Edge clique covers in graphs with independence number two

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1 | INTRODUCTION

Throughout, $G = (V, E)$ refers to a simple loopless graph and n denotes the number of vertices in G when the graph in question is clear. Standard graph theory notation is used throughout, but we make the following definitions explicit.

We denote by $G[X]$ the subgraph of G induced by $X \subseteq V(G)$. A *clique* is a set $Q \subseteq V(G)$ such that $G[Q]$ is a complete graph. The *edge clique cover number* of a graph G, denoted $\text{ecc}(G)$, is the minimum number of complete subgraphs of G whose union contains every edge of G. This parameter, also known as the *intersection number* of a graph, was introduced by Erdős, Goodman, and Posa [1]. In addition to being an interesting parameter in its own right, it is also related to a parameter known as the *competition number* of a graph, introduced by Cohen [2] in the study of food webs.

We begin with the following problem:

Problem 1.1 (Chen, Jacobson, Kézdy, Lehel, Scheinerman, Wan [3]) *If G is a claw free graph, is $\text{ecc}(G) \leq n$?*

As evidence, it was shown in [3] that the answer is "yes" in the special case of quasi-line graphs (graphs where the neighborhood of any vertex can be partitioned into two cliques), a question which was of interest from the point of view of competition numbers of graphs. Recently, Javadi and Hajebi [4] have given an affirmative answer to Problem 1.1 when $\alpha(G) \geq 3$. Their result relies heavily on Chudnovsky and Seymour's structure theorem for claw-free graphs (see [5]), which only applies to graphs with $\alpha(G) \geq 3$. Since their techniques cannot be extended to the case of $\alpha(G) = 2$, Problem 1.1 is reduced to the following conjecture, which is the focus of our study in this paper:

Conjecture 1 *If G is a graph with $\alpha(G) = 2$, then $\text{ecc}(G) \leq n$.*

Our work is organized as follows. Section 2 contains our main results on $\text{ecc}(G)$, where we improve upon the previously best known bound of $\text{ecc}(G) \leq cn^{4/3} \log^{1/3} n$ if $\alpha(G) = 2$, given by Javadi, Maleki, and Omoomi in [6][1]. We also show how our results can be used to obtain even better bounds for fractional edge clique covers. In Section 3, we

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[1]We note that an upper bound of $2(1 - o(1))n$ is cited in [4], which is claimed to have been shown in a manuscript written by one of the authors of that paper. However, this manuscript does not appear to be available in publication and no explicit proof is given elsewhere.

consider the dual problem of packing edges with respect to cliques. In Section 4, we present some basic observations about graphs of independence number 2 and use them to derive properties that any minimal counterexample to Conjecture 1 must have, if one exists. Finally, in Section 5, we examine $\text{ecc}(G)$ for some interesting subclasses of graphs with independence number 2; in particular, we consider forbidden induced subgraphs, the complete joins of graphs, and graphs with bounded diameter.

2 | IMPROVED UPPER BOUNDS

2.1 | Edge clique cover number

We begin with the following simple lemma.

Lemma 2.1 *If $\alpha(G) \leq 2$ and x is a vertex of G, $\overline{N(x)}$ induces a clique in G.*

Given a graph G, we say that an edge $e \in E(G)$ is a *dominating edge* if every $v \in V(G)$ is adjacent to an end of e. The proof of our main result relies on the two following lemmas:

Lemma 2.2 *If G with $\alpha(G) \leq 2$ has no dominating edge, then $\text{ecc}(G) \leq n$.*

Proof The hypothesis on G implies that the ends of any edge of G belongs to some $\overline{N(x)}$. Hence the collection of all non-neighborhoods constitutes a family of n cliques (by Lemma 2.1) that cover all edges of the graph.

Lemma 2.3 *Let G be a graph with $\alpha(G) = 2$. For an edge $e \in E(G)$, $\alpha(G - e) > \alpha(G)$ if and only if e is not a dominating edge.*

Proof Let x and y be the endpoints of e. First we prove that if e is not a dominating edge, then $\alpha(G - e) > \alpha(G)$. Since e is not a dominating edge, there exists a vertex $v \in V(G)$ such that $xv \notin E(G)$ and $yv \notin E(G)$. Thus, in graph $G - e$ the set $\{x, y, v\}$ is an independent set. Therefore, $\alpha(G - e) \geq 3 > 2 \geq \alpha(G)$. Now, we prove that if e is a dominating edge, then $\alpha(G - e) = \alpha(G)$. Since for every $v \in V(G)$, v is adjacent to at least one of x or y, no independent set of size 3 can contain x and y in $G - xy$. Thus, $\alpha(G - e) = \alpha(G)$.

In [6], it is shown that there is a constant c such that $\text{ecc}(G) \leq cn^{4/3} \log^{1/3} n$ if $\alpha(G) = 2$. By applying Lemmas 2.2 and 2.3, we obtain the following general theorem, which implies a linear upper bound for $\text{ecc}(G)$ in terms of n. Recall that the vertex cover number of a graph G, denoted $\beta(G)$, is the cardinality of a smallest set of vertices which is incident to every edge of G.

Theorem 2.4 *Let G be a graph with $\alpha(G) = 2$. If G' is a minimal spanning subgraph with $\alpha(G') = 2$, then $\text{ecc}(G) \leq n + \beta(G - E(G'))$.*

Proof Let $D = E(G) \setminus E(G')$ and let $G_D = (V(G), D)$ (or, equivalently, $G_D = G - E(G')$). As a consequence of Lemma 2.3, every edge of D is a dominating edge in G. By minimality, no edge remaining in G' is a dominating edge of G'. By Lemma 2.2, G' has an edge clique cover C of size at most n. Since the only edges of G which are not covered by C are those in D, we are left to find a set of at most $\beta(G_D)$ cliques which cover D in G.

Let $v \in V(G)$ be any vertex and denote by $D_v = \{vu_1, \ldots, vu_k\}$ the edges of D incident to v. Since $u_1, \ldots, u_k$ are non-neighbors of v in G', they form a clique in G' by Lemma 2.2, and thus form a clique in G as well. Let $K_v = \{v, u_1, \ldots, u_k\}$; clearly K_v covers D_v. If W is a vertex cover for G_D, then every edge of D is incident to some vertex $w \in W$. It then follows that $\{K_w : w \in W\}$ covers D, and so D can be covered by at most $\beta(G_D)$ cliques as desired.

