallel (*parallel operation*); (iii) the graph obtained from two series-parallel graphs by identifying vertex s from one of them with vertex t from the other and removing their labels is series-parallel (*series operation*). Thus, the abbreviation $\mathcal{SP}$ is commonly used to denote the class of K_4-minor-free graphs. We will also use this notation, as well as $\mathcal{SP}_{2k+1}$ to denote the class of K_4-minor-free graphs of odd-girth at least $2k + 1$.

We note that many homomorphism problems are studied on the class of K_4-minor-free graphs in the literature. A notable article is [32], in which Nešetřil and Nigussie prove that the homomorphism order restricted to the class $\mathcal{SP}$ is *universal*, that is, it contains an isomorphic copy of each countable order as an induced sub-order. In other words, the concept of homomorphisms inside the class of K_4-minor-free graphs is far from being trivial, although ordinary vertex-colouring on this class is a simple problem. As we will see next, circular colouring is also a nontrivial problem on K_4-minor-free graphs.

2.3. Circular chromatic number

Given two integers p and q with $gcd(p, q) = 1$, the *circular clique* $C_{p,q}$ is the graph on vertex set $\{0, \ldots, p - 1\}$ with i adjacent to j if and only if $q \leq i - j \leq p - q$. A homomorphism of a graph G to $C_{p,q}$ is called a (p, q)-colouring, and the *circular chromatic number* of G, denoted $\chi_c(G)$, is the smallest rational p/q such that G has a (p, q)-colouring. Since $C_{p,1}$ is the complete graph K_p, we have $\chi_c(G) \leq \chi(G)$.

The (supremum of) circular chromatic number of K_4-minor-free graphs of given odd-girth was completely determined in the series of papers [19, 35, 36]. For the case of triangle-free K_4-minor-free graphs, it is proved in [19] that $\mathcal{SP}_5$ is bounded by the circular clique $C_{8,3}$, also known as the Wagner graph. Furthermore, each graph in $\mathcal{SP}_7$ has circular chromatic number at most 5/2 [35] (equivalently $\mathcal{SP}_7$ is bounded by the 5-cycle). The latter result is shown to be optimal in the following theorem, that we will use in one of our proofs.

Theorem 10 (Pan and Zhu [36])**.** *For any $\epsilon > 0$, there is a graph of $\mathcal{SP}_7$ with circular chromatic number at least $5/2 - \epsilon$.*

Given a graph G, a rational p/q, a positive integer k and a homomorphism h of G to $C_{p,q}$, a *pk-tight cycle* in G with respect to h is a cycle C of length pk such that any two consecutive vertices u_i and u_{i+1} of C satisfy $h(u_{i+1}) - h(u_i) = q \bmod p$. The following proposition is important when studying the circular chromatic number of a graph, and will be useful to us.

Proposition 11 (Guichard [16])**.** *Let G be a graph with $\chi_c(G) = p/q$. Then, in any homomorphism h of G to $C_{p,q}$, there is positive integer k such that G contains a tight pk-cycle with respect to h.*

3. Necessary and sufficient conditions for bounding $\mathcal{SP}_{2k+1}$

In this section, we develop necessary and sufficient conditions under which a graph B of odd-girth $2k + 1$ is a bound for $\mathcal{SP}_{2k+1}$. We first introduce some notions that are important to express these conditions.

3.1. Preliminaries

The aforementioned conditions are derived from a specific family of K_4-minor-free graphs defined below.

Definition 12. *Given any positive integer k and integers p, q, r between 1 and k, $T_{2k+1}(p, q, r)$ is the graph built as follows. Let u, v, w be three vertices. Join u to v by two disjoint paths of length p and $2k+1-p$, u to w by two disjoint paths of length q and $2k+1-q$ and v to w by two disjoint paths of length r and $2k+1-r$.*

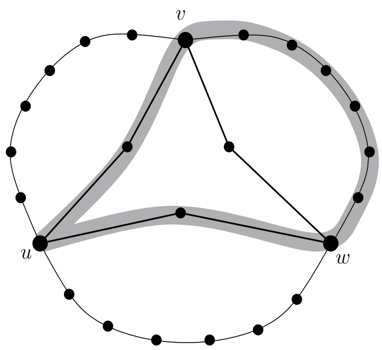


Figure 1: The graph $T_9(2, 2, 2)$; an 11-cycle goes trough u, v, w.

The graph $T_{2k+1}(p, q, r)$ is K_4-minor-free (for any values of k, p, q and r). We are mostly interested in the case where this graph has odd-girth $2k+1$, that is, when there is no odd-cycle going through u, v and w of length less than $2k+1$. A triple $\{p, q, r\}$ is called *k-good* if the odd-girth of $T_{2k+1}(p, q, r)$ is at least $2k+1$. Clearly, the shortest cycle of $T_{2k+1}(p, q, r)$ going through all three of u, v and w has length $p+q+r$ (see Figure 1 for an example). Note that there are exactly eight cycles going through all three of u, v and w, four of even length and four of odd length. Thus, deciding whether $\{p, q, r\}$ is k-good is an easy task. However, we provide an easier necessary and sufficient condition, as follows.

Proposition 13. *Let k be a positive integer and p, q, r be three integers between 1 and k. We have the following.*

(i) If $p+q+r$ is odd, then $\{p, q, r\}$ is k-good if and only if $p+q+r \geq 2k+1$.

(ii) If $p+q+r$ is even, then $\{p, q, r\}$ is k-good if and only if p, q, r satisfy the triangular inequalities ($p \leq q+r$, $q \leq p+r$ and $r \leq p+q$).

Proof. As mentioned before, $p+q+r$ is the length of a shortest cycle of $T_{2k+1}(p, q, r)$ containing all three of u, v and w. Thus, (i) follows directly from the definition of a k-good triple. For (ii) we may assume, without loss of generality, that $p \leq q \leq r$. Then, a shortest odd cycle of $T_{2k+1}(p, q, r)$ containing all three of u, v and w is of length $p+q+2k+1-r$, which is at least $2k+1$ if and only if $p+q \geq r$. □

As a direct consequence of Proposition 13, we obtain the following.

Observation 14. *Let k be a positive integer and let p, q be two integers between 1 and k. Then, $\{p, q, k\}$ is k-good if and only if $p+q \geq k$. In particular, $\{p, k, k\}$ is always a k-good triple.*

The following definition will be central to our work.

Definition 15. *Let k be a positive integer, B be a graph, and $(\widetilde{B}, d_B)$ a k-partial distance graph of B. For an edge xy with $d_B(xy) = p$, we say that a k-good triple $\{p, q, r\}$ is* realized *on xy if there are two vertices z_1 and z_2 of B with $d_B(x, z_1) = q$, $d_B(y, z_1) = r$, $d_B(x, z_2) = r$ and $d_B(y, z_2) = q$. We say that $(\widetilde{B}, d_B)$ has the* all k-good triple property *if $E(\widetilde{B}) \neq \emptyset$, and for each edge xy and each k-good triple $T = \{p, q, r\}$ with $p = d_B(x, y)$, T is realized on xy.*

We observe the following facts with respect to Definition 15.

Observation 16. *If a k-partial distance graph $(\widetilde{B}, d_B)$ of some graph B of odd-girth $2k+1$ has the all k-good triple property, then for every edge xy of $\widetilde{B}$ with $d_B(x,y) \le k$, there is a $(2k+1)$-cycle of B containing both x and y.*

Proof. Let p denote the distance between x and y in B. Our hypothesis is that p is less than or equal to k. Since

$$p + \left\lfloor \frac{2k+1-p}{2} \right\rfloor + \left\lceil \frac{2k+1-p}{2} \right\rceil = 2k+1,$$

Proposition 13(i) tells us that

$$\left\{ p, \left\lfloor \frac{2k+1-p}{2} \right\rfloor, \left\lceil \frac{2k+1-p}{2} \right\rceil \right\} \text{ is a } k\text{-good triple.}$$

Hence there is a vertex z of B such that xz and yz are both edges in $\widetilde{B}$ and

$$d_B(x,z) = \left\lfloor \frac{2k+1-p}{2} \right\rfloor \text{ and } d_B(y,z) = \left\lceil \frac{2k+1-p}{2} \right\rceil.$$

The paths of these lengths connecting z to x and z to y in B, together with a path from x to y of length p, form a closed walk of length $2k+1$ in B. Such a walk must contain an odd cycle, and since B is of odd-girth $2k+1$, this walk is a $(2k+1)$-cycle in which x and y are at distance p. □

Observation 17. *If a k-partial distance graph $(\widetilde{B}, d_B)$ of some graph B has the all k-good triple property, then for each p with $1 \le p \le k$, $(\widetilde{B}, d_B)$ contains an edge of weight p.*

Proof. By definition, $(\widetilde{B}, d_B)$ contains at least one edge e that has weight $d \le k$. By Observation 14, $\{d, k, k\}$ is a k-good triple. Therefore, again by definition, $(\widetilde{B}, d_B)$ contains an edge of weight k. Noting that $\{p, k, k\}$ is also a k-good triple completes the proof. □

3.2. Main theorem

We are now ready to prove the main theorem of this section. Roughly speaking, our claim is that for a graph B of odd-girth $2k+1$, the existence of a k-partial distance-graph of B with the all k-good triple property is both necessary and sufficient for B to be a minimal bound for $\mathcal{SP}_{2k+1}$. The more precise statement is as follows.

Theorem 18. *Let B be a graph and $(\widetilde{B}, d_B)$ be a k-partial distance graph of B. If B is of odd-girth $2k+1$ and $(\widetilde{B}, d_B)$ has the all k-good triple property, then B is a bound for $\mathcal{SP}_{2k+1}$. Furthermore, if B is a minimal bound (in terms of subgraph inclusion) for $\mathcal{SP}_{2k+1}$ and B has odd-girth $2k+1$, then there exists a k-partial distance graph $(\widetilde{B}, d_B)$ of B which has the all k-good triple property and moreover, $E(B) \subseteq E(\widetilde{B})$.*

Proof. For the first part of the claim, suppose that B is of odd-girth $2k+1$ and that $(\widetilde{B}, d_B)$ has the all k-good triple property. We show that every K_4-minor-free graph of odd-girth at least $2k+1$ admits a homomorphism to B.

Let G be a graph of order n in $\mathcal{SP}_{2k+1}$. By Theorem 9, G is a partial 2-tree, hence it is obtained from a 2-tree H by removing some edges. Let H be a 2-tree such that $V(H) = V(G)$ and $E(G)$ is a subset of $E(H)$. The 2-tree structure of H gives us a linear ordering of its vertices $v_0, v_1, \ldots, v_n$ such that the edge v_0v_1 is in H and for any i between 2 and n, v_i has exactly two neighbours among $\{v_j : j < i\}$ in H that form an edge in H. For i between 1 and n, let H_i denote the graph H induced by the vertices $v_0, v_1, \ldots, v_i$.

For any edge xy in H, we define $\omega(xy) = \min\{d_G(x,y), k\}$. We now build a weighted graph homomorphism of (H, ω) to $(\widetilde{B}, d_B)$. For this, we build a weighted graph homomorphism of (H_i, ω) to $(\widetilde{B}, d_B)$ for each i from 1 to n. When i is stricly less than n, ω is understood as its restriction to $V(H_i)$.

Since $(\widetilde{B}, d_B)$ has the all k-good triple property, by Observation 17, there exists an edge xy of weight $\omega(v_0v_1)$. Let us map v_0 and v_1 to x and y respectively. Then (H_1, ω) admits a weighted graph homomorphism to $(\widetilde{B}, d_B)$.

Suppose that for some i between 1 and $n-1$, (H_i, ω) has a weighted graph homomorphism ϕ to $(\widetilde{B}, d_B)$. We shall extend it by selecting an adequate image for v_{i+1}. Let x and y be the two neighbours of v_{i+1} in H_{i+1}. They form an edge in H_i and thus $d_B(\phi(x), \phi(y)) = \omega(xy)$. Let us define $p = \omega(xy)$, $q = \omega(v_{i+1}x)$ and $r = \omega(v_{i+1}y)$.

We claim that $\{p, q, r\}$ is a k-good triple. To prove our claim we consider three possibilities:

- At least two of p, q and r are equal to k: this case follows from Observation 14.
- Exactly one of p, q and r is equal to k: let us say $p = k$ without loss of generality. It means that the distance in G between x and y is greater than or equal to k while q and r are the actual distances between v_{i+1} and x and y. The triangular inequality is satisfied by the distances in G so that $q + r \geq d_G(x, y) \geq k = p$. By Observation 14, $\{p, q, r\}$ is a k-good triple.
- None of p, q and r is equal to k: then p, q and r are the actual distances and verify the triangular inequalities. Moreover, if $p + q + r$ is odd, this sum is at least $2k + 1$ since the odd-girth of G is at least $2k + 1$. By Proposition 13, $\{p, q, r\}$ is a k-good triple.

Then, since $(\widetilde{B}, d_B)$ has the all k-good triple property, there exists a vertex z in $V(B)$ such that $d_B(z, \phi(x)) = q$ and $d_B(z, \phi(y)) = r$. Let $\phi(v_{i+1})$ be this vertex z. Now ϕ is a homomorphism of (H_{i+1}, ω) to $(\widetilde{B}, d_B)$.

By the end of the process, we have proved the existence of (and built) a homomorphism of (H, ω) to $(\widetilde{B}, d_B)$. The edges of G are exactly those edges of H with weight 1. Since ϕ sends these edges to edges of $(\widetilde{B}, d_B)$ with weight 1, it means that ϕ induces a homomorphism of G to B.

We now prove the second claim of the statement: let B be a minimal bound of odd-girth $2k + 1$ for $\mathcal{SP}_{2k+1}$; our aim is to build a k-partial distance graph of B with the all k-good triple property.

Let $\mathcal{C}$ be the class of k-partial distance graphs $(\widetilde{G}, d_G)$ satisying:
(i) $G \in \mathcal{SP}_{2k+1}$;
(ii) $\widetilde{G}$ is a 2-tree;
(iii) $(\widetilde{G}, d_G)$ is a k-partial distance graph of G;
(iv) for every edge uv of $\widetilde{G}$, u and v lie on a common $(2k + 1)$-cycle of G.

It is clear that $\mathcal{C}$ is nonempty, for example for $G = C_{2k+1}$ it is easy to construct a corresponding k-partial distance graph satisfying (i)–(iv).

Our aim is to show that if B^* ($B \subseteq B^*$) is minimal such that (B^*, d_B) bounds $\mathcal{C}$, then (B^*, d_B) has the all k-good triple property. We first need to show that such a B^* exists. To this end, we show that $(K_{|V(B)|}, d_B)$ is a bound for $\mathcal{C}$.