Corollary 2.5 *If G is a graph with $\alpha(G) = 2$, then $\mathrm{ecc}(G) \leq n + \delta(G)$.*

Proof Let G', D, and G_D be as in the proof of Theorem 2.4. Recalling that every edge in D is a dominating edge in G, it follows that, for any vertex $v \in V(G)$, every edge of D is incident to a vertex of $N_{G'}[v]$. Let v be a vertex of minimum degree. If some edge of D is incident to v, $d_{G'}(v) \leq \delta(G) - 1$ and so $N_{G'}[v]$ is a vertex cover for G_D of cardinality at most $(\delta(G) - 1) + 1 = \delta(G)$. On the other hand, if no edge of D is incident to v, then in fact $N_{G'}(v)$ suffices for our vertex cover, which also has cardinality at most $\delta(G)$.

Corollary 2.6 *If G is a graph with $\alpha(G) = 2$, then $\mathrm{ecc}(G) \leq 2n - c\sqrt{n \log n}$ for some absolute constant $c > 0$.*

Proof Again, let G', D, and G_D be as in the proof of Theorem 2.4. If the vertices x, y, z form a triangle in G_D, then $\{x, y, z\}$ is an independent set in G'. By our choice of G' to have $\alpha(G') = 2$, it follows that G_D is triangle-free. It is known that every triangle-free graph has an independent set of size $\Omega(\sqrt{n \log n})$ [7]. Since the complement of an independent set is a vertex cover, we get an upper bound of $n - c\sqrt{n \log n}$ for the size of a vertex cover of G_D, and so G has an edge clique cover of size at most $2n - c\sqrt{n \log n}$.

Corollary 2.6 makes explicit use of the lower bound known on the Ramsey number $R(3, t)$ (the minimum value of n such that every n-vertex graph contains either a clique of size 3 or an independent set of size t). It is worth noting that this bound was shown to be tight by a famous theorem of Kim [8], and so any significant improvement to the bound given in Corollary 2.6 will almost certainly require a different approach than that of Theorem 2.4.

2.2 | Fractional edge clique covers

Consider the fractional version of the problem. A graph G has a *fractional edge clique cover* of size k if there exists a set of cliques Q and a real-valued weight function w such that

1. $w(Q) \geq 0$ for all $Q \in Q$,
2. $\displaystyle\sum_{Q\,:\,e \in E(G[Q])} w(Q) \geq 1$ for each $e \in E(G)$, and
3. $\displaystyle\sum_{Q \in Q} w(Q) = k$.

The *fractional edge clique cover number* of a graph G, denoted $\mathrm{ecc}_f(G)$ and first studied in [9], is the smallest k such that G has a fractional edge clique cover of size k. The proof of Theorem 2.4 is easily modified to give an upper bound on $\mathrm{ecc}_f(G)$ which is an improvement on its integer counterparts.

Theorem 2.7 *If G is a graph with $\alpha(G) = 2$, then $\mathrm{ecc}_f(G) \leq \frac{3}{2}n$.*

Proof Let G, G', and D be as in the proof of Theorem 2.4. As before, we begin with an edge clique cover of G' of size at most n. Instead of taking a vertex cover for D, we take the set $\{K_v \mid v \in V(G)\}$. Note that each edge of D is covered twice by $\{K_v\}$. Thus, by assigning weight 1 to the cliques covering G' and weight $\frac{1}{2}$ to each K_v, we obtain the desired result.

3 | EDGE PACKING WITH RESPECT TO CLIQUES

A set of edges $E' \subseteq E(G)$ is called an *edge packing with respect to cliques* (which we will simply call a *packing* for the remainder of this work) if for any two distinct edges $e_1, e_2 \in E'$, no clique of G contains both e_1 and e_2. The *packing number* of G, denoted $\text{pack}(G)$, is the maximum size of a packing of G. Since no clique of G can cover two edges of a packing, $\text{ecc}(G) \geq \text{pack}(G)$. Indeed, it is easy to see that if we consider the edge clique cover problem as an integer program, then the edge packing problem is its dual. It thus easily follows that $\text{pack}(G) \leq \text{pack}_f(G) = \text{ecc}_f(G) \leq \text{ecc}(G)$, where $\text{pack}_f(G)$ is the value of an optimal solution to the linear relaxation of the edge packing integer program.

We begin with two simple observations.

Proposition 3.1 *If F is a packing of G and $v \in V(G)$, then v is incident to at most $\alpha(G)$ edges that belong to F.*

Proof Denote by $G_F = (V, F)$ the graph G restricted to F. The neighborhood of v in G_F must form an independent set in G, as otherwise two packed edges would belong to a common triangle.

This leads us to an easy to obtain upper bound on $\text{pack}(G)$ in the case where $\alpha(G) = 2$.

Proposition 3.2 *If G is a graph with $\alpha(G) = 2$, then $\text{pack}(G) \leq n$.*

Proof By Proposition 3.1, each vertex v has at most 2 incident packed edges. By a handshaking argument, $\text{pack}(G) \leq n$.

One may tempted to conjecture that, should a counterexample exists to Conjecture 1, it should have very large packing number. The following theorem shows, perhaps somewhat surprisingly, that graphs with $\alpha = 2$ and maximum possible packing number actually satisfy Conjecture 1.

Theorem 3.3 *If $\alpha(G) = 2$ and $\text{pack}(G) = n$, then $\text{ecc}(G) = n$.*

Proof Let F be a packing of n edges. By Proposition 3.1, every vertex is incident to exactly two edges of F. For $u, v \in V$, denote $N_{u,v} = N_G[u] \cap N_G[v]$. We prove that for every $uv \in F$, $N_{u,v}$ is a clique, and that the collection of cliques $\{N_{u,v} : uv \in F\}$ covers $E(G)$.

Let $v \in V(G)$ be an arbitrary vertex, and let $v_1, v_2 \in V(G)$ be such that $vv_1, vv_2 \in F$ (note that this implies that $v_1 v_2 \notin E(G)$). If $vz \in E$ for some z, then z must be a neighbor of at least one of v_1 or v_2, for if not, $\{v_1, v_2, z\}$ form an independent set. We now argue that z is adjacent to exactly one of v_1 and v_2. Suppose that both $v_1 z, v_2 z \in E$. Let $z_1, z_2 \in V(G)$ be such that $zz_1, zz_2 \in F$. Note that z_1, z_2 must be distinct from v, as well as from v_1, v_2 – if $v_i = z_j$ for $i, j \in \{1, 2\}$, then vv_i, zv_i are both packed edges in the $\{v, v_i, z\}$ triangle, a contradiction. If z_1 is a neighbor of v, then as above, it must be also neighbor of v_1 or v_2. Suppose w.l.o.g. that $z_1 v_1 \in E$. Then $\{v_1, v, z, z_1\}$ form a clique, contradicting that vv_1 and zz_1 are both packed edges. Therefore, vz_1, and by the same argument vz_2, cannot be edges, implying that $\{v, z_1, z_2\}$ is an independent set, a contradiction. Thus, no neighbor of v is a common neighbour to both v_1 and v_2; in other words, $N_{v,v_1} \setminus \{v\}$ are all non-neighbors of v_2. This implies that $N_{v,v_1} \setminus \{v\}$ is a clique (since $\alpha(G) = 2$), and thus so is N_{v,v_1} as claimed.

It remains to show that these cliques cover all edges of G. If $vz \in E(G)$, where $z \notin \{v_1, v_2\}$, then exactly one of $v_1 z$ and $v_2 z$ is an edge of G, as argued above. In other words, every edge incident to v must lie either in the clique N_{v,v_1} or the clique N_{v,v_2}. Thus $\{N_{u,v} : uv \in F\}$ covers $E(G)$.

We note that infinite families of graphs satisfying the conditions of Theorem 3.3 exist. Javadi and Hajebi [4] show that the pth power of the cycle C_{3p+1} has independence number 2 and edge clique cover number $3p + 1$; it is easy to check that pairs of vertices at distance p around such a cycle are adjacent in C_{3p+1} and these edges form a packing. This also shows that the bound in Proposition 3.2 cannot be improved upon.

It may seem tempting to consider this as evidence in favour of the validity of Conjecture 1, as graphs with low packing number might seem to be easier to cover with few cliques. However, $\text{ecc}(G)$ can be much greater than $\text{pack}(G)$ – the complete multipartite graph $K_{2,2,\ldots,2}$ with n parts satisfies $\alpha(G) = 2$, $\text{pack}(G) = 4$, and $\text{ecc}(G) = \theta(\log n)$ for large enough n [9].

4 | TOWARD A MINIMUM COUNTEREXAMPLE TO CONJECTURE 1

We now establish a number of lemmas, stated in their most general form, which will allow us to simply state a number of properties that any vertex-minimal counterexample to Conjecture 1 must possess, if such a graph exists.

Lemma 4.1 *If G is a graph with $\alpha(G) = 2$ and $u \in V(G)$ is an arbitrary vertex, then*

1. $\text{ecc}(G) = \text{ecc}(G - u) + k$ *if $d(u) = n - 1$, where k denotes the number of isolated vertices in $G - u$;*
2. $\text{ecc}(G) \leq \text{ecc}(G - u) + 1$ *if u is simplicial (i.e. if $N(u)$ is a clique);*
3. $\text{ecc}(G) \leq \text{ecc}(G[N(u)]) + \ell + d(u) + 1$ *if $1 < d(u) < n - 1$, where ℓ denotes the number of isolated vertices in $G[N(u)]$;*
4. $\text{ecc}(G) \leq 2d(u) + 1$ *if $\text{ecc}(G[N(u)]) \leq d(u)$;*
5. $\text{ecc}(G) \leq n$ *if $\delta(G) \leq \frac{n-1}{2}$ and $\text{ecc}(G[N(u)]) \leq d(u)$ where u is a vertex of minimum degree.*

Proof The proofs of the first two claims are left to the reader. To prove the third, let C be a minimal clique cover of $G[N(u)]$ for some vertex u satisfying the given condition. By adding u to each $C \in C$, we obtain a collection of cliques which covers all edges in $G[N[u]]$. By Proposition 2.1, $\overline{N(u)}$ is a clique, and so we are left only to cover those edges from $N(u)$ to $\overline{N(u)}$. To do this, for each $x \in N(u)$ we add the clique $\{x\} \cup (N(x) \cap \overline{N(u)})$. Altogether we have at most $\text{ecc}(G[N(u)]) + d(u) + 1$ cliques, as claimed. The fourth claim follows from the third, and the fifth from the fourth.

Proposition 4.2 *Let G be a graph with $\alpha(G) = 2$. If G contains distinct vertices $x, y \in V(G)$ such that $N[x] \subseteq N[y]$, then $\text{ecc}(G) \leq \text{ecc}(G - y) + 1$.*

Proof If there exists such x and y, note that $Z = N(y) \setminus N(x)$ is a clique. The family of at most $\text{ecc}(G - y)$ cliques covering the edges of $G - y$ can easily be extended to the desired collection of $\text{ecc}(G - y) + 1$ cliques by adding y to every clique that contains x, and adding a new clique $\{y\} \cup Z$ to the collection.

Proposition 4.3 *Let G be a graph with $\alpha(G) = 2$. If G contains non-adjacent vertices $x, y \in V(G)$ such that $N(x) \subsetneq N(y)$ and $\text{ecc}(G - y) \leq n - 1$, then $\text{ecc}(G) \leq n$.*

Proof Since $\alpha(G) = 2$ and x and y are non adjacent, every other vertex is adjacent to either x or y. Since $N(x) \subseteq N(y)$, this gives that $V(G) = \{x, y\} \cup N(y)$. Note also that if there is an isolated vertex in the graph induced by $N(y)$, then $N(y)$ is just a clique plus an isolated vertex. With the addition of x and y, G can be edge covered with four

cliques. Since we may easily assume that G has at least 4 vertices, we need only consider the case when every vertex in $N(y)$ has a neighbor in $N(y)$.