Indeed, since B bounds $\mathcal{SP}_{2k+1}$, there exists a homomorphism $f : G \to B$. We claim that f is also a weighted graph homomorphism of $(\widetilde{G}, d_G)$ to $(K_{|V(B)|}, d_B)$. Clearly, f preserves edges of weight 1. Now, let uv be an edge of weight $p \geq 2$ in $\widetilde{G}$. By Property (iv), u and v lie on a common $(2k + 1)$-cycle of G. By Observation 7, we have $d_G(u, v) = d_B(f(u), f(v))$, that is, uv is mapped to an edge of weight p which shows the claim for f.

Now, consider a minimal graph B^* with $B \subseteq B^*$ such that (B^*, d_B) bounds the class $\mathcal{C}$. By Property (iv), every edge of a weighted graph in $\mathcal{C}$ has weight at most k, therefore this is also the case for B^*. In other words, (B^*, d_B) is a k-partial distance graph of B. We will show that (B^*, d_B) has the all k-good triple property.

Clearly, we have $E(B^*) \neq \emptyset$. Therefore, assume by contradiction that for some edge xy with $d_B(x, y) = p$ and some k-good triple $\{p, q, r\}$, there is no vertex z in B^* with $xz, yz \in E(B^*)$, $d_B(x, z) = q$ and $d_B(y, z) = r$. By minimality of B^*, there exists a weighted graph $(\widetilde{G_{xy}}, d_{G_{xy}})$ of $\mathcal{C}$ such that for any homomorphism f of $(\widetilde{G_{xy}}, d_{G_{xy}})$ to (B^*, d_B), there is an edge ab of $\widetilde{G_{xy}}$ of weight p, $f(a) = x$ and $f(b) = y$.

We now build a new weighted graph from $(\widetilde{G_{xy}}, d_{G_{xy}})$ as follows. Let $\widehat{T}$ be a 2-tree completion of the graph $T = T_{2k+1}(p, q, r)$ where $\widehat{T}$ contains the triangle uvw; this triangle has weights p, q and r in $(\widehat{T}, d_T)$. Then, for each edge ab of $\widetilde{G_{xy}}$ with $d_{G_{xy}}(a, b) = p$, we add a distinct copy of $(\widehat{T}, d_T)$ to $(\widetilde{G_{xy}}, d_{G_{xy}})$ by identifying the edge ab with the edge uv of $(\widehat{T}, d_T)$ (that both have weight p). It is clear that the resulting weighted graph, that we call $(\widetilde{G'_{xy}}, d_{G'_{xy}})$, belongs to the class $\mathcal{C}$. Moreover, $(\widetilde{G_{xy}}, d_{G_{xy}})$ is a subgraph of $(\widehat{G'_{xy}}, d_{G'_{xy}})$ (indeed the new vertices added in the construction have not altered the distances between original vertices of G_{xy}). Thus, there exists a homomorphism ϕ of $(\widehat{G'_{xy}}, d_{G'_{xy}})$ to (B^*, d_B), whose restriction to the subgraph $(\widetilde{G_{xy}}, d_{G_{xy}})$ is also a homomorphism. Therefore, by the choice of G_{xy}, at least one pair a, b of vertices of $\widehat{G'_{xy}}$ with $d_{G'_{xy}}(a, b) = p$ is mapped by ϕ to x and y, respectively. But then, the copy of $\widehat{T_{2k+1}}(p, q, r)$ added to G for this pair forces the existence of the desired triangle on edge xy in B^*, which is a contradiction and completes the proof. $\square$

3.3. Some properties of minimal bounds

We now show that a minimal bound must satisfy some simple structural conditions. The following lemmas are examples of such conditions that are useful in the theoretical investigation of minimal bounds, as we will see in Section 6.

Lemma 19. *Let k be strictly greater than 1. If B is a minimal bound (in terms of subgraph inclusion) of odd-girth $2k+1$ for $\mathcal{SP}_{2k+1}$, then any degree 2-vertex belongs to a 6-cycle.*

Proof. Let u be a vertex of degree 2 in B, with v, w its two neighbours. By the second part of Theorem 18, we can assume that the partial distance graph $(\widetilde{B}, d_B)$ of B with the all k-good triple property contains all edges of B, in particular, the edge uv. The triple $\{1, 1, 2\}$ is k-good and hence, it must be realizable on edge uv, which implies that the edge vw of weight 2 belongs to $(\widetilde{B}, d_B)$. But then, the only way to realize the k-good triple $\{2, 2, 2\}$ on vw is if u is part of a 4-cycle or a 6-cycle. But if u is part of a 4-cycle, then B is not a core, contradicting its minimality. $\square$

In the following lemma, we remark that the claim holds with respect to minimality in terms of *induced* subgraph inclusion, which is stronger than the minimality condition of Theorem 18 and Lemma 19 (which is just about subgraph inclusion).

Lemma 20. *Let k be strictly greater than 1. If B is a minimal bound (in terms of induced subgraph inclusion) of odd-girth $2k+1$ for $\mathcal{SP}_{2k+1}$, then the set of degree 2-vertices of B forms an independent set.*

Proof. By the hypothesis, B has a spanning subgraph B' that is a minimal bound in terms of subgraph inclusion. Then B' is a core and, therefore, has minimum degree at least 2. If two adjacent vertices of B have degree 2, then they must have degree 2 in B'. By Lemma 19, they must be part of the same 6-cycle. But then B' is not a core, a contradiction. $\square$

The following lemma requires an even stronger minimality condition than the one in Lemma 20.

Lemma 21. *Let k be striclty greater than 1. If B is a minimal bound (in terms of induced subgraph inclusion) of odd-girth $2k+1$ for $\mathcal{SP}_{2k+1}$ that has no homomorphism to any smaller graph of odd-girth $2k+1$, then any 6-cycle C of B can contain at most two degree 2 vertices. If furthermore C contains two such vertices, then they must be at distance 3 in C (and in B).*

Proof. Let u be a degree 2 vertex of B belonging to a 6-cycle $C : uvwxyz$. By Lemma 20, v and z must have degree at least 3. Assume for a contradiction that y or w (say w) has degree 2. Then, u and w must belong to a common $(2k+1)$-cycle (otherwise, identifying u and w would give a homomorphism to a smaller graph of odd-girth $2k+1$). Moreover, since $deg(u) = deg(w) = 2$, this $(2k+1)$-cycle uses four edges of C. Thus, replacing them with the two other edges of C yields a $(2k-1)$-cycle, a contradiction. $\square$

4. A polynomial-time algorithm to check whether a given graph of odd-girth $2k+1$ bounds $\mathcal{SP}_{2k+1}$

In this section, we show that the characterization of Theorem 18 is sufficiently strong to imply the existence of a polynomial-time algorithm that checks whether a given graph of odd-girth $2k+1$ bounds $\mathcal{SP}_{2k+1}$. We describe this algorithm as Algorithm 1.

Algorithm 1. *Deciding whether a graph of odd-girth $2k+1$ bounds $\mathcal{SP}_{2k+1}$.*

Input: An integer k, a graph B.

1: Compute the odd-girth g of B.
2: **if** $g \neq 2k+1$ **then**
3:     **return** NO *# (B is not a bound)*
4: **end if**
5: Compute the distance function d_B of B.
6: Let $(\widetilde{B}, d_B)$ be the k-partial distance graph of B obtained from the complete distance graph of B by removing all edges of weight more than k.
7: Compute the set $\mathcal{T}_k$ of k-good triples.
8: **for** $e = xy$ in $E(\widetilde{B})$ with $d_B(xy) \leq k$ **do**
9:     $p \leftarrow d_B(xy)$
10:     **for** each k-good triple $t = \{p, q, r\} \in \mathcal{T}_k$ containing p **do**
11:         **if** there is no pair z, z' of $V(B)$ with $d_B(xz) = d_B(yz') = q$ and $d_B(yz) = d_B(xz') = r$ **then** *# (the edge e fails for the all k-good triple property)*
12:             **if** $d_B(xy) \geq 2$ **then**
13:                 $\widetilde{B} \leftarrow (\widetilde{B} - uv)$
14:                 Restart the loop (Step 8).
15:             **else** *# (uv is an edge of B)*
16:                 **return** Algorithm 1(k,$B - uv$). *# (Recursive call with a smaller graph.)*
17:             **end if**
18:         **end if**
19:     **end for**
20: **end for**
21: **return** YES *# ($\widetilde{B}$ is a certificate)*

end

We now analyze Algorithm 1.

Theorem 22. *Algorithm 1 checks in time $O(mn^5k^3) = O(mn^8)$ whether a given graph B of odd-girth $2k+1$ with n vertices and m edges bounds $\mathcal{SP}_{2k+1}$.*

Proof. We prove that Algorithm 1 is correct, that is, it returns "YES" if and only if B is a bound for $\mathcal{SP}_{2k+1}$.

Assume first that Algorithm 1 returns "YES". Then, it has found a k-partial distance graph of some subgraph B' of B and a k-partial distance graph $(\widetilde{B'}, d_{B'})$ of B such that each edge of $\widetilde{B'}$ has passed the check of Step 11. Therefore, $(\widetilde{B'}, d_{B'})$ has the all k-good triple property and by Theorem 18, B' is a bound — and so is B.

Assume now that B is a bound for $\mathcal{SP}_{2k+1}$. Then, it contains a minimal subgraph B' that is also a bound. By Theorem 18, there is a k-partial distance graph $(\widetilde{B'}, d_{B'})$ of B' having the all k-good triple property.

We will now show that no edge xy of $\widetilde{B'}$ will ever be deleted by the algorithm (that is, xy will always succeed the check of Step 11). By contradiction, assume that xy is the first edge of $\widetilde{B'}$ that does not succeed the check of Step 11. By Observation 16, for each edge uv of $\widetilde{B'}$, there is a $(2k+1)$-cycle C of B' (and hence B) containing both x and y (recall that by definition $d_{B'}(uv) \leq k$). By Observation 7, $d_{B'}(xy) = d_B(xy) \leq k$. By our assumption on xy being the first edge to be deleted, all edges of $\widetilde{B'}$ are still present in $\widetilde{B}$ at this step of the execution. Furthermore, the distances in the copy of B' in B are the same as the distances in B' and hence the edges of $\widetilde{B'}$ in $\widetilde{B}$ have the same weights as in $\widetilde{B'}$. Hence, xy cannot fail the check of Step 11, which is a contradiction.

By the previous paragraph, Algorithm 1 will never delete any edge of $\widetilde{B'}$ from $\widetilde{B}$. Therefore, even in a possible recursive call at Step 16, the input graph will always have odd-girth $2k+1$ and the k-partial distance graph will never become empty. Thus, Algorithm 1 will never return "NO". Therefore, at some step, there will be a k-partial distance graph (of some subgraph of B) having the all k-good triple property, and Algorithm 1 will return "YES". Thus, Algorithm 1 is correct.

Now, for the running time, note that there are $O(m)$ recursive calls to the algorithm (Step 16) since each call corresponds to the deletion of one edge of B. For each call, we have the computation of the odd-girth of B at Step 1, which can be done in time $O(n(m + n \log n)) = O(n^3)$ (see for example the literature review in [23]). Algorithm 1 computes all distances at Step 5 and then creates $\widetilde{B}$, which can be done in $O(n^3)$ steps using Dijkstra's algorithm. Then, the loop of Step 8 is over $O(n^2)$ pairs, and for each pair, we have $O(k^3)$ k-good triples to check; each check needs to go through $O(n)$ vertices of B. Hence, one iteration of the loop takes $O(nk^3)$ steps. However, the loop may be restarted $O(n^2)$ times at Step 13 (at most once for each of the $O(n^2)$ edges of $\widetilde{B}$), hence there may be $O(n^4)$ total iterations of the loop. Therefore, each recursive call to the algorithm may take $O(n^5k^3)$ time, and the total running time is $O(mn^5k^3)$. This is also $O(mn^8)$, indeed $k = O(n)$ because B must contain a $(2k+1)$-cycle (to be more precise, if this is not the case, the algorithm stops at Step 3). □

Given an input graph B of odd-girth $2k+1$, when our algorithm returns "NO", it would be interesting to produce, as an explicit NO-certificate, a K_4-minor-free graph of odd-girth $2k+1$ which does not admit a homomorphism to B. Algorithm 1 could provide such a certificate, the rough ideas are as follows. Since B is a NO-instance, Algorithm 1, after deleting a sequence $e_1, e_2, \dots, e_m$ of weighted edges, has considered a graph B_m which is either bipartite or has odd-girth larger than $2k+1$. Thus, C_{2k+1} is a member of $\mathcal{SP}_{2k+1}$ which does not admit a homomorphism to B_m. Let $\widehat{C_{2k+1}}$ be a 2-tree containing C_{2k+1} as a spanning subgraph and let $G_m = (\widehat{C_{2k+1}}, d_{C_{2k+1}})$ be the corresponding k-partial distance graph of C_{2k+1}. Starting from this, and in reverse order from $i = m-1$ to $i = 1$, we build a weighted graph G_i from G_{i+1} as follows. Assume that in Algorithm 1, the edge e_i of weight p has been deleted because of no realized k-good triple $\{p, q, r\}$ on e_i. Let $(\widehat{T}, d_T)$ be a weighted 2-tree completion of the graph $T_{2k+1}(p, q, r)$ where $\widehat{T}$ includes the edge uv of weight p of $T_{2k+1}(p, q, r)$. Then, on each edge xy of weight p of the weighted graph G_{i+1}, glue two disjoint copies of $(\widehat{T}, d_T)$ (one copy identifing u with x and v with y, and one copy identifying u with y and v with x). At the final step, we obtain a weighted graph G_1 whose subgraph induced by the edges of weight 1 does not admit a homomorphism to B. Note however that the order of this graph could be super-polynomial in $|V(B)|$; we do not know if one can create such a NO-certificate of order polynomial in $|V(B)|$.

We conclude this section by asking for a similar (not necessarily polynomial-time) algorithm as Algorithm 1 for the case of planar graphs. Let $\mathcal{P}_{2k+1}$ be the class of planar graphs of odd-girth at least $2k+1$.

Problem 23. *For a fixed k, give an explicit algorithm which decides whether an input graph of odd-girth $2k+1$ bounds $\mathcal{P}_{2k+1}$.*

For $k = 1$ and by the virtue of the Four-Colour Theorem, one only needs to check whether B contains K_4 as a subgraph. We observe that the existence of an algorithm that does not use the Four-Colour Theorem is related to Problem 2.1 of the classic book on graph colouring by Jensen and Toft [21].

For other values of k, we do not know of any explicit algorithm. Note however, that a hypothetic algorithm exists: for a given graph B of odd-girth $2k+1$ not bounding $\mathcal{P}_{2k+1}$, let $f_k(B)$ be the smallest order of a graph in $\mathcal{P}_{2k+1}$ with no homomorphism to B, and let $f_k(n)$ be the maximum of f_k over all such graphs of order n. Then, given B with odd-girth $2k+1$, one may simply check, for all graphs in $\mathcal{P}_{2k+1}$ of order at most $f_k(|V(B)|)$, whether it maps to B. Since f_k is well-defined, this is a finite-time algorithm, but it relies on the knowledge of f_k. Note that part of Problem 2.1 of [21] consists in giving an upper bound on $f_1(K_4)$ without using the Four-Colour Theorem, which is already a difficult problem.

5. General families of bounds

In this section, we exhibit three bounds of odd-girth $2k+1$ for the class $\mathcal{SP}_{2k+1}$.

5.1. Projective hypercubes

Projective hypercubes are well-known examples of very symmetric graphs. Property *(ii)* of the following lemma is to claim that projective hypercubes are *distance-transitive*, which is a well-known fact. This is needed for a proof of Property *(iii)*, hence we include a proof for the sake of completeness.

Lemma 24. *The projective hypercube $PC(2k)$ has the following properties.*

(i) For each pair x, y of vertices of $PC(2k)$, $d(x, y) \leq k$; furthermore, x and y belong to at least one $(2k+1)$-cycle.

(ii) *If $d(u,v) = d(x,y)$ for some vertices u, v, x, y of $PC(2k)$, then there is an automorphism of $PC(2k)$ which maps u to x and v to y (in other words, $PC(2k)$ is distance-transitive).*

(iii) *Let $\{p, q, r\}$ be a k-good triple with $1 \leq p, q, r \leq k$. Suppose x and y are two vertices of $PC(2k)$ at distance p. Let $\phi(u) = x$ and $\phi(v) = y$ be a mapping of two (main) vertices of $T_{2k+1}(p, q, r)$. Then, ϕ can be extended to a homomorphism of $T_{2k+1}(p, q, r)$ to $PC(2k)$.*

Proof. We use the Cayley representation of the projective hypercube.

(i). Let D be the set of coordinates at which x and y differ, and $\overline{D}$ be the complement of D, that is, the set of coordinates at which x and y do not differ. Then $x = y + \sum_{i \in D} e_i = y + J + \sum_{i \in \overline{D}} e_i$. Let P_1 be the path connecting x and y by adding elements of D to x in a consecutive way. Similarly, let P_2 be the path connecting x and y by adding elements of $\overline{D} \cup \{J\}$. The smaller of P_1 and P_2 provides the distance between x and y and the union of the two is an example of a $(2k+1)$-cycle containing both x and y.

(ii). We need a more symmetric representation of $PC(2k)$. Let S' be the set of $2k+1$ elements of $\mathbb{Z}_2^{2k+1}$, each with exactly two 1's which are consecutive in the cyclic order. Then, the Cayley graph $(\mathbb{Z}_2^{2k+1}, S')$ has two connected components, each isomorphic to $PC(2k)$. It is now easy to observe that any permutation of S' (equivalently, any permutation of S in the original form) induces an automorphism of $PC(2k)$. Thus, to map uv to xy, we could first map u to x by the automorphism $\phi(t) = t + x - u$. Then, composing ϕ with a permutation of S' which maps $\phi(v)$ to y will induce an automorphism that maps u to x and v to y.

(iii). Let a, b and c be chosen so that $a \in \{p, 2k+1-p\}, b \in \{q, 2k+1-q\}$, $c \in \{r, 2k+1-r\}$ and that $a+b+c$ is the length of a shortest odd-cycle going through all u, v and w. First, we want to give a homomorphism of $T_{2k+1}(p, q, r)$ to $PC(2k)$. Without loss of generality, we may assume that $a \leq b \leq c$.

Since $a+b+c$ is odd and $2k+1 \leq a+b+c \leq 3k+1$, there is an integer t such that $a+b+c = 2k+1+2t$ (thus $0 \leq t \leq k-1$). We first note that $b+c \leq 2k+1$, as otherwise $a+(2k+1-b)+(2k+1-c) < a+b+c$, contradicting the choice of a, b, c. Thus we have $2t \leq a$.

Let S_1, S_2, S_3 be a partition of $S = \{e_1, \ldots, e_{2k}, J\}$ with $|S_1| = a$, $|S_2| = b - t$ and $|S_3| = c - t$. Let S_1' be a subset of size t of S_1. Let $S_2^+ = S_2 \cup S_1'$ and $S_3^+ = S_3 \cup S_1'$. Let $x = \vec{0}$ (the 0 vector in $\mathbb{Z}_2^{2k}$), $y = \sum_{v \in S_1} v$, and $z = \sum_{v \in S_2^+} v = y + \sum_{v \in S_3^+} v$. It is now easy to check that $d(x,y) \in \{a, 2k+1-a\}$, $d(x,z) \in \{b, 2k+1-b\}$ and $d(z,y) \in \{c, 2k+1-c\}$. Furthermore by Part (i), each pair of vertices of $PC(2k)$ is in a $(2k+1)$-cycle. Thus, the mapping of $\phi(u) = x$, $\phi(v) = y$ and $\phi(w) = z$ can be extended to a homomorphism of $T_{2k+1}(p, q, r)$ to $PC(2k)$.

This shows that there is a homomorphism ϕ of $T_{2k+1}(p, q, r)$ to $PC(2k)$ which maps u to $\vec{0}$. Since both these graphs are of odd-girth $2k+1$, by Observation 8, ϕ must preserve the distance (in $T_{2k+1}(p, q, r)$) between pairs in $\{u, v, w\}$. Now, the proof is completed using Part (ii) (distance-transitivity of $PC(2k)$). □

Theorem 25. *The projective hypercube $PC(2k)$ bounds $\mathcal{SP}_{2k+1}$.*

Proof. We show that $(K_{2^{2k}}, d_{PC(2k)})$ has the all k-good triple property, which by Theorem 18 will prove our claim. By Lemma 24(i), $(K_{2^{2k}}, d_{PC(2k)})$ is a k-partial distance graph of $PC(2k)$. By Lemma 24(ii), $PC(2k)$ is distance-transitive. It remains to prove that for each edge xy of weight p ($1 \leq p \leq k$) of $(K_{2^{2k}}, d_{PC(2k)})$, every k-good triple $\{p, q, r\}$ is realized on the edge xy. Consider the graph $T = T_{2k+1}(p, q, r)$. We define a mapping ϕ of T to $PC(2k)$ by first mapping the two vertices u and v of T with degree 4 and at distance p in T to x and y. By Lemma 24(iii), we can extend ϕ to the whole of T. By Observation 7, and by considering the three $(2k+1)$-cycles of T, we have $d_{PC(2k)}(x,z) = q$ and $d_{PC(2k)}(y,z) = r$. This completes the proof. □

Theorem 25 has applications to edge-colourings, that will be discussed in Section 7.

5.2. Kneser graphs

It was recently shown [31] that if $PC(2k)$ bounds the class of planar graphs of odd-girth $2k+1$ (that is, if Conjecture 4 holds), then it is an optimal bound of odd-girth $2k+1$ (in terms of the order). However, for K_4-minor-free graphs, $PC(2k)$ is far from being optimal.

Consider the hypercube $H(2k+1)$; its vertices can be labeled by subsets of a $(2k+1)$-set (call it U) where X and Y are adjacent if $X \subset Y$ and $|X|+1 = |Y|$. In this notation, antipodal pairs are complementary sets. Recall that $PC(2k)$ is obtained from $H(2k+1)$ by identifying antipodal pairs of vertices. Thus, the vertices of $PC(2k)$ can be labeled by pairs of complementary subsets of U, or simply by a subset of size at most k (the smaller of the two).

It is not difficult to observe the following (where $S \triangle T$ denotes the symmetric difference of sets S and T).

Observation 26. *For any set $A \subseteq U$, the mapping f_A defined by $f_A(\{X, \overline{X}\}) = \{A \triangle X, A \triangle \overline{X}\}$ is an automorphism of $PC(2k)$.*

Using the above presentation of $PC(2k)$, the set of vertices at distance k from $\emptyset$ are the k-subsets of U, and two such vertices are adjacent if they have no intersection (because then one is a subset of the complement of the other and the difference of sizes is 1). Thus, the Kneser graph $K(2k+1,k)$ (also known as *odd graph*) is an induced subgraph of $PC(2k)$. These graphs are well-known examples of distance-transitive graphs (indeed the distance between two vertices is determined by the size of their intersection). In particular, $K(5,2)$ is the Petersen graph. We refer to [13] for more details on this family of graphs.

The following theorem is then a strengthening of Theorem 25.

Theorem 27. *The Kneser graph $K(2k+1,k)$ bounds $\mathcal{SP}_{2k+1}$.*

Proof. We use Theorem 18 by showing that $(K_n, d_{K(2k+1,k)})$ with $n = |V(K(2k+1,k))| = \binom{2k+1}{k}$ is a k-partial distance graph of $K(2k+1,k)$ and has the all k-good triple property. Since $K(2k+1,k)$ is distance-transitive, it suffices to prove that for any k-good triple $\{p,q,r\}$, there are three vertices in $K(2k+1,k)$ whose pairwise distances are p, q and r. We will use Theorem 25 and the above-mentioned presentation of $K(2k+1,k)$ as an induced subgraph of $PC(2k)$. In this presentation, each vertex of $PC(2k)$ is a pair $(S,\overline{S})$ of subsets of a $(2k+1)$-set U, where $|S| \leq k$. We may use any of S or $\overline{S}$ to denote the vertex $(S,\overline{S})$.

Let $\{p,q,r\}$ be a k-good triple. By Theorem 25, there are three vertices A, B and C of $PC(2k)$ with $|A| \leq k$, $|B| \leq k$ and $|C| \leq k$ such that their pairwise distances are p, q and r. Our goal is to select a set X of elements of U, such that the sizes of their symmetric differences $A \triangle X$, $B \triangle X$ and $C \triangle X$ are k or $k+1$. Then, all these three new vertices belong to an induced subgraph of $PC(2k)$ isomorphic to $K(2k+1,k)$. Since by Observation 26 the operation is an automorphism of $PC(2k)$, we have proved the claim for the triple $\{p,q,r\}$. In fact, we will construct X in four steps.

First, let X_1 be the set of elements of U that belong to at least two of the sets A, B, and C, and let $A_1 = A \triangle X_1$, $B_1 = B \triangle X_1$ and $C_1 = C \triangle X_1$. Then, A_1, B_1 and C_1 are pairwise disjoint.

From A_1, B_1 and C_1, we build three sets A_2, B_2 and C_2 such that the difference of the sizes of any two of them is at most 1. To do this, assuming $|A_1| \geq |B_1| \geq |C_1|$, we first consider a set X_2 of $\lfloor(|B_1| - |C_1|)/2\rfloor$ elements of B_1 (none of them belongs to C_1 since B_1 and C_1 are disjoint), and consider again the symmetric differences of the three sets with X_2: $A_2 = A_1 \triangle X_2$, $B_2 = B_1 \triangle X_2$ and $C_2 = C_1 \triangle X_2$. Now, we have $|B_2| \geq |C_2| = \lfloor(|B_1|+|C_1|)/2\rfloor$ and $|B_2| - |C_2| \leq 1$, moreover $A_2 \cap B_2 = \emptyset$. We repeat the operation with A_2 and C_2: let X_3 be a set of $\lfloor(|A_2| - |C_2|)/2\rfloor$ elements of $A_2 \setminus (B_2 \cup C_2)$ (such a set exists because $\lfloor(|A_2| - |C_2|)/2\rfloor = \lfloor(|A_1| - |C_1|)/2\rfloor \leq |A_1| = |A_2 \setminus (B_2 \cup C_2)|$). We let $A_3 = A_2 \triangle X_3$, $B_3 = B_2 \triangle X_3$ and $C_3 = C_2 \triangle X_3$. Now, the size of each of A_3, B_3 and C_3 is s or $s+1$, with $s = \lfloor(|A_1| + |B_1| + |C_1|)/3\rfloor$.

Moreover, we have $A_3 \cup B_3 = A_1 \cup B_1$. Since A_1 and B_1 are disjoint and $|A_1| \leq k$ and $|B_1| \leq k$, we have $s \leq k$. If $s = k$, we are done. Otherwise, consider the set X_4 of elements which are in none of A_3, B_3 and C_3. If $|X_4| \geq k-s$, selecting a subset X_4' of X_4 of size $k-s$ and adding X_4' to all three of A_3, B_3 and C_3, we are done. Otherwise, we have $|X_4| < k-s$. Nevertheless, let $A_4 = A_3 \triangle X_4 = A_3 \cup X_4$, $B_4 = B_3 \triangle X_4 = B_3 \cup X_4$ and $C_4 = C_3 \triangle X_4 = C_3 \cup X_4$. Note that now we have $A_4 \cup B_4 \cup C_4 = U$, and the size of each of A_4, B_4 and C_4 is t or $t+1$, with $t < k$.

Let n_a be the number of elements that are in A_4 but not in B_4 nor C_4, that is, $n_a = |A_4 \setminus (B_4 \cup C_4)|$; n_b and n_c are defined in the same way. Similarly, let $n_{ab} = |(A_4 \cup B_4) \setminus C_4|$, and we define n_{bc} and n_{ac} in the same way. We claim that $n_a \geq k-t$. Note that $n_a + n_b + n_{ab} = |U| - n_c$. Hence, since $n_c \leq q+1 \leq k$, we have $n_a + n_b + n_{ab} \geq k+1$. But since $n_b + n_{ab} \leq |B_4| \leq t+1$, we have $n_a \geq k-t$. The same argument shows that $n_b \geq k-t$ and $n_c \geq k-t$.

Therefore, we can select three sets of size $k-t$ in each of $A_4 \setminus (B_4 \cup C_4)$, $B_4 \setminus (A_4 \cup C_4)$ and $C_4 \setminus (A_4 \cup B_4)$; let X_5 be the union of these three sets. Then, the sets $A_5 = A_4 \triangle X_5$, $B_5 = B_4 \triangle X_5$ and $C_5 = C_4 \triangle X_5$ all have size k or $k+1$, which completes the proof. □

By the definition of fractional chromatic number, the following is an immediate corollary of Theorem 27.

Corollary 28. *For every graph G in $\mathcal{SP}_{2k+1}$, we have $\chi_f(G) \leq 2 + \frac{1}{k}$.*

This bound is tight as $\chi_f(C_{2k+1}) = 2 + \frac{1}{k}$.

During the writing of this paper, it has come to our attention that Theorem 27 and Corollary 28 were obtained, independently, in a recent preprint of Feder and Subi [9] and also by Goodard and Xu [12].

5.3. Augmented toroidal grids

In this subsection, we provide a bound of odd-girth $2k+1$ and of order $4k^2$ for $\mathcal{SP}_{2k+1}$.

For any pair of integers (a, b), let $T(a, b)$ denote the cartesian product $C_a \square C_b$. This graph can be seen as the *toroidal grid of dimension* $a \times b$. Figure 2 depicts a representation of $T(24, 24)$.