Let $A = N(y) \setminus N(x)$, which is non-empty by assumption. Denote by B the set of vertices in $N(x)$ which have a neighbor in A, and let $C = N(x) \setminus B$. Since A is non-empty and $\alpha(G) = 2$, C is a clique. Let $C_1, C_2, \ldots C_p$, with $p \leq n-1$, be the collection of cliques which covers all edges of $G \setminus y$. For every i such that C_i does not contain x, we add y to the clique C_i (it is still a clique since y dominates every vertex but x). We also add a new clique $\{y\} \cup C$. We claim that this collection of at most n cliques is an edge cover of G. Every vertex a in A is adjacent to some edge in $N(y)$, so it must be contained in some clique C_i that cannot contain x (since x and a are not adjacent), and therefore the edge ya will be covered by the new clique $C_i \cup \{y\}$. Every vertex $b \in B$ is adjacent to some vertex a in A so the edge ab must be contained in some clique that cannot contain x, so similarly the new clique with y will contain the edge yb. Since we added the clique $\{y\} \cup C$ the collection is an edge cover as claimed.

Proposition 4.4 *Let G be a graph with $\alpha(G) = 2$. If G contains non-adjacent vertices x, y such that $N(x) = N(y)$ and $\mathrm{ecc}(G - \{x, y\}) \leq n - 2$, then $\mathrm{ecc}(G) \leq n$.*

Proof As in the proof of Proposition 4.3, $V(G) = \{x, y\} \cup N(y)$ and we may assume that every vertex in $N(y)$ has a neighbor in $N(y)$. Let G' be the graph induced by $N(y)$. If G' is complete then G satisfies the theorem. Assume then that x' and y' are non adjacent vertices in G'. Now apply induction to G' and get a family of cliques $C_1, C_2, \ldots C_p$, with $p \leq n - 2$ that cover all edges of G'. We add x to every clique that contains x', and y to every clique that contains y' (note that no clique contain both x' and y'). Denote by X (resp. Y) be the set of neighbors of x' (resp. y') in G'. Because $\alpha(G) = 2$, $X \setminus Y$ and $Y \setminus X$ induce cliques. It is easy to check that adding the cliques $\{x\} \cup (Y \setminus X)$ and $\{y\} \cup (X \setminus Y)$ gives a collection of at most n cliques that cover every edge.

Lemma 4.5 *Let G be a graph with $\alpha(G) = 2$. If X is a clique such that $|X| \geq \frac{n+1}{2}$ and $\mathrm{ecc}(G - X) \leq |G - X|$, then $\mathrm{ecc}(G) \leq n$.*

Proof Let $|X| = k$. If $v \in V(G - X)$ has a neighbor in X, let G_v be the graph induced by v and all its neighbors in X. The collection of all G_v's covers all edges from X to $G - X$, and so G has an edge clique cover of size at most $(n - k) + (n - k) + 1 \leq 2n - 2\frac{n+1}{2} + 1 = n$.

Lemma 4.6 *If G is a graph with $\alpha(G) = 2$, $X \subset V(G)$ is a vertex cut, and $\mathrm{ecc}(G[X]) \leq |X| < \frac{n-1}{3}$, then $\mathrm{ecc}(G) \leq n$.*

Proof Let $|X| = k$. Since $\alpha(G) \leq 2$, $G - X$ must have exactly 2 connected components (call them A and B) and each much be a complete graph. Let $X = \{x_1 \ldots, x_k\}$, let $A_i = G[x_i \cup (N(x_i) \cap V(A))]$, and let $B_i = G[x_i \cup (N(x_i) \cap V(B))]$. The collection of A_i's and B_i's are complete graphs which cover all edges from X to $A \cup B$. By assumption, the edges of $G[X]$ can be covered by at most k complete subgraphs. Finally, we must cover A and B. Note that if x_1 is nonadjacent to a vertex in B, then it must be complete to A (and vice versa). Thus, either A or B (possibly both) has already been covered, and at most one more complete subgraph is needed. Thus $E(G)$ can be covered by $3k + 1 \leq n$ complete subgraphs.

Lemma 4.6 implies that any minimal counterexample to Conjecture 1 must have large connectivity. The following lemma implies that in any minimal counterexample, if it exists, any two non-adjacent vertices must have a large common neighbourhood, and the one following it shows that the connectivity bound can be improved if one considers only connected separators.

Lemma 4.7 *Let G be a graph with $\alpha(G) = 2$ and let $u, v \in V(G)$ be a pair of non-adjacent vertices. If $\mathrm{ecc}(G[N(u) \cap N(v)]) \leq |N(u) \cap N(v)|$ and $|N(u) \cap N(v)| \leq \frac{n-2}{5}$, then $\mathrm{ecc}(G) \leq n$.*

Proof For two non-adjacent vertices $u, v \in V$, let $N_u = N(u) \setminus N(v)$, $N_v = N(v) \setminus N(u)$ and $N_{uv} = N(u) \cap N(v)$. We denote the cardinalities of these sets by $n_u = |N_u|$, $n_v = |N_v|$, $n_{uv} = |N_{uv}|$. By Lemma 2.1, $G[N_u \cup \{u\}]$ and $G[N_v \cup \{v\}]$ are cliques in G. By assumption, n_{uv} cliques cover $G[N_{uv}]$. Consider the remaining edges to be coverered – (N_u, N_v), (N_u, N_{uv}), (N_v, N_{uv}), $(N_{uv}, \{u, v\})$ – where (A, B) denotes the set of edges between two sets $A, B \subset V(G)$. Let C_x^y denote the clique induced by $\{x, y\} \cup (N(x) \cap N_y)$, where $x \in N_{uv}$, $y \in \{u, v\}$. There are n_{uv} such cliques for each of $y = u$ and $y = v$, and the collection of these cliques cover the edges (N_u, N_{uv}), (N_v, N_{uv}), and $(N_{uv}, \{u, v\})$. To cover the remaining edges of (N_u, N_v), we choose the smaller of N_u and N_v (without loss of generality N_u), and for each $z \in N_u$ use the clique induced by $\{z\} \cup (N(z) \cap N_v)$. In total, the number of cliques used to cover $E(G)$ is $2 + n_{uv} + 2n_{uv} + \min\{n_u, n_v\} \leq 2 + \frac{n-2}{5} + \frac{2(n-2)}{5} + \frac{1}{2}\left(n - \frac{n-2}{5} - 2\right) \leq n$.