Figure 2: A representation of the 24×24 toroidal grid.

The graph $T(2a, 2b)$ is of diameter $a + b$ and, furthermore, given a vertex v, there is a unique vertex at distance $a + b$ of v which is therefore called *antipodal* of v; we denote it by $\overline{v}$.

The *augmented toroidal grid* of dimensions $2a$ and $2b$, denoted $AT(2a, 2b)$ is the graph obtained from $T(2a, 2b)$ by adding an edge between v and $\overline{v}$ for each vertex v. We will restrict ourselves to augmented toroidal grids of equal dimensions. More formally, for any positive integer k, let $AT(2k, 2k)$ be the graph defined on the vertex set $\{0, 1, \ldots, 2k-1\}^2$ such that a pair $\{(i_1, j_1), (i_2, j_2)\}$ is an edge if

$i_1 = i_2$ and $|j_1 - j_2| \in \{1, 2k-1\}$ (vertical edges),

or $j_1 = j_2$ and $|i_1 - i_2| \in \{1, 2k-1\}$ (horizontal edges),

or $i_2 - i_1 + k \in \{0, 2k\}$ and $j_2 - j_1 + k \in \{0, 2k\}$ (antipodal edges).

Figure 3 gives a representation of the graph $AT(6, 6)$.

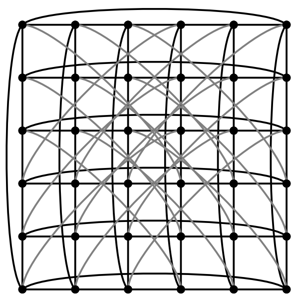


Figure 3: The augmented toroidal grid $AT(6, 6)$. Gray edges belong to $AT(6, 6)$ but not to $T(6, 6)$.

Observe that after the removal of parallel edges, the graph $AT(2, 2)$ is isomorphic to $PC(2)$ (that is K_4). Similarly, $AT(4, 4)$ is isomorphic to $PC(4)$. Indeed, $T(4, 4)$ is $C_4 \square C_4$ that is $(K_2 \square K_2) \square (K_2 \square K_2)$

which is isomorphic to $H(4)$ (and antipodal vertices of $T(4,4)$ are antipodal vertices of $H(4)$). The previous observation is not true for other values of k. Nevertheless, in general, $AT(2k,2k)$ is a subgraph of $PC(2k)$. This property and others are gathered in the following lemma.

Lemma 29. *For any positive integer k, let AT denote the graph $AT(2k,2k)$ and T the graph $T(2k,2k)$. The following statements are true.*

(i) AT is a subgraph of $PC(2k)$.

(ii) AT is vertex-transitive.

(iii) Any two vertices of AT belong to a common $(2k+1)$-cycle, hence AT has diameter at most k.

(iv) AT has odd-girth $2k+1$, hence AT has diameter exactly k.

(v) Any vertex v in $V(AT)$ can be seen as a vertex in $V(T)$ and, for any two vertices u and v in $V(AT)$,

$$d_{AT}(u,v) = \min\{d_T(u,v), 2k+1-d_T(u,v)\}.$$

Moreover, for any vertex u in $V(AT)$ and any integer d between 1 and k, the neighbourhood of u at distance d in AT is the set

$$N^d_{AT}(u) = N^d_T(u) \cup N^{d-1}_T(\overline{u}).$$

Proof. (i). One may label the edges of AT with canonical vectors $e_1, e_2, \ldots, e_m$ of $\{0,1\}^{2k}$ and J as follows (indices are now to be understood modulo $2k$):

$\{(i-1,j),(i,j)\}$ with label e_i if $i \leq k$ and e_{i-k} otherwise,

$\{(i,j-1),(i,j)\}$ with label e_{k+j} if $j \leq k$ and e_j otherwise,

$\{(i,j),(i+k,j+k)\}$ with label J.

Note that the binary sum of the labels of the edges along any cycle of AT is the all-zero vector. Reciprocally, if the sum of labels along a path is the all-zero vector, then this path is closed. Then for any path from vertex $(0,0)$ to some vertex v of AT, the binary sum of the labels is the same. Thus, we may define the mapping ϕ from the vertices of AT to the vertices of $PC(2k)$ such that for any vertex v of AT, $\phi(v)$ is the binary sum of the labels along any path from $(0,0)$ to v. The mapping ϕ is an injective homomorphism from AT to $PC(2k)$. Its image is isomorphic to AT which, in turn, is a subgraph of $PC(2k)$.

(ii). Let v_1 and v_2 be two vertices of AT. There are integers i_1, i_2, j_1 and j_2 between 0 and $2k-1$ such that $v_1 = (i_1, j_1)$ and $v_2 = (i_2, j_2)$. It is easy to observe that the mapping $h : (i,j) \mapsto (i+i_2-i_1, j+j_2-j_1)$ (operations are modulo $2k$) is an automorphism of AT mapping v_1 to v_2.

(iii). Since AT is vertex-transitive, we may assume that one of these two vertices is the origin $(0,0)$. Let i and j be two integers between 0 and $2k-1$. We need to prove that $(0,0)$ and (i,j) are in a common $(2k+1)$-cycle. By the symmetries of AT, we may assume that i and j are both smaller than or equal to k. If we forget about the antipodal edges, we have the toroidal grid T. In this graph, there is a shortest path from $(0,0)$ to (k,k) going through (i,j). Together with the antipodal edge $\{(0,0),(k,k)\}$, it forms a $(2k+1)$-cycle in AT going through $(0,0)$ and (i,j).

(iv). This is a consequence of (i), (iii) and the fact that $PC(2k)$ has odd-girth $2k+1$.

(v). Let u and v be two vertices of AT. In (iii), we described a $(2k+1)$-cycle going through both vertices. This cycle uses exactly one antipodal edge. Since AT is of odd-girth $2k+1$, by Observation 7 the distance between u and v is given by this cycle. On this cycle, we may distinguish the path from u to v not using the antipodal edge; it has length $d_T(u,v)$ (see our description in (iii)). The other path uses an antipodal edge and has length $2k+1-d_T(u,v)$. The distance in AT between u and v must be the smaller of these quantities. Therefore, the first part of (v) is proven.

Let u be a vertex of AT. Then, a vertex v can be at distance d from u for two possible reasons. If $d = d_T(u,v)$ then v is in $N^d_T(u)$. Otherwise, $d = 2k+1-d_T(u,v)$. In such case, an edge of colour J is used in any shortest path connecting u to v. Following the previous proof, one such shortest path can be built starting with the edge $u\overline{u}$ corresponding to J. The path from $\overline{u}$ to v is then of length $d-1$ and it is a shortest path connecting the two, thus v is in $N^{d-1}_T(\overline{u})$. Reciprocally, any vertex in $N^d_T(u)$ is in

$N^d_{AT}(u)$ because of the distance formula and the fact that $1 \leq d \leq k$. For a vertex v in $N^{d-1}_T(\overline{u})$, any $(2k+1)$-cycle going through u, $\overline{u}$ and v uses the edges of a shortest $(d-1)$-path from $\overline{u}$ to v. Since $d \leq k$, we may derive that v is in $N^d_{AT}(u)$. $\square$

An illustration of the set $N^4_{AT}((0,0))$ for $k=7$ described in Lemma 29(v), is given in Figure 4. In this figure, towards a simpler presentation, edges connecting top to bottom, right to left and antipodal edges are not depicted.

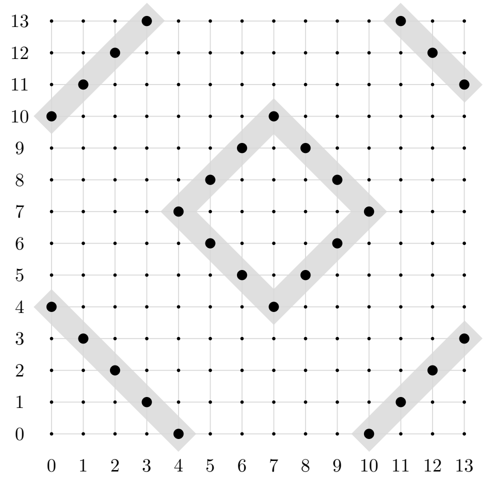


Figure 4: Illustration of the set $N^4_{AT}((0,0))$ for $k=7$.

Theorem 30. *For any positive integer k, the augmented toroidal grid $AT(2k,2k)$ bounds $\mathcal{SP}_{2k+1}$.*

Proof. Let k be some fixed positive integer and let AT and T denote respectively the graphs $AT(2k,2k)$ and $T(2k,2k)$. We shall prove that the complete distance graph of AT has the all k-good triple property. The edge set is clearly non-empty. Let p be some integer between 1 and k and let u and v be two vertices at distance p in AT. Let q and r be two integers such that $\{p,q,r\}$ is a k-good triple. We want to find a vertex w at distance r of u and at distance q of v.

Since AT is vertex-transitive, we may assume that $u=(0,0)$. We may also assume, without loss of generality, that $q \leq r$. Moreover, thanks to vertical and horizontal symmetries of AT, we may assume that $v=(a,b)$ where a and b are between 0 and k. Finally, since the vertical axis and the horizontal axis play the same role, we may assume that $a \leq b$. With these assumptions made, we shall restrict ourselves to the set of vertices S^* consisting of vertices (i,j) with i and j between 0 and k and prove that a suitable vertex w lies in S^*.

For any integer d between 0 and k, we define $\mathrm{Diag}(d)$ to be the set of vertices (i,j) in S^* such that $i+j=d$. By Lemma 29(v),

$$N^d_{AT}(u) \cap S^* = \mathrm{Diag}(d) \cup \mathrm{Diag}(2k+1-d). \tag{1}$$

Claim 30.A. *Let (i,j) be a vertex in S^* and d an integer between 1 and k. Let (x,y) be a vertex in S^*. If $i+j-d \leq x+y \leq i+j+d$ and $x-y$ equals $i-j-d$ or $i-j+d$, then (x,y) is in $N^d_{AT}((i,j))$.*

Proof of claim. Let (x,y) be a vertex in S^* such that $i+j-d \leq x+y \leq i+j+d$ and $x-y=i-j-d$ (the case when $x-y=i-j+d$ is analogous). We have $y=x-i+j+d$ then $x+y=2x-i+j+d$. Since $x+y \leq i+j+d$, we deduce that $x \leq i$. Similarly, we may prove that $y \geq j$. Thus $|x-i| = i-x$ and $|y-j| = y-j$ and $d_T((i,j),(x,y)) = i-x+y-j$. But since $x-y=i-j-d$, we have $d_T((i,j),(x,y)) = d$. As $d \leq k$, Lemma 29 allows us to conclude that $d_{AT}((i,j),(x,y)) = d$. This ends the proof of the claim. ($\square$)

An illustration of the subset of $N^4_{AT}((2,4)) \cap S^*$ described in Claim 30.A (for $k=9$) is shown in Figure 5.

Equation (1) tells us that v is either in $\mathrm{Diag}(p)$ or in $\mathrm{Diag}(2k+1-p)$. In both cases, we may derive that $p \geq b-a$. Indeed, if v is in $\mathrm{Diag}(p)$, we have $p=a+b$ and the conclusion is easy. If v is in

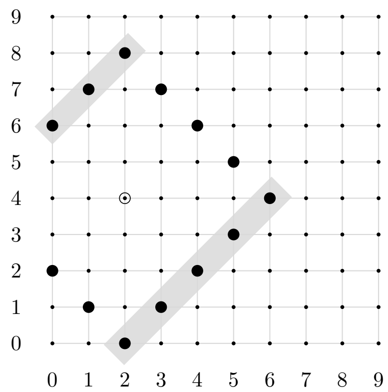


Figure 5: Illustration of the subset of $N^4_{AT}((2,4)) \cap S^*$ (in gray) described in Claim 30.A for $k = 9$. The large dots are the vertices of $N^4_{AT}((2,4)) \cap S^*$.

$\mathrm{Diag}(2k+1-p)$, we have $p+a+b = 2k+1$ but since $b \le k$, we get that $p+a-b \ge 0$. Since p, q and r are distances in the graph $T_{2k+1}(p,q,r)$ we may use the triangular inequality and affirm that $r+q \ge p$. In the end, we always have

$$r + q \ge b - a. \tag{2}$$

Similarly we show that

$$r - q \le a + b. \tag{3}$$

Indeed, if $p = a+b$, it is just the triangular inequality $r \le p+q$. Otherwise, $a+b = 2k+1-p$ and the quantity $2k+1-p-r+q$ cannot be negative (p and r are both at most k).

Another inequality we will need is

$$2k + 1 - r \ge a + b - q. \tag{4}$$

Once again, if $p = a+b$, it follows easily from the fact that $2k+1-p-r \ge 0$ (because p and r are at most k). If $p = 2k+1-a-b$, it is just the expression of the triangular inequality $r \le p+q$.

To prove our main claim, which is to show the existence of a vertex w at distance r of u and distance q of v, we distinguish two cases with respect to the parity of $a+b+q+r$.

Case 1. Suppose that $a+b+q+r$ is even. Then, let $y = \frac{1}{2}(b+r-a-q)$ and $x = r-y = \frac{1}{2}(r+q+a-b)$. First of all, $b+r-a-q$ and $a+b+q+r$ have the same parity where the latter is assumed to be even. Hence y is an integer, and therefore, so is x. We have assumed that $a \le b$ and $q \le r$, therefore $y \ge 0$. Since r and b are both smaller than or equal to k, we get that $y \le k$. Since $y \ge 0$ and $r \le k$, we observe that $x \le k$. Inequality (2) ensures that $x \ge 0$. We claim that $w = (x,y)$ works. First of all note that w is in S^*. Second, since $x+y = r$, w is in $\mathrm{Diag}(r)$ and by Equation (1), it is in $N^r_{AT}(u)$.

It remains to show that $d_{AT}(w,v) = q$. We have $x-y = a-b+q$ and $x+y = r$. Inequality (3) implies that $x+y \le a+b+q$. We consider two possibilities based on whether the p-path connecting u and v uses an edge of colour J or not. Assume v is in $\mathrm{Diag}(p)$, in such a case we have $p = a+b$ and the triangular inequality $p \le q+r$ allows us to derive that $a+b-q \le x+y$. Otherwise v is in $\mathrm{Diag}(2k+1-p)$, and then $a+b = 2k+1-p$. Hence $p+q+r$ is odd and, therefore, must be greater than or equal to $2k+1$. Therefore $a+b-q \le r$. In both eventualities, Claim 30.A ensures that w is also in $N^q_{AT}(v)$.