Lemma 4.8 *Let G be a graph with $\alpha(G) = 2$. If G contains a minimal vertex cut S such that $G[S]$ is disconnected, then $\mathrm{ecc}(G) \leq \frac{n}{2} + 4$.*

Proof Suppose a subset S of $V(G)$ separates G into two cliques and is disconnected. If S induces a disconnected graph, then this graph has exactly two connected components which must be cliques. Then G consists of four pairwise disjoint cliques, say S_0, S_1, S_2, S_3, with $S = S_0 \cup S_2$ and $V(G) \setminus S = S_1 \cup S_3$ and $(S_i, S_{i+2}) = \varnothing$ (addition modulo 4). Observe that $S_1 \cup S_3$ is also a separating set of G and one of the two separating sets has at most half of the vertices of G. Without loss of generality, assume it is S. Further, each of the cliques S_i is the union (as described above) of two sets of vertices, those adjacent to all of S_{i+1} and those adjacent to all of S_{i-1}. Let us denote these sets by S_i^+ and S_i^-. Clearly the sets $S_i \cup S_{i-1}^+$ and $S_i \cup S_{i+1}^-$ induce cliques in G and cover most of the edges of G, in particular all the edges of the cliques induced by the S_i. It remains to cover the edges between S_i and S_{i+1}^+ and between S_i and S_{i-1}^-. This is quite easy since each vertex of S_i only needs to cover the edges to at most one of S_{i+1}^+ and S_{i-1}^-. The cliques consisting each of a vertex $x \in S_j^* \setminus (S_j^+ \cap S_j^-)$ for $j = 0, 2$, and its neighbors in S_{j*1} ($* \in \{+, -\}$) cover the edges that need to be covered and there are at most $\frac{n}{2}$ of them. Thus, $\mathrm{ecc}(G) \leq \frac{n}{2} + 4$.

From Lemmas 4.1 to 4.8, we may deduce the following:

Theorem 4.9 *If G is a minimal counterexample to Conjecture 1, then*

1. $\Delta(G) < n - 1$;
2. $\delta(G) \geq \frac{n}{2} - 1$;
3. G contains no vertices x, y such that $N(x) \subseteq N(y)$ or such that $N[x] \subseteq N[y]$;
4. $\kappa(G) \geq \frac{n}{3}$ and any minimal vertex cut induces a connected subgraph of G;
5. $\omega(G) \leq \frac{n}{2}$;
6. for any pair u, v of nonadjacent vertices, $n_{uv} > \frac{n-2}{5}$;

Furthermore, since the proofs of Lemmas 4.1 to 4.8 are all inductive, Theorem 4.9 holds when restricted to any class $\mathcal{G}$ of graphs having some hereditary property $\mathcal{P}$ (in particular, for H-free graphs, which we consider in Section 5.3).

5 | COVERINGS IN PARTICULAR GRAPH CLASSES

This final section considers $\mathrm{ecc}(G)$ for particular classes of graphs when restricted to those satisfying $\alpha(G) = 2$.

5.1 | Bounded vertex cover number

A graph G is called *tame* if there exists a connected claw-free graph H with $\alpha(H) \geq 3$ such that G is an induced subgraph of H. In [4], it is shown that if G is a tame graph, then $\mathrm{ecc}(G) \leq n + 1$, and further that $\mathrm{ecc}(G) \leq n$ if G is not the union of 3 cliques. They note, specifically, that graphs with independence number 2 which are the union of 3 cliques are tame. Corollary 5.2 below provides a similar result for graphs which are the union of 3 cliques but which may have arbitrary independence number, with only a small additional cost in the number of cliques required to cover the edges. We begin by first proving a simple but more general lemma.

Lemma 5.1 *If $V(G)$ can be covered by k disjoint cliques of size $n_1 \geq n_2 \geq \ldots \geq n_k$, then $\mathrm{ecc}(G) \leq k + \sum_{i=1}^{k}(i-1)n_i$.*

Proof We proceed by induction on k. The statement is trivial when $k = 1$, so assume that $k > 1$ and that the statement holds for all positive integers less than k. Let Q_i be the clique of size n_i in the vertex cover given by the statement of the theorem, and let $G' = G - Q_k$. Since G' can be covered by $k - 1$ cliques, $\mathrm{ecc}(G') \leq k - 1 + \sum_{i=1}^{k-1}(i-1)n_i$. We extend this clique cover as follows. First, for each $q \in Q_k$ and $i \in \{1, \ldots, k-1\}$, add the clique $q \cup (N_G(q) \cap Q_i)$. This adds $(k-1)n_k$ cliques which cover all edges with exactly one end in Q_k. Adding the clique Q_k gives our desired clique cover of size $k + \sum_{i=1}^{k}(i-1)n_i$. ∎

In the special case when $k = 3$, we obtain the following corollary, again noting that in the special case where the graph is also claw-free, then a stronger upper bound of $n + 1$ is given in [4]. We make use of this result in Section 5.3 for a specific class of graphs whose vertices can be covered by three cliques.

Corollary 5.2 *If $\chi(\overline{G})$ can be properly 3-coloured with colour classes of size $p \geq q \geq r$, then $\mathrm{ecc}(G) \leq 2r + q + 3$ and, in particular $\mathrm{ecc}(G) \leq n + 3$.*

Note that a further corollary to Corollary 5.2 is that if G is a minimal counterexample to Conjecture 1, then either $\chi(\overline{G}) \geq 4$ or every proper 3-colouring of $\overline{G}$ has colour classes of roughly equal size.

In light of the bound given in Corollary 2.6, we also note the following can be proven by a similar argument:

Corollary 5.3 *If $\chi(\overline{G}) = 4$, then $\mathrm{ecc}(G) \leq \frac{3}{2}n + 4$.*

5.2 | Fixed diameter

We show that any future work can be restricted to those graphs having diameter exactly 2.

Proposition 5.4 *If G is a graph with $\alpha(G) \leq 2$, then G has diameter at most 3.*

Proof By contradiction, suppose that there is a vertex u and vertices v_i, $i = 1, 2, 3, 4$ such that v_i is at distance i from u. The independent set u, v_2, v_4 contradicts the assumption of $\alpha(G) = 2$. ∎

Theorem 5.5 *If G is a graph with $\alpha(G) = 2$ and $\mathrm{diam}(G) = 3$, then $\mathrm{ecc}(G) \leq \lceil \frac{n}{2} \rceil + 1$.*

Proof Let $u, v \in V(G)$ be at distance 3. Let $N_i(u)$ denote the set of vertices at distance i from u in G. By Lemma 2.1, the sets $N_i(u)$ induce complete graphs for $i = 1, 2, 3$, since there are no edges between $N_i(u)$ and $N_{i+2}(u)$ for $i = 0, 1$. Further, the set $N_2(u) \cup N_3(u)$ induces a complete graph since if $u_3 \in N_3(u)$ and $u_2 \in N_2(u)$ are not adjacent, the set $\{u, u_2, u_3\}$ is independent. Thus $N_2(u) \cup N_3(u)$ and $N[u]$ are two cliques which cover all edges of G except those having one end in $N_1(u)$ and the other in $N_2(u)$. Let A be the smaller of $N_1(u), N_2(u)$ and let B be the other. Clearly $|A| \leq \frac{n-2}{2}$ and the cliques C_x induced by $\{x\} \cup (N(x) \cap B)$ for $x \in A$ cover the remaining edges. Thus $E(G)$ is covered by at most $2 + \lceil \frac{n-2}{2} \rceil = \lceil \frac{n}{2} \rceil + 1$. Note that $\lceil \frac{n}{2} \rceil + 1 \leq n$ whenever $n \geq 2$, and that this is the case in a graph of diameter 3.

5.3 | Forbidden induced subgraphs

Recall that a graph G is called H-free if no induced subgraph of G is isomorphic to H. In this section, we give bounds on $\mathrm{ecc}(G)$ for a variety of H-free graphs, ultimately showing that Conjecture 1.1 holds when restricted to H-free graphs if H is any graph on four vertices. To this end, we remind the reader of the graphs on four vertices which do not contain three independent vertices, given in Figure 1.
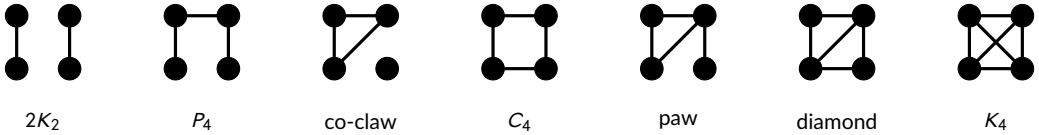

$2K_2$ P_4 co-claw C_4 paw   diamond K_4

FIGURE 1 4-vertex graphs with $\alpha \leq 2$

Proposition 5.6 *If G is $2K_2$-free and $\alpha(G) = 2$, then $\mathrm{ecc}(G) \leq n$.*

Proof Let x and y be non-adjacent vertices. Since $\alpha(G) = 2$, $\{x, y\}$ must dominate the graph. Let $X = N(x) \setminus N(y)$, $Y = N(y) \setminus N(x)$, and $Z = N(x) \cap N(y)$. By Theorem 4.9 and the observation at the end of Section 4 that the theorem applies to H-free graphs, we may assume that X and Y are non-empty. Since $\alpha(G) = 2$, each of X and Y is a clique; since G is $2K_2$-free, X is complete to Y. Thus $X \cup Y$ is a clique. Let n' denote the order of $G - \{x, y\} - X - Y$. By induction, $\mathrm{ecc}(G - \{x, y\} - X - Y) \leq n'$. The remaining edges may be covered by the cliques $x \cup X$, $y \cup Y$, and $X \cup Y$, and so the edges of G may be covered by at most $n' + 3 < n$ cliques.

Proposition 5.7 *If G is P_4-free and $\alpha(G) = 2$, then $\mathrm{ecc}(G) \leq n$.*

Proof It is well known (see, e.g., [10]) that any P_4-free graph contains adjacent twins (that is, two vertices with identical closed neighbourhoods), and so the bound follows immediately from Lemma 4.2.

Proposition 5.8 *If G is co-claw-free and $\alpha(G) = 2$, then $\mathrm{ecc}(G) \leq n$.*

Proof The assumptions on G imply that $\overline{G}$ is the disjoint union of isolated vertices, paths, and cycles of length at least 4. If $\overline{G}$ has an isolated vertex, then G has a universal vertex and the result follows from Lemma 4.1.1 by an inductive argument. If some connected component of $\overline{G}$ is a path, then let x be an end of the path and y a neighbour; in this case, $N_G(y) \subsetneq N_G(x)$, and the result follows from Proposition 4.3. The final case to consider is that $\overline{G}$ is a collection

of disjoint cycles. It follows that $\chi(\overline{G}) \leq 3$, and $\overline{G}$ can be 3-coloured with colour classes of size $p \geq q \geq r$, where r equals the number of odd cycles in $\overline{G}$. If $2r + q + 3 \leq n$, then $\mathrm{ecc}(G) \leq n$ by Lemma 5.2. Suppose that $2r + q + 3 > n$. Since $q \leq \frac{1}{2}(n - r)$, it follows that $3r + 6 > n$. Combining this bound with the fact that $n \geq 5r$, we have that $n < 15$ and so $r < 3$. If $r = 2$, then $\overline{G}$ must be the disjoint union of two 5-cycles. If $r = 1$, then $n \leq 8$ and $\overline{G}$ is a 5-cycle or 7-cycle, and if $r = 0$ then $\overline{G}$ is a 4-cycle. It is easy to check that each of these three graphs satisfy $\mathrm{ecc}(G) \leq n$, completing the proof.