Case 2. Suppose now that $a+b+q+r$ is odd. Then let $y = \frac{1}{2}(b-a-q-r+2k+1)$ and $x = 2k+1-r-y = \frac{1}{2}(a-b+q-r+2k+1)$. Then, $b-a-q-r+2k+1$ must be even and y is an integer. In turn, x is also an integer. Since q and r are at most k, we know that $2k+1-q-r \ge 0$. Since $a \le b$, we know that $b-a \ge 0$ so that $y \ge 0$. Moreover, by Inequality (2), we get $2k+1+b-a-q-r \le 2k+1$ so that $y \le k + \frac{1}{2}$. As y is an integer, we have $y \le k$. Concerning x, the quantity $2k+1+a+q-b-r$ cannot be negative since b and r are at most k. Thus $x \ge 0$. By assumption, $a \le b$ and $q \le r$ so that $a-b+q-r+2k+1 \le 2k+1$. Since x is an integer, it has to be at most k. Therefore, if we call w the vertex (x,y), we have w in S^*.

Since $x+y = 2k+1-r$, w is in $\mathrm{Diag}(2k+1-r)$ and by Equation (1), it is in $N^r_{AT}(u)$.

Finally, we have $x-y=a-b+q$ and $x+y=2k+1-r$. By Equation (4), we know that $a+b-q \leq x+y$. For the remaining inequality, we distinguish whether $a+b=p$ or $a+b=2k+1-p$. If $a+b=p$ then $p+q+r$ is odd and must be at least $2k+1$ so that $2k+1-r \leq a+b+q$. If $a+b=2k+1-p$, we just need to use the triangular inequality $p \leq q+r$ to derive that $2k+1-r \leq a+b+q$. In both cases, Claim 30.A ensures that w is also in $N^q_{AT}(v)$. This completes the proof of Theorem 30. □

6. Optimal bounds for $\mathcal{SP}_5$ and $\mathcal{SP}_7$

For $k=1$, the triangle (K_3) is the best bound one can find for $\mathcal{SP}=\mathcal{SP}_3$. This is also best possible in the sense that $K_3 \in \mathcal{SP}$. For $k \geq 2$, in contrast with the case of planar graphs, not only the projective hypercube $PC(2k)$ and the Kneser graph $K(2k+1,k)$ are not optimal bounds, but even the augmented square toroidal grid $AT(2k,2k)$ seems to be non-optimal. In general, we do not know the optimal bounds and leave this an open question, but in this section we describe optimal bounds for the cases $k=2$ and $k=3$. For for $k=1,2,3$, the three optimal bounds also provide the optimal bound for the fractional and circular chromatic numbers of graphs in $\mathcal{SP}_{2k+1}$. This suggests that perhaps in general, the optimal bound of odd-girth $2k+1$ for $\mathcal{SP}_{2k+1}$ provides, simultaneously, the best bound for both fractional and circular chromatic numbers of graphs in $\mathcal{SP}_{2k+1}$. Thus, any such result can be seen as a strengthening of both theorems about these two notions.

6.1. Bounding $\mathcal{SP}_5$

For $k=2$, Theorem 25 implies that the Clebsch graph (of order 16) bounds $\mathcal{SP}_5$. By Theorem 27, the Petersen graph (of order 10) also bounds $\mathcal{SP}_5$. One can further check (using Algorithm 1) that some other graphs such as the Dodecahedral graph (of order 20), the Armanios-Wells graphs (a distance-regular graph of order 32 [1, 44]) and the Grötzsch graph (of order 11) bound $\mathcal{SP}_5$. From results on circular colouring (see [19]) we know that $C_{8,3}$ (also known as the Wagner graph) bounds $\mathcal{SP}_5$. In the next theorem we show that the (unique) optimal bound is obtained from $C_{8,3}$ by removing two edges. More precisely, let C_8^{++} be the graph obtained from an 8-cycle by adding two disjoint antipodal edges (see Figure 6). Alternatively, C_8^{++} is obtained from the Petersen graph $K(5,2)$ by removing two adjacent vertices. We show next that C_8^{++} is a bound of odd-girth 5 for $\mathcal{SP}_5$ and that it is the unique optimal such bound, both in terms of order and size.

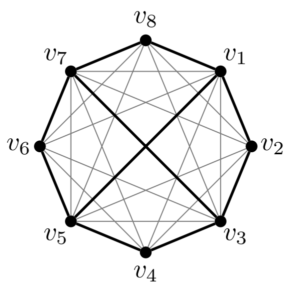


Figure 6: The graph C_8^{++} bounding $\mathcal{SP}_5$ (black edges). The gray edges are the weight 2-edges in its corresponding partial distance graph having the all 2-good triple property.

Theorem 31. *The graph C_8^{++} bounds $\mathcal{SP}_5$. Furthermore, this is the unique bound of odd-girth 5 for $\mathcal{SP}_5$ that is optimal both in terms of order and size.*

Proof. We will use the labeling of Figure 6. Thus, the vertices of the 8-cycle are labeled $v_1, v_2, \ldots, v_8$ in the cyclic order. Edges v_1v_5 and v_3v_7 are the two added antipodal edges. To see that this is a bound, by Theorem 18, it is sufficient to show that some 2-partial distance graph of C_8^{++} has the all 2-good triple property. The 2-partial distance graph we will consider is the graph which has all edges but the two missing diagonals, that is $K_8 - \{v_2v_6, v_4v_8\}$, see Figure 6 (black edges have weight 1 and gray edges have weight 2). The list of 2-good triples is $\{1,1,2\}$, $\{1,2,2\}$, $\{2,2,2\}$. This graph is highly symmetric and using these symmetries, it is enough to check the properties only for the following edges: v_1v_2, v_1v_5 (both of weight 1) and v_1v_3, v_1v_4, v_2v_4 (of weight 2). This is a straightforward task.

Next, we show that any bound of odd-girth 5 for $\mathcal{SP}_5$ must have at least eight vertices. Let B be a minimal bound of odd-girth 5. Then B must be a core, has no triangle, and should contain a 5-cycle. Moreover B cannot be isomorphic to C_5 because the graph $T_5(2,2,2)$ from $\mathcal{SP}_5$ does not admit a homomorphism to C_5. It can be easily checked that there are no triangle-free cores on six vertices, so by contradiction we may assume that B has seven vertices. Let $v_1, \dots, v_5$ be the vertices of a 5-cycle in B and let v_6 and v_7 be the other two vertices. Then, since B is triangle-free, v_6 is adjacent to at most two vertices of the 5-cycle. Suppose that it is adjacent to exactly two, without loss of generality it is adjacent to v_1 and v_3. But then, one of v_2 and v_6 has its neighbourhood included in the other's (because their only other possible neighbour is v_7). Hence identifying v_2 and v_6 is a homomorphism of B to a proper subgraph of itself, which contradicts with B being a core. Thus, each of v_6 and v_7 is adjacent to at most one vertex in the 5-cycle; since they cannot be of degree 1, they must be adjacent and then we have two adjacent vertices of degree 2 in B, contradicting Lemma 20. Thus, B has order at least 8.

Now, we want to prove that B must have at least ten edges. By contradiction, suppose B has at most nine edges. Then, B must have order exactly 8, as otherwise either it contains a vertex of degree 1 or it contains two adjacent vertices of degree 2. Again, we know that B must contain a 5-cycle and we label the vertices of B with $v_1, \dots, v_8$ such that $v_1, \dots, v_5$ induces a 5-cycle. If for some $i \in \{6, 7, 8\}$ we have $N(v_i) \subset \{v_1, \dots, v_5\}$, then B is not a core. Thus, by symmetry, we may assume that v_6v_7 and v_6v_8 are edges of B (hence v_7 is not adjacent to v_8 as B is triangle-free).

Based on the adjacencies of v_6 in $\{v_1, \dots, v_5\}$, we consider two cases.

Case 1. Assume that v_6 is of degree 2. In this case, by Corollary 20, v_7 and v_8 each must have two neighbours among $v_1, \dots, v_5$, which leaves us with a minimum of eleven edges, a contradiction.

Case 2. Assume that v_6 has degree at least 3. Without loss of generality assume that v_1 is adjacent to v_6. In this case, each of v_7 and v_8 has at least one neighbour in $\{v_1, \dots, v_5\}$. Thus we have a minimum of ten edges.

Now, if there are exactly ten edges and eight vertices, we also have a 5-cycle $v_1 \dots v_5$ and edges v_6v_7 and v_6v_8. Then each of v_6, v_7, v_8 has exactly one neighbour in $\{v_1, \dots, v_5\}$. We claim that neither v_2 nor v_5 can be such a neighbour. Otherwise, by symmetries, we may assume that v_7 is adjacent to v_2, but identifying v_7 and v_1 produces a homomorphism of B to a proper subgraph of itself, which contradicts with B being a core. Finally, since by Lemma 20 there cannot be two adjacent vertices of degree 2 in B, v_3 and v_4 are the neighbours of v_7, v_8. This graph is isomorphic to C_8^{++} (v_4v_5 and v_1v_6 being the two diagonal edges). □

Since C_8^{++} is a proper subgraph of both $C_{8,3}$ and the Petersen graph $K(5,2)$, Theorem 31 is a common strengthening of both the result on the circular chromatic number from [35] (which states that all graphs in $\mathcal{SP}_5$ map to $C_{8,3}$) and the case $k = 2$ of Corollary 28 on the fractional chromatic number (that all graphs in $\mathcal{SP}_5$ map to $K(5,2)$).

6.2. Bounding $\mathcal{SP}_7$

Again, by Theorem 25, Theorem 27 and Theorem 30, the projective hypercube $PC(6)$ (of order 64), the Kneser graph $K(7,3)$ (of order 35) and $AT(6,6)$ (of order 36) all bound $\mathcal{SP}_7$. Among other bounds are the Coxeter graph and the 16-vertex graph X_{16} of Figure 7. The Coxeter graph is a subgraph of $K(7,3)$ of order 28, more precisely it is obtained from $K(7,3)$ by removing lines of a Fano plane. More noticeably, it is of girth 7. The graph X_{16} is the smallest induced subgraph of $PC(6)$ whose *complete* distance graph has the all 3-good triple property (this was only verified by a computer check). Next, we introduce a graph X_{15} of odd-girth 7 which bounds $\mathcal{SP}_7$. We then show that 15 is the smallest order of such a bound.

The graph X_{15} is built from a 10-cycle whose vertices are labeled (in cyclic order) $v_0, v_1, \dots, v_9$. Then, a set of five vertices $x_0, x_1, \dots, x_4$ is added, with x_i being adjacent to v_i and v_{i+5}. See Figure 8 for two drawings of X_{15}. This graph is an induced subgraph of the Kneser graph $K(7,3)$ (and, therefore, also of $PC(6)$), and has circular chromatic number $5/2$.

We will show that X_{15} is an optimal bound of odd-girth 7 for $\mathcal{SP}_7$ in terms of the order. In other words, we show no such bound on 14 or less vertices exists. To prove this, we use the following result of ours on the circular chromatic number of small graphs of odd-girth 7. Since we wish to include this result in a forthcoming publication [2], for the sake of completeness, the proof is deferred to AppendixA.

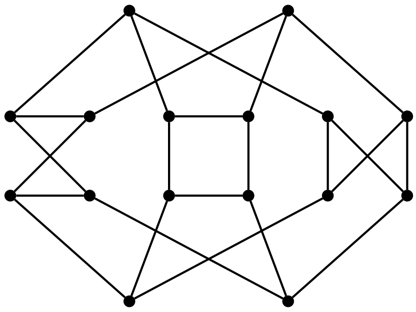


Figure 7: A 16-vertex bound for $\mathcal{SP}_7$.

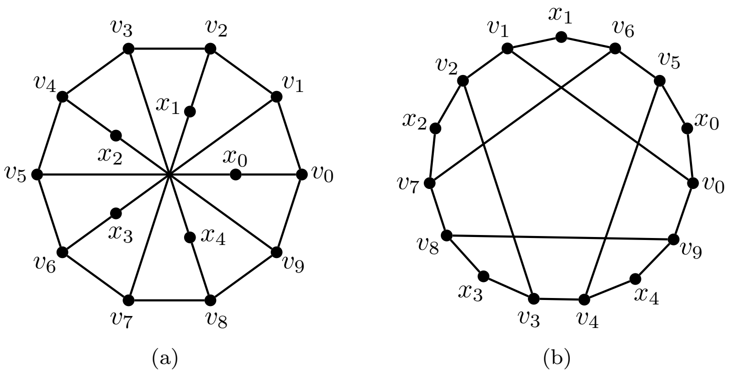


Figure 8: Two drawings of the graph X_{15}, an optimal 15-vertex bound for $\mathcal{SP}_7$.

Theorem 32 ([2]). *Any graph G of order at most 14 and odd-girth at least 7 admits a homomorphism to C_5.*

We note that the value 14 is optimal in the statement of Theorem 32 (see [2] for details).

Theorem 33. *The graph X_{15} bounds $\mathcal{SP}_7$. Furthermore, it is a bound of odd-girth 7 for $\mathcal{SP}_7$ that is optimal in terms of the order.*

Proof. To see that X_{15} bounds $\mathcal{SP}_7$, observe that the partial distance graph $(\widetilde{X_{15}}, d_{X_{15}})$, where $\widetilde{X_{15}}$ contains all pairs of vertices except for the pairs x_ix_j, has the all 3-good triple property. The list of 3-good triples is $\{1,1,2\}$, $\{1,2,3\}$, $\{1,3,3\}$, $\{2,2,2\}$, $\{2,2,3\}$, $\{2,3,3\}$, $\{3,3,3\}$. Considering the symmetries of X_{15}, it is enough to check the existence of all possible triangles on the following edges: v_0v_1, v_0x_0 (of weight 1), v_0v_2, v_0x_3, v_0v_5 (weight 2), v_0v_3, v_0v_4, v_0x_3,x_0, x_3 (weight 3). Once again this is straightforward.

It remains to show that no graph of odd-girth 7 on 14 vertices (or less) bounds $\mathcal{SP}_7$. Towards a contradiction, assume that B is such a bound and moreover assume B is a minimal such bound, that is, no smaller graph (in terms of the order and size) of odd-girth 7 bounds $\mathcal{SP}_7$.

Thus, in particular, B is a core and, therefore, has minimum degree 2. Moreover, for any pair u, v of nonadjacent vertices of B, there is a 5-walk connecting u and v (otherwise, identifying u and v and leaving all other vertices untouched produces a bound of smaller order which is also of odd-girth 7).

By Lemma 19, each vertex of degree 2 is part of a 6-cycle and by Lemma 20, B has no pair of adjacent vertices of degree 2. Furthermore, by Theorem 10 and Theorem 32, the circular chromatic number of B is exactly $5/2$.

Let ψ be a $\frac{5}{2}$-circular colouring of B, that is, a mapping of B to the 5-cycle C_5 with $V(C_5) = \{0,1,2,3,4\}$ (although this labeling of C_5 does not correspond to the definition of a $\frac{5}{2}$-circular colouring, for simplicity we let the edges of C_5 be of the form $i(i+1 \bmod 5)$). By Proposition 11 there should be a tight $5k$-cycle in B for some $k \geq 1$; since B has at most 14 vertices and odd-girth 7 we have $k = 2$. Let C be this tight 10-cycle, and label its vertices with $v_0, v_2, \ldots, v_9$, in cyclic order. Then we can assume that $\psi(v_i) = i \bmod 5$. From this colouring, it is clear that C does not contain any chord, for otherwise there would be a 5-cycle in B.

Let x be a vertex in $V(B) \setminus V(C)$ and assume by symmetry that $\psi(x) = 1$. Then, considering the colouring ψ, the only possible neighbours of x in C are v_0, v_2, v_5 and v_7. Vertex x cannot be adjacent to both v_2 and v_5 as otherwise B has a 5-cycle. Similarly it cannot be adjacent to both v_0 and v_7. Next we show that it cannot be adjacent to both v_0 and v_2. By contradiction suppose v is adjacent to v_0 and v_2. Then, x cannot be adjacent to v_1 and there must be a 5-walk connecting x and v_1. Since there is also a 2-path connecting them, the 5-walk must be a 5-path. Let its vertices be labeled $x, x_0, x_4, x_3, x_2, v_1$ in the order of the path. Furthermore, since its vertices are part of a 7-cycle and $\psi(x) = \psi(v_1) = 1$, ψ must map this path onto C_5. By symmetry of x and v_1, we may assume $\psi(x_i) = i$. Since any tight 10-cycle must be chordless, and since replacing x with v_1 in C would result in a new tight 10-cycle, x_0 and x_2 are distinct from vertices of C. But overall, we have at most 14 vertices, thus one of x_3 or x_4 must be on C. By symmetry of these two vertices (with respect to the currently forced and coloured structure) we may assume x_3 is a vertex of C. Considering the colouring ψ either we have $x_3 = v_3$ in which case $v_3x_4x_0xv_2$ is a 5-cycle of B, or $x_3 = v_8$ in which case $v_8v_9v_0v_1x_2$ is a 5-cycle of B, a contradiction in both cases. In conclusion we obtain the following claim.

Claim 33.A. *Any vertex x in $V(B) \setminus V(C)$ is adjacent to at most two vertices of C and if adjacent to two vertices, then those vertices are antipodal in C.*

Since there are at most four vertices not in $V(C)$, and since there are five antipodal pairs in C, for one such pair, say v_0, v_5, both vertices are of degree 2 in B. Thus, by Lemma 19, v_0 belongs to a 6-cycle; call it C_{v_0}. Then, v_9 and v_1 are necessarily vertices of C_{v_0}. Since C is chordless and by Claim 33.A, C_{v_0} shares three or four (consecutive) vertices with C.

We first show that they cannot share four vertices. By symmetry, assume v_8 is the fourth vertex in common. Let u and v be the other two vertices of C_{v_0}, assuming u is adjacent to v_1. It follows that $\psi(u) = 0$ and $\psi(v) = 4$, therefore replacing v_0 by u and v_9 by v in C results in another tight 10-cycle. Since u is not adjacent to v_0, there must be a 5-walk connecting u and v_0, but since v_0 and u are at distance 2, this 5-walk is a 5-path. It must have v_9 as a vertex. Label the other vertices x_1, x_2 and x_3. It follows from Claim 33.A and the fact that tight 10-cycles are chordless that all these three vertices must be new vertices, contradicting the fact that B has at most 14 vertices.

Therefore, we may assume that $v_9v_0v_1xyz$ is a 6-cycle containing v_0 and x, y, z are all distinct from vertices of C. Let t be the last possible vertex (when $|V(B)| = 14$). The possible colours for y are 1 or 4, and by symmetry of these two colours we may assume $\psi(y) = 4$. Thus y is not adjacent to v_2 (as $\psi(v_2) = 2$), an edge between y and v_3 would result in a 5-cycle, and neither x nor z can be adjacent to v_2 or v_3 by Claim 33.A. Thus, by Lemma 20, t is adjacent to one of v_2 or v_3. On the other hand, since v_5 is of degree 2 and again by Lemma 20, both v_4 and v_6 must have neighbours in $V(B) \setminus V(C)$. By Claim 33.A and the value of $\psi(y)$, the only possibility is that x is adjacent to v_6 and z is adjacent to v_4. Now y cannot be adjacent to v_7 or v_8 since it would create a 5-cycle. Thus, by Lemma 20 for pairs v_2, v_3 and v_7, v_8, by Claim 33.A, and by the symmetry of v_2, v_7 and v_3, v_8 we may assume t is adjacent to v_2 and v_7. The only other edge that can now be added without creating a shorter odd-cycle or contradicting ψ is the edge xt. But then, v_3 is a degree 2 vertex which does not belong to any 6-cycle. This contradiction completes the proof. □

Again, observe that X_{15} has circular chromatic number $5/2$ and is a proper subgraph of the Kneser graph $K(7, 3)$. Hence, Theorem 33 can be seen as a common strengthening of both the result on the circular chromatic number of graphs in $\mathcal{SP}_7$ (from [19]) and the case $k = 3$ of Corollary 28 on the fractional chromatic number of this family of graphs.

7. Applications to edge-colourings

In this section, we present edge-colouring results for K_4-minor-free multigraphs that follow from our previous results.

Let G be an r-regular multigraph. If G contains a set X of an odd number of vertices such that at most $r - 1$ edges connect X to $V(G) - X$, then G cannot be r-edge-coloured. An r-regular multigraph without such a subset of vertices is called an *r-graph*. Seymour's result from [42] implies in particular that every K_4-minor-free r-graph is r-edge-colourable. Here, we show that for odd values of r, this claim is an easy consequence of Theorem 25, thus giving an alternative proof for the odd cases of Seymour's result of [42]. Our results in Sections 5 and 6 are therefore strengthenings of this fact, and we present stronger edge-colouring applications of these results.

Theorem 34. *For every K_4-minor-free $(2k+1)$-graph G, $\chi'(G) = 2k+1$.*

Proof. The proof is the same as the proof for planar graphs of [29]. We give the main idea. Consider a planar embedding of G and let G^* be the dual of G with respect to this embedding. Note that G^* is also K_4-minor-free. Furthermore, it is not difficult to verify (see [29] for details) that the hypothesis on G is equivalent to the fact that G^* has odd-girth $2k+1$. Thus, by Theorem 25, G^* admits a homomorphism, say ϕ, to $PC(2k)$. On the other hand, $PC(2k)$ has a natural $(2k+1)$-edge-colouring by its definition as a Cayley graph. In such a colouring, each $(2k+1)$-cycle receives all $2k+1$ different colours. This edge-colouring of $PC(2k)$ induces an edge-colouring ψ of G^* using the homomorphism ϕ. While ψ is not necessarily a proper edge-colouring of G^*, it inherits the property that each $(2k+1)$-cycle of G^* is coloured with $2k+1$ different colours. Thus, if we colour each edge of G with the colour of its corresponding edge in G^*, we obtain a proper edge-colouring. □

Consider the edge-colouring of C_8^{++} induced by $PC(4)$ viewed as the Cayley graph $(\mathbb{Z}_2^4, S = \{e_1, e_2, e_3, e_4, J\})$, where each edge is coloured with the vector of S corresponding to the difference between its endpoints (See Figure 9 for an illustration).

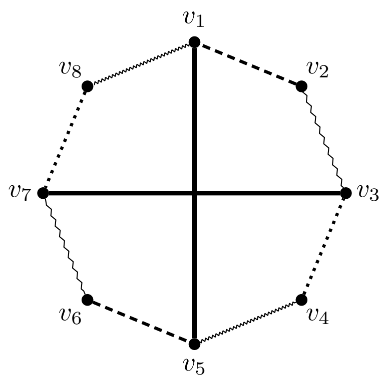


Figure 9: The 5-edge-colouring of C_8^{++} induced by the canonical edge-colouring of $PC(4)$.

Note that there are only four 5-cycles in C_8^{++}, and there are only two different cyclic orders of edge-colours induced by these four 5-cycles. Then, using the technique of the proof of Theorem 34, we can prove the following.

Theorem 35. *Let G be a plane K_4-minor-free 5-graph. Let $\{c_1, c_2, c_3, c_4, c_5\}$ be a set of five colours. Then, one can colour the edges of G such that at each vertex, the cyclic order of colours is either c_1, c_2, c_3, c_4, c_5 or c_1, c_4, c_5, c_2, c_3.*

Proof. The proof is the same as the one of Theorem 34, except that using Theorem 31 we can consider a homomorphism ϕ of G to C_8^{++} and the 5-edge-colouring of C_8^{++} induced by $PC(4)$ which is depicted in Figure 9. □

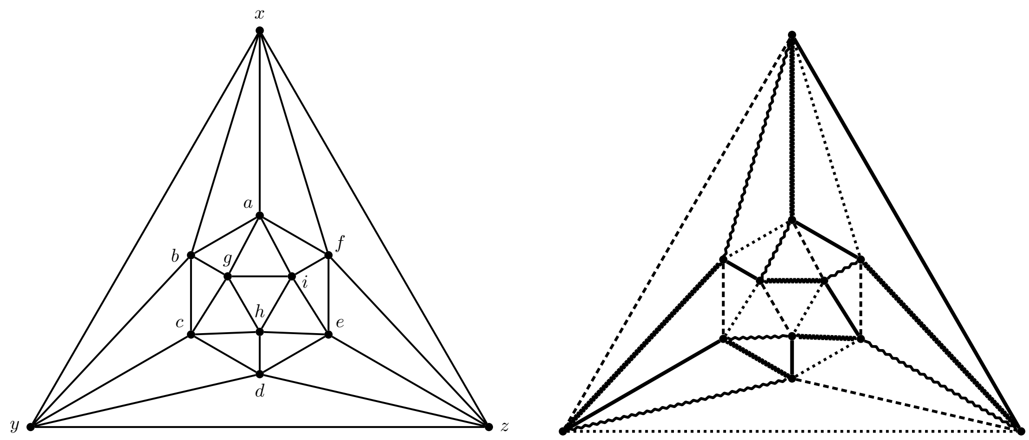
As an application, consider a 5-colourable plane 5-graph G. Furthermore, suppose that the set of colours available to us are green, dark blue and light blue, dark red and light red. Then, one wishes to 5-edge-colour G so that at each vertex x, in a circular ordering of the edges incident to x, the colours light blue and dark blue (light red and dark green, respectively) do not appear consecutively. We call a proper edge-colouring satisfying this constraint, a *super proper edge-colouring.* We obtain the following direct consequence of Theorem 35.

Corollary 36. *Every plane K_4-minor-free 5-graph admits a super proper 5-edge-colouring.*

The next proposition shows that the Icosahedral graph (which has a unique planar embedding, see Figure 10), and which is a 5-edge-colourable planar 5-regular graph (see the colouring of Figure 10(b)), admits no super proper 5-edge-colouring. Therefore Corollary 36 cannot be extended to planar 5-graphs.

Proposition 37. *The plane Icosahedral graph is not super properly* 5*-edge-colourable.*

Proof. We use the labeling of Figure 10(a). Assume for a contradiction that we have a super proper 5-edge-colouring. By the uniqueness of the planar embedding of the Icosahedral graph and by its symmetries, we may assume, without loss of generality, that the edge xa is coloured green. Then, again without loss of generality, we can assume that xb and xz are coloured blue (light or dark) and that xy and xf are coloured red (light or dark). Note that the edges of any triangular face must have a green, a blue and a red edge. Therefore, considering the xfz-triangle fz is green and considering the triangle xaf, af is blue. Now, at vertex f, the edge fi must be red. And then in the triangle afi, ai must be green, a contradiction since a has two incident green edges. □



(a) A planar embedding.

(b) A 5-edge-colouring.

Figure 10: The Icosahedral graph and a 5-edge-colouring of it.

The bound X_{15} of $\mathcal{SP}_7$ implies a similar edge-colouring result for plane K_4-minor-free 7-graphs as the one of Theorem 35. To express it, we first describe the 7-edge-colouring of X_{15} induced by the canonical edge-colouring of $PC(6)$. In this edge-colouring, each edge v_iv_{i+1} of C_{10} is coloured i mod 5 (thus a total of five colours is used on the edges of C_{10}, each twice) Moreover, each pair of edges incident with a vertex x_j ($j = 1, 2, \dots, 5$) is coloured with colours 6 and 7. In X_{15}, any 7-cycle contains (exactly) one of the x_j's and its two incident edges, and then a continuous set of five edges from C_{10}. These five edges therefore always induce the same cyclic order. Thus we obtain the following.

Theorem 38. *Every plane K_4-minor-free 7-graph G can be edge-coloured with colours $1, 2, \dots, 7$ such that at each vertex, the set of colours $1, 2, \dots, 5$ induces this cyclic order (in either clockwise or anticlockwise order).*

Finally, a general result on edge-colouring plane K_4-minor-free $(2k+1)$-graphs can be obtained from Theorem 30 (that $AT(2k, 2k)$ bounds $\mathcal{SP}_{2k+1}$). Note that a $(2k+1)$-cycle of $AT(2k, 2k)$ uses exactly k horizontal edges, k vertical edges and an antipodal edge. Furthermore, the set of horizontal edges, in their order of appearance on the cycle, induces a cyclic order of $e_1, e_2, \dots, e_k$ and similarly the set of vertical edges induces a cyclic order of $e_{k+1}, e_{k+2}, \dots, e_{2k}$. Thus, we can derive the following definition of special $2k+1$-edge-colourings.

Given k, let $B = b_1, b_2, \dots, b_k$ be a sequence of k distinct colours in the family of blue colours and let $R = r_1, r_2, \dots, r_k$ be a sequence of k distinct colours in the family of red colours. Given a $(2k+1)$-regular plane multigraph G, we say that G is (B, R)-edge-colourable if it can be properly edge-coloured using colours from B, R and a unique green colour such that at each vertex v, the cyclic ordering of the blue colours (respectively red) around v always induces the same cyclic order as in B (in R, respectively).

Theorem 39. *Let B and R be two sequences of blue and red colours such that $|B| = |R| = k$. Then, every plane K_4-minor-free $(2k+1)$-graph is (B, R)-edge-colourable.*

8. Concluding remarks

We conclude the paper with some remarks and open problems.

1. As observed before, we have $\chi_c(\mathcal{SP}) = \chi_c(K_3)$, $\chi_c(\mathcal{SP}_5) = \chi_c(C_8^{++})$, $\chi_c(\mathcal{SP}_7) = \chi_c(X_{15})$ and $\chi_f(\mathcal{SP}) = \chi_f(K_3)$, $\chi_f(\mathcal{SP}_5) = \chi_f(C_8^{++})$, $\chi_f(\mathcal{SP}_7) = \chi_f(X_{15})$ (see [19, 35, 36] and Corollary 28 for the values of $\chi_c(\mathcal{SP}_{2k+1})$ and $\chi_f(\mathcal{SP}_{2k+1})$). We expect that this will generally be the case, that is, the optimal bound of odd-girth $2k+1$ for $\mathcal{SP}_{2k+1}$ should be a subgraph of $PC(2k)$ whose existence improves simultaneously the results on the circular and fractional chromatic numbers of $\mathcal{SP}_{2k+1}$.