Rather than considering the cases of C_4, the paw, and the diamond separately, we prove a stronger result on $\overline{P_2 \cup P_3}$-free graphs (where $\overline{P_2 \cup P_3}$ denotes the complement of the disjoint union of a 2-vertex path and a 3-vertex path), noting that $\overline{P_2 \cup P_3}$ contains each of C_4, the paw, and the diamond as an induced subgraph.
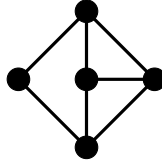

FIGURE 2 $\overline{P_2 \cup P_3}$

Theorem 5.9 *If G is $\overline{P_2 \cup P_3}$-free and $\alpha(G) = 2$, then $\mathrm{ecc}(G) \leq n$.*

Proof Let G be a minimal counterexample. As mentioned at the end of Section 4, Theorem 4.9 holds when restricted to H-free graphs; for the rest of this proof we let $H = \overline{P_2 \cup P_3}$. In particular, we may assume that G has no universal vertex, and that G contains no pair of vertices u, v such that $N(u) \subseteq N(v)$ or $N[u] \subseteq N[v]$. We may also assume, by Lemma 2.2, that G has a dominating edge xy. Let us denote by X, Y, Z the sets $X = N(x) \setminus N(y)$, $Y = N(y) \setminus N(x)$, and $Z = N(x) \cap N(y)$. Note that X and Y are cliques.

It is easy to see that because the graph is H-free, every vertex in X has at most one neighbour in Y and vice versa. Moreover if a vertex $u \in X$ has no neighbour in Y then $N[u] \subseteq N[x]$, a contradiction. Thus, X and Y have same order and the edges between X and Y form a perfect matching $x_1 y_1, x_2 y_2, \ldots, x_p y_p$.

Let z be a vertex in Z. It is easy to see that if z is not adjacent to y_i then it is not adjacent to x_i, since the graph is H-free. Furthermore, z must be adjacent to every other x_j and y_j since $\alpha(G) = 2$. This implies that Z can be partitioned $(Z_0, Z_1 \ldots Z_p)$, where Z_0 is the set of vertices in Z adjacent to every vertex in $X \cup Y$ and Z_i the set of vertices in Z adjacent to every vertex in $X \cup Y$ except x_i and y_i. Note, moreover, that for every $i \geq 1$, Z_i is a clique since $\alpha(G) = 2$ and x_i is a non neighbour of the vertices in Z_i. Figure 3 represents the graph at this point of the proof; we proceed by cases on possible values of p.

Case 1: $p \geq 4$

We claim that Z is a clique. Let i and j be distinct integers between 0 and p, and let r, s be distinct integers in $\{1, \ldots, p\} \setminus \{i, j\}$. The vertices $\{x, y_r, x_s\}$ induce a $\overline{P_3}$ which dominates $Z_i \cup Z_j$, and so $Z_i \cup Z_j$ is a clique, and thus so is Z. We may then cover the edges of G with all cliques of the form $\{x_i, y_i\} \cup Z \setminus Z_i$ together with the three cliques $\{x, y\} \cup Z$, $\{x\} \cup X$ and $\{y\} \cup Y$.

Case 2: $p = 3$

Let i be an integer between 1 and p, and let r, s be distinct integers in $\{1, \ldots, p\} \setminus \{i\}$. The vertices $\{x, y_r, x_s\}$ induce
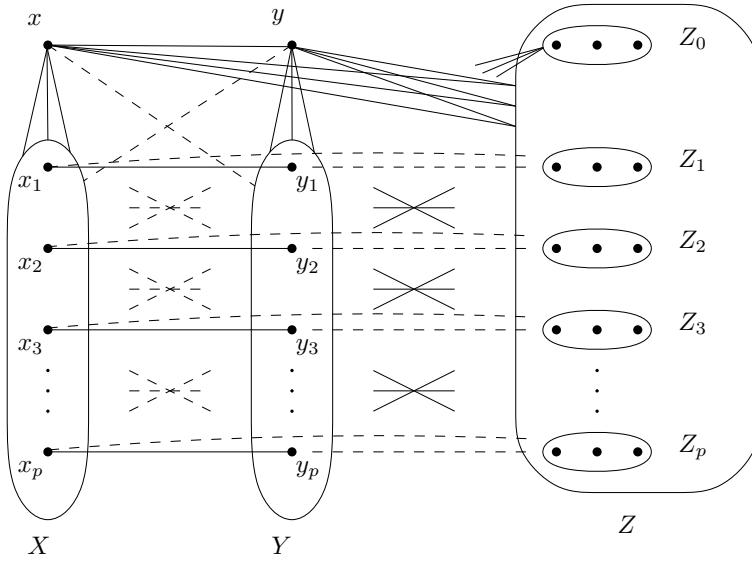
FIGURE 3 The structure of a minimal counterexample to Theorem 5.9.

a $\overline{P_3}$ which dominates $Z_i \cup Z_0$, and so (as in Case 1), $Z_i \cup Z_0$ is a clique. In particular, if Z_0 is non-empty, then each element of Z_0 is a universal vertex in G, a contradiction. Thus, we may assume that $Z_0 = \varnothing$.

Let $t = n - 8$ be the number of vertices in Z. If $G[Z]$ has no isolated vertex, then it is possible to cover all the edges of $G[Z]$ with t cliques by induction. We add the edge xy to every clique to get a family of t cliques that cover all edges within the graph induced by $Z \cup \{x, y\}$. On the other hand, if Z has an isolated vertex u, then it has exactly one since $\alpha(G) = 2$. We can then cover the edges of $G[Z]$ by $t - 1$ cliques by induction, add the edge xy to each of those cliques, and add the triangle xyu to get a family of t cliques that cover all edges within the graph induced by $Z \cup \{x, y\}$. We finally add the two cliques $\{x\} \cup X$ and $\{y\} \cup Y$ and the following six cliques to obtain our desired edge cover: $\{x_1, y_1\} \cup Z_2$, $\{x_1, y_1\} \cup Z_3$, $\{x_2, y_2\} \cup Z_1$, $\{x_2, y_2\} \cup Z_3$, $\{x_3, y_3\} \cup Z_1$, $\{x_3, y_3\} \cup Z_2$.