2. The *generalized level k-Mycielski graph of C_{2k+1}*, denoted $M_k(C_{2k+1})$, is constructed as follows. We have $V(M_k(C_{2k+1})) = V_1 \cup \ldots \cup V_k \cup \{v\}$, where $V_i = \{u_0^i, \ldots, u_{2k}^i\}$ for $1 \leq i \leq k$. The first level, V_1, induces a $(2k+1)$-cycle $u_0^1, \ldots, u_{2k}^1$. For each level V_i with $2 \leq i \leq k$, vertex u_j^i is adjacent to the two vertices $u^{i-1}_{(j-1) \bmod 2k+1}$ and $u^{i-1}_{(j+1) \bmod 2k+1}$ in the level V_{i-1}. Finally, vertex v is adjacent to all vertices in V_k. Thus, $M_2(C_5)$ is simply the classic Mycielski construction for C_5, that is, the Grötzsch graph. For every $k \geq 1$, $M_k(C_{2k+1})$ has odd-girth $2k+1$, is 4-chromatic and is a subgraph of $PC(2k)$ [37]. Note that $M_2(C_5)$ contains C_8^{++} as a subgraph and therefore bounds $\mathcal{SP}_5$. Similarly, $M_3(C_7)$ contains X_{15} as a subgraph and hence bounds $\mathcal{SP}_7$. Thus, we conjecture that $M_k(C_{2k+1})$ bounds $\mathcal{SP}_{2k+1}$. (Using Algorithm 1, we have verified this conjecture by a computer check for $k \leq 10$.) Since $M_k(C_{2k+1})$ has order $2k^2+k+1$, a confirmation of this conjecture would provide a family of smaller bounds than the augmented square toroidal grids (that have order $4k^2$).

3. Using the notion of walk-power, it is shown in [17] that any bound of odd-girth at least $2k+1$ for $\mathcal{SP}_{2k+1}$ is of order at least $\binom{k+2}{2}$. Thus, the bounds of Theorem 30 are optimal up to a factor of 8.

4. As a strengthening of Proposition 37, does there exist a plane k-graph G such that in any k-edge-colouring c of G and for any cyclic permutation of the k colours, there is a vertex of G where c induces this permutation of colours?

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AppendixA. Proof of Theorem 32

Theorem 32 ([2]). *Any graph G of order at most 14 and odd-girth at least 7 admits a homomorphism to C_5.*

Proof. Let $V(C_5) = \{0, 1, 2, 3, 4\}$ where the numbering follows the cyclic order. By contradiction, we consider a counterexample G of minimum order. In particular, G does not map to a smaller graph of odd-girth 7 and G is a core (hence has minimum degree 2). Then, there is a 5-walk connecting each pair of vertices of G.

We say that a vertex v of G is an (i, j)-vertex if $deg(v) = i$ and there are exactly j vertices at distance 2 of v. Similarly, it is an $(\geq i, \geq j)$-vertex if $deg(v) \geq i$ and $|N^2(v)| \geq j$. An $(i, \geq j)$-vertex, and $(\leq i, \leq j)$-vertex and an $(i, \leq j)$-vertex are defined analogously. We start with proving the following.

Claim 32.A. *Every vertex of G is an $(\leq 3, \leq 4)$-vertex.*

Proof of claim. Let u be a vertex of maximum degree in G. Among all vertices of maximum degree in G, we select u such that the size of the second neighbourhood $N^2(u)$ of u is also maximized. By contradiction, assume that u is a $(3, \geq 5)$-vertex or a $(4, i)$-vertex for some i. In the latter case, we must have $i \geq 4$, for otherwise some neighbour of u has its neighbourhood included in the one of another neighbour of u, contradicting the fact that G is a core. Letting $Z = V(G) \setminus (N[u] \cup N^2(u))$, this implies that $|Z| \leq 5$, and therefore the subgraph $G[Z]$ induced by Z is bipartite. Assume we can 3-colour Z with colours 0, 1 and 3 as follows.
(i) Each isolated vertex of $G[Z]$ receives colour 3.
(ii) The remaining vertices are properly 2-coloured with colours 0 and 1.
(iii) No vertex of $N^2(u)$ is adjacent to two vertices coloured 0 and 1, respectively.

Then, we could extend this 3-colouring of Z to a homomorphism of G to C_5: colour $N^2(u)$ with colours 2 and 4, and $N(u)$ with colour 3. Therefore, since G is a counterexample, no such colouring exists.

In particular, $G[Z]$ must be disconnected and have at least two components of size at least 2. Indeed, if $G[Z]$ has only one component of size at least 2, for any 3-colouring satisfying (i) and (ii), we also have (iii) (otherwise there is a triangle or a 5-cycle in G).

Similarly, some connected component of $G[Z]$ has at least three vertices. Indeed, if each connected component of $G[Z]$ has at most two vertices, we 3-colour Z such that (i) and (ii) hold. If (iii) does not hold, there is a vertex of $N^2(u)$ with a neighbour z_1 coloured 0 and a neighbour z_2 coloured 1; so z_1 and z_2 must belong to distinct components (otherwise there is a triangle in G). Let z_3 be the neighbour of z_1 in $G[Z]$. Then, we switch the colours of z_1 and z_3. Now, if some vertex of $N^2(u)$ is adjacent to two vertices coloured 0 and 1, we would necessarily have a 5-cycle in G, which is impossible. Therefore (i)–(iii) hold for the new colouring of Z, a contradiction.

Therefore, $G[Z]$ has one component of size 3 (say a 2-path $z_1z_2z_3$) and one component of size 2 (a 1-path z_4z_5). This implies that u is either a $(3,5)$-vertex, or a $(4,4)$-vertex. Consider a 2-colouring of the 2-path with z_1, z_3 coloured 1 and z_2 coloured 0, and both possible 2-colourings of the 1-path. Then, since these colourings should not satisfy (i)—(iii), by symmetry there must be a common neighbour of z_1 and z_5 (say y_1) and a common neighbour of z_3 and z_4 (say y_2). Now, we define a 5-colouring h of Z as follows: $h(z_1) = h(z_3) = 3$, $h(z_2) = 4$, $h(z_4) = 1$, $h(z_5) = 0$. Now, note that for each vertex y of $N^2(u)$, the set $h(N(y) \cap Z)$ of colours of neighbours of y is a subset of one of $\{0, 3\}$, $\{1, 3\}$ and $\{4\}$ (otherwise we have a triangle or a 5-cycle in G). We extend h to $N^2(u)$ as follows. For each vertex y in $N^2(u)$, if $h(N(y) \cap Z) = \{0\}$ or $h(N(y) \cap Z) = \{0, 3\}$, then $h(y) = 4$. If $h(N(y) \cap Z) \subseteq \{1, 3\}$, then $h(y) = 2$. Otherwise, $h(N(y) \cap Z) = \{4\}$ and we let $h(y) = 0$. If no vertex of $N(u)$ is adjacent to two vertices coloured 0 and 4, respectively, we could colour the vertices of $N(u)$ with colours 1 and 3, let $h(u) = 2$, and h would be a homomorphism of G to C_5, a contradiction. Therefore, some vertex x_1 of $N(u)$ is adjacent to two vertices of $N^2(u)$ coloured 0 and 4, respectively. One of them must be a vertex y_4 (distinct from y_1 and y_2) of $N^2(u)$ coloured 0 (and y_4 is a neighbour of z_2). Then, x_1 is not a neighbour of y_1 (otherwise we have a 5-cycle), and x_1 is adjacent to a new vertex y_3 of $N^2(u)$ coloured 4 (and y_3 is a neighbour of z_5). Note that, by the definition of the colouring h and because G has odd-girth 7, the vertices of $\{y_1, y_2, y_3, y_4\}$ have no further neighbours in Z.

If $|N^2(u)| = 4$ (that is, u is a $(4,4)$-vertex), then we claim that $G \to C_5$. Indeed, define h' as follows: $h'(y_1) = h'(y_3) = 4$, $h'(z_1) = 3$, $h'(z_2) = h'(y_2) = 2$, $h'(z_3) = h'(z_4) = h'(y_4) = 1$, and $h'(z_5) = 0$. Now, observe that a vertex of $N(u)$ cannot be adjacent to both y_2 and y_4 (otherwise there is a 5-cycle in G). Hence, we can extend h': for any vertex x of $N(u)$, if x is adjacent to a subset of $\{y_1, y_2, y_3\}$ (that are coloured 2 or 4), we let $h'(x) = 3$. Otherwise, the set of colours of the neighbours of x is a subset of $\{1, 4\}$, and we let $h'(x) = 0$. Finally, we let $h'(u) = 4$, and h' is a homomorphism of G to C_5, a contradiction.

Hence, u must be a $(3,5)$-vertex. Let $N(u) = \{x_1, x_2, x_3\}$ (recall that x_1 is adjacent to y_3 and y_4). Since y_1 cannot be adjacent to x_1 (which has already three neighbours), we may assume that y_1 is a neighbour of x_2. Similarly, the third neighbour of y_2 must be x_3. Then, observe that y_4 has degree 2, because if it is adjacent to x_2 or x_3, G contains a 5-cycle. Hence, the remaining edges of G can only be edges from y_3 and y_5 to $\{x_2, x_3, z_1, z_3, z_4\}$. But the graph G' obtained by adding all these possible edges (although it is of odd-girth 5) admits a homomorphism f to C_5 as follows: x_2, x_3, z_1, z_3 are mapped to 0; y_2, y_5, z_2 are mapped to 1; y_4 and z_4 are mapped to 2; x_1 and z_5 are mapped to 3; u, y_1, y_3 are mapped to 4. This contradiction completes the proof of Claim 32.A. (□)

Claim 32.B. *G does not contain any thread with three vertices of degree 2.*

Proof of claim. By contradiction, consider such a thread T between vertices x and y: $xt_1t_2t_3y$, where for $i = 1, 2, 3$, t_i has degree 2 in G. By minimality of G, $G - \{t_1, t_2, t_3\}$ admits a homomorphism h to C_5. But then it is easy to extend h to G, a contradiction. (□)

Since G is not homomorphic to $C_5 = C_{5,2}$, we have $\chi_c(G) = p/q > 5/2$. Moreover, by Claim 32.A, G has maximum degree 3 and hence by Brook's theorem, $p/q \le \chi(G) \le 3$. By Proposition 11, G must contain a pk-cycle for some $k \ge 1$, therefore we have $p \le pk \le 14$. These facts imply that $p/q \in \{14/5, 13/5, 11/4, 8/3\}$. We now distinguish between these four cases.

Case 1.. Assume that $\chi_c(G) = 14/5$. Let c be a 14/5-circular colouring of G. By Proposition 11, G contains a (spanning) 14-tight cycle with respect to c, that is, a cycle $C : v_0, \ldots, v_{13}$ with $c(v_i) = 5i \bmod 14$. Then, by c, vertex v_i may have only $v_{i-4 \bmod 14}$, $v_{i+4 \bmod 14}$ and $v_{i+7 \bmod 14}$ as additional neighbours. But the two former would create a 5-cycle, hence v_i may only be adjacent to $v_{i+7 \bmod 14}$. But then, G must be bipartite and therefore maps to C_5, a contradiction.

Case 2.. Assume that $\chi_c(G) = 13/5$. Let c be a 13/5-circular colouring of G. By Proposition 11, G contains a 13-tight cycle with respect to c, that is, a cycle $C : v_0, \ldots, v_{12}$ with $c(v_i) = 5i \bmod 13$. Again, by c, v_i may only have $v_{(i-4) \bmod 14}$ and $v_{(i+4) \bmod 14}$ as additional neighbours in C, but any of these edges would create a 5-cycle. Hence, C must be chordless and G has order 14. Let x be the last vertex of G. By Claim 32.A, x has degree at most 3, but then we have a thread with at least three vertices of degree 2 on C, contradicting Claim 32.B.

Case 3.. Assume that $\chi_c(G) = 11/4$. Let c be an 11/4-circular colouring of G. By Proposition 11, G contains a 11-tight cycle with respect to c, that is, a cycle $C : v_0, \ldots, v_{10}$ with $c(v_i) = 4i \bmod 11$. As in the previous cases, C is chordless. Let x_1, x_2 and x_3 be the (at most) three remaining vertices of G. Note that a vertex x_i coloured j by c may have the vertices coloured $(j + \ell) \bmod 11$ as neighbours, for $\ell = 4, 5, 6, 7$. Because G has no 5-cycle, x_i cannot be adjacent to the two vertices coloured with the pairs of colours $\{(j + \ell) \bmod 11, (j + \ell + 1) \bmod 11\}$ for $\ell = 4, 5, 6$. In other words, x_i can have at most two neighbours on C, and if it has two, then these neighbours must be at distance 2 or 5 in C.

Next, we show that x_i cannot have two neighbours at distance 2 in C. Assume to the contrary that (without loss of generality) vertex x_1 is coloured 0 and is adjacent to v_1 and v_{10}. The cycle $x_1, v_0, \ldots, v10$ is another 11-tight cycle. There must be a 5-walk (in fact, a 5-path) $P : v_0t_1t_2t_3t_4x_1$ connecting v_0 and x_1. First, note that t_4 cannot be a vertex of C since t_4 is neither v_1 nor v_{10} and x_1 has only two neighbours in C. Moreover, t_1 is not a vertex of C either, since C has no chord. Therefore, $\{t_1, t_4\} = \{x_2, x_3\}$. Hence, t_2 and t_3 must be vertices of C (but not any of v_0, v_1, v_{10}). Since we have two 11-tight cycles, x_2 and x_3 can only be neighbours with v_2, v_9, v_5 or v_6. If t_2 or t_3 is v_2 or v_9, we obtain a 5-cycle in G. Therefore, without loss of generality, $t_2 = v_6$ and $t_3 = v_5$. No edge can be added to this graph, so that there is a thread of three consecutive degree 2 vertices in G, contradicting Claim 32.B.

Further, we prove that there must be a pair of consecutive vertices of degree 2 in C. Assume the contrary. Then, G is fixed (up to isomorphism) and each vertex x_i has two neighbours in C; without loss of generality, x_1 is adjacent to v_1 and v_6; x_2 is adjacent to v_3 and v_8; x_3 is adjacent to v_5 and v_{10}. Moreover, there are no further edges in G (otherwise we obtain a 5-cycle). But then, there is no 5-walk connecting v_0 and v_2, a contradiction.

Therefore, assume without loss of generality that v_1 and v_2 both have degree 2. Then, by Claim 32.B (again without loss of generality), v_0 is adjacent to x_1 and v_3 is adjacent to x_2. Assume now that x_3 is adjacent to both v_4 and v_{10}. Vertex x_1 can only be the neighbour of one vertex among v_5, v_6 and v_9 while x_2 can only be the neighbour of one vertex among v_5, v_8 and v_9. Thus, v_7 has degree 2. Claim 32.B ensures us that v_8 or v_6 has degree 3. Without loss of generality, we assume that v_6 has degree 3. Then it must be the neighbour of x_1. Similarly, v_8 or v_9 must have degree 3 but the only possible way is for v_8 to be a neighbour of x_2. Then, there are no further edges in G, otherwise we get a 5-cycle or contradict Claim 32.A. But then, the following mapping is a homomorphism to C_5: v_0 to v_{10} are mapped (in the cyclic order) to 3, 2, 3, 2, 1, 2, 1, 0, 4, 3, 4 and x_1, x_2, x_3 are mapped to 2, 3 and 0, respectively. This is a contradiction.

Hence, we can assume without loss of generality that v_4 has degree 2. Then, v_2 and v_4 are connected by a 5-walk that must go through v_1, v_0 and v_5 and hence x_1 must be adjacent to v_5. A 5-walk connecting v_1 and v_{10} has to go through v_2 and v_3, but not v_4 (otherwise, since v_4 has degree 2, it must go through v_5 but then v_5 and v_{10} are adjacent, a contradiction). Hence we have a 2-path connecting x_2 and v_{10}; this 2-path has to use x_3 or v_9. If it uses x_3, then vertices v_6 and v_7 must have degree 2. By Claim 32.B, vertex v_8 must have degree 3. This can only be achieved if x_2v_8 is an edge but it creates a C_5. Hence, the 2-path between v_{10} and x_2 goes through v_9. Claim 32.B ensures that v_6, v_7 or v_8 is a neighbour of x_3. In any of these cases, x_3 cannot be a neighbour of v_{10} at the same time. Therefore v_{10} must have degree 2. Then, the 5-walk from v_8 to v_{10} has to go through v_0 and x_1. That is, x_1 and v_8 are connected by a 3-path. Then x_3 must be adjacent to x_1 and to v_7. But then x_1 is a $(3, \geq 5)$-vertex, contradicting Claim 32.A. This completes the proof of Case 3.

Case 4.. Assume that $\chi_c(G) = 8/3$. Let c be an 8/3-circular colouring of G. By Proposition 11, G contains an 8-tight cycle with respect to c, that is, a cycle $C : v_0, \dots, v_7$ with $c(v_i) = 3i \bmod 8$. Similarly as before, C must be chordless. Let $X = \{x_i, 1 \le i \le |V(G)| - 8\}$ be the (at most six) remaining vertices of G.

First of all, we claim that any vertex x_i of X may have at most one neighbour in C. The proof is similar to the similar statement in Case 3. Assume by contradiction that x_1 has two neighbours in C. Without loss of generality, we assume that x_1 is coloured with colour 5 by c. Then, the neighbours of x_1 may be the ones coloured 0, 1 and 2, that is, v_0, v_3 and v_6. But x_1 cannot be adjacent to both v_0 and v_3 or v_3 and v_6 (otherwise we have a 5-cycle in G). Hence, x_1 is adjacent to v_0 and v_6 (and takes part in another 8-tight cycle $x_1v_0v_1v_2v_3v_4v_5v_6$). Now, we have a 5-walk (in fact, a 5-path) $P_1 : v_7t_1t_2t_3t_4x_1$ connecting x_1 and v_7. We now show that no inner-vertex t_i of this 5-walk belongs to C. Clearly, this is the case for t_1 and t_4 since neither v_7 nor x_1 has a third neighbour in C. Without loss of generality, assume by contradiction that t_2 is a vertex of C (a symmetric argument holds for t_3). Then t_1 would have two neighbours in a 8-tight cycle so that t_2 must be v_1 or v_5. In both cases, this leads to a 5-cycle in G. Therefore, all inner-vertices in P_1 are distinct vertices of X. Now, consider the 5-path P_2 that must connect v_0 and v_6. By Claim 32.A, v_0 and v_6 cannot have new neighbours. So that P_2 starts by v_0, v_1 and ends with v_5, v_6. Then, there must be a 3-path between v_1 and v_5. This path cannot go through v_2 or v_4 because it would create a C_5 in G. Moreover, the neighbour of v_1 in P_2 cannot be in P_1 otherwise it vould create a short odd cycle. Then it must be x the last vertex of X. But the same reasoning tells us that the neighbour of v_7 in P_2 is also x. Then P_2 is not a 5-walk, a contradiction. Hence we have proved that any vertex x_i may have at most one neighbour in C. Since X has size at most 6, it implies that there are at least two degree 2 vertices in C.

Next, we show that there cannot be two adjacent degree 2 vertices in C. Assume by contradiction that v_1 and v_2 are both of degree 2. The 5-walk connecting v_2 and v_4 must go through v_1 and v_0. Furthermore, it cannot go through v_5 or v_7 (otherwise we have a 5-cycle). Hence it must go through two adjacent vertices x_1 and x_2 forming a 3-path between v_0 and v_4. Using the same argument for the 5-walk connecting v_1 and v_7, we get a disjoint 3-path uding x_3 and x_4 connecting v_3 and v_7. But then, there can be no further vertex in G (otherwise one of the degree 3-vertices of C is a $(3, 5)$-vertex, contradicting Claim 32.A. Furthermore the only additional edges that might exist are x_1x_3 and x_2x_4. But then, G admits a homomorphism to C_5 with the vertices v_0 to v_7 mapped to 4, 0, 1, 2, 1, 0, 1, 0 and x_1 to x_4 are mapped to 4, 3, 3, 2 — a contradiction. Hence, the set of degree 2 vertices of C forms an independent set. This implies that there are at most four degree 2 vertices in C.

Therefore, there are either two, three or four vertices of degree 2 in C.

Subcase 4a: there are exactly two vertices of C with degree 2. If these two vertices are at distance 2 on C, we assume without loss of generelity that they are v_0 and v_6, and that we have the edges v_7x_6 and v_ix_i for $1 \le i \le 5$. Then, because by Claim 32.A there is no $(3, \ge 5)$-vertex, the only other possible neighbours of x_i for $i = 2, 3, 4$ are x_{i-1} and x_{i+1}. Since there is no vertex of degree 2 in a 4-cycle (otherwise G is not a core), we have the edges x_1x_2, x_2x_3, x_3x_4 and x_4x_5. By the same argument, we need a third neighbour for x_1, but then we get either a 5-cycle or a degree 4 vertex, a contradiction. Next, assume that the two vertices of C with degree 2 are at distance 3 on C; without loss of generality, they are v_0 and v_5 and we have the edges v_ix_i for $1 \le i \le 4$, v_6x_5 and v_7x_6. As before, we must have the edges x_1x_2, x_2x_3 and x_3x_4. If x_1x_4 is an edge, then x_5 and x_6 either have degree 1 or degree 2 in a 4-cycle (and thus G is not a core). Then the third neighbours of x_1 and x_4 must be x_5 and x_6, respectively. The last possible edge is x_5x_6. But in any case, there is no 5-walk connecting v_0 and v_6, a contradiction.

Therefore, we may assume that the two vertices of degree 2 of C are at distance 4 on C. Assume they are v_0 and v_4; again we have the edges v_ix_i for $i = 1, 2, 3$ and v_ix_{i-1} for $i = 5, 6, 7$. By the same argument as before, we have the edges x_1x_2, x_2x_3, x_4x_5 and x_5x_6. Again, x_1 must have a third neighbour, which can be only x_3, x_4 or x_6, since all other vertices cannot have further neighbours. But each possibility either creates a short odd cycle, or x_1 becomes a $(3, \ge 5)$-vertex, a contradiction in each case.

Subcase 4b: there are exactly three vertices of C with degree 2. Then, there must be two such vertices at distance 2 on C, say v_0 and v_6.

Assume first that the third vertex of degree 2 on C is v_4. Then, we assume that we have the edges v_ix_i for $i = 1, 2, 3$, v_5x_4 and v_7x_5. By similar arguments as before, x_2 has degree 3 and is a neighbour of x_1 and x_3. Both x_1 and x_3 need a third neighbour, that can be one of x_4, x_5 and x_6. But both cannot be adjacent to x_6 (otherwise we get a 5-cycle), so without loss of generality we may assume that we have the edge x_1x_4 (note that x_1x_5 would produce a 5-cycle). If x_3 is adjacent to x_5, then x_6 does not exist (otherwise it must be adjacent to x_4 and x_5 but then x_1 is a $(3,5)$-vertex, contradicting Claim 32.A. Then, we have the following homomorphism to C_5: v_0 to v_7 are mapped to 5, 1, 2, 1, 5, 4, 3, 4 in the cyclic order, and x_1 to x_5 are mapped to 2, 1, 2, 3, 3 — a contradiction. Therefore, x_3 must be adjacent to x_6. Then, the last edge of G is x_5x_6. But then, v_6 and x_5 are not connected by any 5-walk, a contradiction.

Hence, by the symmetries of the previous case, the third vertex of C with degree 2 must be v_3. Assume we have the edges v_ix_i for $i = 1, 2$, v_ix_{i-1} for $i = 4, 5$, and the edge v_7x_5. Consider the 5-path P that must connect v_0 and v_2; the neighbour of v_0 on P must be v_7, and the neighbour of v_7 on P must be x_5 (otherwise it is v_6, and the next vertex of P is v_5; then we have a 2-path connecting v_2 and v_5, creating a 5-cycle). Similarly, the neighbour of v_2 on P cannot be v_3 (otherwise the next vertex of P must be v_4 and we have a 2-path connecting v_4 and v_7, creating a 5-cycle). Hence, x_2 is the neighbour of v_2 on P. The only possible common neighbour of x_2 and x_5 being x_6, we have the edges x_2x_6 and x_5x_6. The same argument for the 5-path connecting v_6 and v_7 implies the existence of edge x_3x_6. We cannot have the edge x_1x_3, otherwise x_3 is a $(3, \geq 5)$-vertex, contradicting Claim 32.A (by symmetry we do not have the edge x_2x_4 either). Hence, the further possible neighbours of x_1 are x_2 and x_4. We must have the edge x_1x_4, otherwise G is not a core. The last possible edges are x_1x_2 and x_3x_4, but the existence of any of these edges implies that either x_2 or x_4 becomes a $(3,5)$-vertex, contradicting Claim 32.A. Then, x_2 and v_3 are not connected by any 5-walk, a contradiction.

Subcase 4c: there are exactly four vertices of C with degree 2. Without loss of generality these are v_0, v_2, v_4 and v_6. For any two of these vertices, the only way for them to be connected by a 5-walk is to have in G two vertex-disjoint 3-paths $v_1x_1x_2v_5$ and $v_3x_3x_4v_7$. If there is no further vertex, then there is no further edge either, and we can map G to C_5, a contradition (by mapping v_0 to v_7, in the cyclic order, to 1, 0, 4, 3, 2, 3, 4, 0, and the vertices of X to 1 and 2). Hence, we have a vertex x_5; it can be adjacent to at most two previously considered vertices (which must be in X).

If x_5 has two such neighbours, assume without loss of generality that these are x_1 and x_3 (then x_5 cannot be adjacent to x_2 or x_4). If there is no vertex x_6, it is easy to extend the previous mapping to C_5 by mapping x_5 to 1 or 2. Hence, there is a vertex x_6, which must be adjacent to one of x_2 and x_4 (say x_2). But then, x_1 is a $(3, \geq 5)$-vertex, contradicting Claim 32.A.

Therefore, each remaining vertex of X has at most one neighbour among x_1, x_2, x_3, x_4. Assume without loss of generality that x_5 is a neighbour of x_1. Then, it must have another neighbour, which must be x_6; x_6 also needs a second neighbour, which must be x_3 or x_4 (by symmetry, say x_3), for otherwise G has no further edges and is not a core. But then there is no 5-walk connecting x_2 and x_5, a contradiction. □

AppendixB. A graph in $\mathcal{SP}_5$ that does not map to any triangle-free graph of order at most 7

The next proof is basically saying that the 3-walk chromatic number of $T_5(2,2,2)$ is 8.

We show that the triangle-free and K_4-minor-free graph $T_5(2,2,2)$ cannot map to a triangle-free graph of order less than 8. We use the labels from Figure B.11. By contradiction, assume there is such a mapping $h : V(T_5(2,2,2)) \to V(B)$, where B is a triangle-free graph of order 7. Note that two vertices connected by a 5-walk in $T_5(2,2,2)$ cannot have the same image in B by h (otherwise B would have a triangle). Therefore, each of the vertices a, b and c cannot have the same image by h as any other vertex of $T_5(2,2,2)$. For the remaining vertices, using the same argument the only set of pairs of vertices that may have the same image by h is

$$S = \{\{x_1,z_2\},\{x_1,y_2\},\{x_1,z\},\{y_1,x_2\},\{y_1,z_2\},\{y_1,x\},\{z_1,y_2\},\{z_1,x_2\},\{z_1,y\},\{x_2,y\},\{y_2,z\},\{z_2,x\}\}.$$

Define the graph $H = (V(T_5(2,2,2)), S)$ on $V(T_5(2,2,2))$, where the pairs in S are the edges of H. This identification graph is formed by three isolated vertices and a 6-cycle where for three independent edges, the two endpoints have a common private neighbor of degree 2. Now, the homomorphism h corresponds to a partition of the vertices of H into at most seven vertex-disjoint cliques (where a clique corresponds to a set of vertices having the same image in B). Any such partition of $V(H)$, as indicated before, contains three cliques of size 1 ($\{a\}$, $\{b\}$ and $\{c\}$). In order to cover the remaining vertices, we must use at least one 3-clique of H, for otherwise we would need at least five additional cliques, a contradiction. We claim that we cannot use more than one of the 3-cliques of H. Without loss of generality, for contradiction, assume we have used the two 3-cliques $\{x_1, y_2, z\}$ and $\{y_1, z_2, x\}$ in the partition. Then, together with the image of c by h, the images of x and z by h form a triangle in B, a contradiction. Therefore, the only possibility for the clique cover is to contain one triangle, say, $\{x_1, y_2, z\}$ together with a matching of the uncovered subgraph of H, i.e. $\{x, z_2\}$, $\{y_1, x_2\}$, $\{y, z_1\}$. But then, the images of a, x and y by h again form a triangle in B, a contradiction.

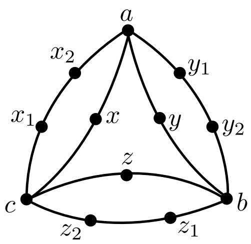

Figure B.11: The graph $T_5(2, 2, 2)$, which does not map to any triangle-free graph of order at most 7.