Case 3: $p = 2$

By a similar argument as in Case 2, we use the $\overline{P_3}$ induced by $\{x, y_1, x_2\}$ to deduce that Z_0 is a clique. Also by a similar argument to that in Case 2, it is possible to cover the edges of $G[Z \cup \{x, y\}]$ with $t = |Z| = n - 6$ cliques. We add the two cliques $\{x\} \cup X$ and $\{y\} \cup Y$ and the following four cliques to obtain our desired edge cover: $\{x_1, y_1\} \cup Z_0$, $\{x_1, y_1\} \cup Z_2$, $\{x_2, y_2\} \cup Z_0$, $\{x_2, y_2\} \cup Z_1$.

Case 4: $p = 1$

Again, we apply an inductive argument to get a family of cliques of size $|Z| = n - 4$ which covers all edges of $G[Z \cup \{x, y\}]$. Now, since x and y_1 are non-adjacent vertices which are complete to Z_0, there can be no P_3 in the complement of $G[Z_0]$ as G is H-free. In other words, Z_0 must be a clique minus a matching. In particular, Z_0 is the union of two cliques A and B. Now we add the cliques $A \cup \{xx_1\}$, $A \cup \{yy_1\}$, $B \cup \{x_1y_1\}$ $B \cup \{xy\}$ to obtain the desired edge cover.

Case 5: $p = 0$

In this case, x and y are universal vertices, a contradiction.

Finally, we note that any K_4-free graph with $\alpha(G) = 2$ has at most 8 vertices, as $R(3, 4) = 9$. It is easily checked that these graphs, presented as their graph6 strings in Figure 4 (see [11] for their complements), satisfy Conjecture 1.

@	BW	CL	DJk	DL{	Dbk	EJqw	ELrw	E`NG	FJvdw	F`v`w	FkYXw
A?	Bw	CN	DK[	DNw	EJaG	EKlw	ELv_	E`NW	FL]uW	Fb]lg	G}UczW
A_	CJ	C]	DK{	D]{	EJeg	EK~o	EL~o	E`]o	FLr~o	Fbg}w	G]^VLo
BG	CK	C^	DLo	D`K	EJnW	ELpw	E]~o	Eklw	FLvvO	FjaHw	GJnTUK

FIGURE 4 The graph6 strings of those graphs satisfying $\omega \leq 3$ and $\alpha \leq 2$.

The results of this section imply the following:

Theorem 5.10 *Let G be a graph with $\alpha(G) = 2$, and let H be any graph on four vertices. If G is H-free, then $\mathrm{ecc}(G) \leq n$.*

5.4 | Graph joins

Let G and G' be two graphs. the *complete join* of G and G', denoted $G \vee G'$ is the graph obtained by taking the disjoint union of G and G' and adding all edges in between the two.

It is clear that $\alpha(G \vee G') = \max(\alpha(G), \alpha(G'))$ and in particular $\alpha(G \vee G') = 2$ if $\alpha(G) = \alpha(G') = 2$. Therefore, in view of the conjectures examined here, one natural question is to ask whether $\mathrm{ecc}(G \vee G') \leq \mathrm{ecc}(G) + \mathrm{ecc}(G')$.

The following theorem answers when G' is a particular graph. If H is a spanning subgraph of H', then we say that H' is a *spanning supergraph* of H.

Theorem 5.11 *Let G be a graph such that $\alpha(G) = 2$. If H is an induced subgraph of G and H' is any spanning supergraph of H, then $\mathrm{ecc}(G \vee H') \leq \mathrm{ecc}(G) + |V(H')|$*

Proof Let $x_1, x_2, \ldots x_n$ be the vertices of G such that $x_1, .., x_k$ are the vertices of H in G. Let $x'_1, \ldots, x'_k$ the vertices of H' labeled such that $x_i x_j \in E(H) \rightarrow x'_i x'_j \in E(H')$. Let $K_1, \ldots K_p$ be an optimal edge covering set of cliques for G. For $i = 1 \ldots p$, consider the following set of vertices in $G \vee H'$:

$$K'_i = K_i \cup \{x'_j \text{ such that } x_j \in K_i\}.$$

Similarly for $i = 1 \ldots k$ define

$$K''_i = \{x'_i\} \cup (G \setminus N[x_i]) \cup \{x'_j \text{ such that } x'_i x'_j \in E(H') \text{ and } x'_i x'_j \notin E(H)\}$$

We show that K''_i is a clique. By Lemma 2.1, $(G \setminus N[x_i])$ induces a clique. Also, $\{x'_i\} \cup \{x'_j \text{ such that } x'_i x'_j \in E(H') \text{ and } x'_i x'_j \notin E(H)\}$ induces a clique, because if for some j'_1 and j'_2, $x'_{j'_1}$ and $x'_{j'_2}$ are non-neighbors, then $x_i, x_{j'_1}$ and $x_{j'_2}$ form an independent set in H, a contradiction. Hence, K''_i is a clique. This gives $(p + k)$ cliques which cover all edges of $G \vee H'$.

6 | CONCLUSION

In this work we studied a conjecture that, if $\alpha(G) = 2$, then the edge clique cover number of G is at most $|V(G)| = n$. We have provided the first known linear bound of $2n - \theta(\sqrt{n \log n})$, and showed that under additional structural conditions the conjecture holds. However, a deeper look into our proof of Theorem 2.4 suggests that perhaps the upper bound provided by this theorem is nearly tight. The linear bound given in Corollary 2.6 relies on the lower bound for the Ramsey number $R(3, t)$, which is known to be tight (ignoring smaller order terms). By taking the complement of a graph which attains this bound, we have a graph G with $\alpha(G) = 2$ and lowest possible clique number. By adding a linear number of dominating edges, one may hope to construct an example where the proof technique of Theorem 2.4 is the optimal way of finding an edge clique cover. Currently, the only known technique for the construction of such ingredient graphs is the random triangle-free process given by Kim [8]. The exploration of the details of this technique and its applications to Conjecture 1 is left as an open direction for future research.

7 | ACKNOWLEDGEMENTS

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