The optimal routing of augmented cubes.☆

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Abstract

A routing in a graph G is a set of paths connecting each ordered pair of vertices. Load of an edge e is the number of times it appears on these paths. The edge-forwarding index of G is the smallest of maximum loads over all routings. Augmented cube of dimension n, AQ_n, is the Cayley graph $(\mathbb{Z}_2^n, \{e_1, e_2, \dots, e_n, J_2, \dots, J_n\})$ where e_i's are the vectors of the standard basis and $J_i = \sum_{j=n-i+1}^{n} e_j$. S.A. Choudum and V. Sunitha showed that the greedy algorithm provides a shortest path between each pair of vertices of AQ_n. Min Xu and Jun-Ming Xu claimed that this routing also proves that the edge-forwarding index of AQ_n is 2^{n-1}. Here we disprove this claim, by showing that in this specific routing some edges are repeated nearly $\frac{4}{3}2^{n-1}$ times (to be precise, $\lfloor \frac{2^{n+1}}{3} \rfloor$ for even values of n and $\lceil \frac{2^{n+1}}{3} \rceil$ for odd values of n). However, by providing other routings, we prove that 2^{n-1} is indeed the edge-forwarding index of AQ_n.

Key words: Augmented cubes; Edge forwarding index; HMS-routing; Optimal routing; Interconnection networks

1. Introduction

Heydemann et al. [4] introduced the notation of the edge-forwarding index. Given a connected graph G of order n, a *routing* R is a set of $n(n-1)$ elementary paths $R(u, v)$ specified for every ordered pair (u, v) of vertices of G. The *load* $\pi(G, R, e)$ of an edge e with respect to R is defined as the number of paths of R going through e. The edge-forwarding index of G with respect to R, denoted $\pi(G, R)$, is the maximum load of all edges of G. The minimum edge-forwarding index over all possible routings is denoted by $\pi(G)$ and is called the *edge-forwarding index* of G. A routing for which $\pi(G)$ is attained is called *optimal*. If each path in R is a shortest path connecting its two ends, then the routing R is said to be a *minimal routing*.

In [1], Choudum and Sunitha introduced the augmented cubes and studied them for application in routing and broadcasting procedures. They provided several equivalent definitions of these graphs. Here we present two definitions, an inductive definition and a definition as a Cayley graph. Let $\mathbb{Z}_2^n$ be the n-dimensional binary group and let $x \oplus_2 y$

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denote the binary sum of vectors x and y in $\mathbb{Z}_2^n$. The inductive definition of the augmented cubes is as follows.

Definition 1. *The augmented cubes of dimensions 1 and 2 are simply the complete graphs on two and four vertices, respectively. For $n \geq 2$ the augmented cube of dimension n, denoted AQ_n, is a graph on $\mathbb{Z}_2^n$ built from two copies AQ_{n-1}^0 and AQ_{n-1}^1 of AQ_{n-1} as follows: vertices of AQ_{n-1}^0 (respectively AQ_{n-1}^1) are viewed as elements of $\mathbb{Z}_2^n$ by adding a 0 (a 1) as the first coordinate. A vertex $0x$ in AQ_{n-1}^0 is adjacent, furthermore, to vertices $1x$ and $(0x \oplus_2 J)$ in AQ_{n-1}^1 where J is the all-1 vector in $\mathbb{Z}_2^n$.*

Let Γ be an additive group, and let S be a symmetric subset of Γ (i.e., $-S = S$) such that 0 is not in S. The *Cayley graph* (Γ, S) is the graph whose vertices are elements of Γ where two vertices x and y are adjacent if $x - y \in S$.

When the binary group $\mathbb{Z}_2^n$ is considered, then for any subset X we have $X = -X$. Let $e_1^n, e_2^n, \ldots, e_n^n$ be the vectors of the standard basis of $\mathbb{Z}_2^n$, thus e_i^n is the binary vector of length n whose i^{th}-coordinate is 1 and all other coordinates are 0. For $i \geq 2$, let J_i^n be the binary vector of length n where the last i coordinates are 1 and the first $n-i$ coordinates are 0, i.e., $J_i^n = \sum\limits_{j=n-i+1}^{n} e_j^n$. Then the augmented cube of dimension n defined above is known to be isomorphic to the following Cayley graph.

Proposition 2. *[1] For every $n \geq 1$, AQ_n is isomorphic to the Cayley graph $(\mathbb{Z}_2^n, S_n)$, where $S_n = \{e_1^n, e_2^n, \ldots, e_n^n, J_2^n, \ldots, J_n^n\}$.*

When it is clear from the context, we write $e_1, e_2, \ldots, e_n$ and $J_2, \ldots, J_n$ in place of $e_1^n, e_2^n, \ldots, e_n^n$ and $J_2^n, \ldots, J_n^n$.

Based on Cayley graph presentation of AQ_n a minimal routing R_n of AQ_n, originally proposed in [1], is as follows: Given vertices X and Y to find a shortest path from X to Y we find the first coordinate (i) at which X and Y differ. If X and Y also differ in the following coordinate ($i+1$), then we define $X_1 = X + J_{n-(i-1)}$ (i.e., we change the values at coordinates i and after), otherwise we define $X_1 = X + e_i$ (i.e., we change the values only at coordinate i). Continuing this process on the newly obtained vertex, we find a shortest path to Y.

In [5], the authors studied the edge-forwarding index of the augmented cubes. Among other results, they claimed that $\pi(AQ_n) = 2^{n-1}$ and that this value is obtained by the minimal routing R_n. Here we show that the latter claim is not correct, we show that the edge-forwarding index of AQ_n with respect to R_n is nearly $\frac{4}{3}2^{n-1}$ (to be precise, $\lfloor \frac{2^{n+1}}{3} \rfloor$ for even values of n and $\lceil \frac{2^{n+1}}{3} \rceil$ for odd values of n). We then introduce a new optimal routing that prove the claim $\pi(AQ_n) = 2^{n-1}$.

To present our work, we first present in Section 2 the notion of an HMS-routing which was defined by Gauyacq [3]. We show that the minimal routing R_n [1] is an HMS-routing. But its edge-forwarding index is $\lfloor \frac{2^{n+1}}{3} \rfloor$ for even values of n and $\lceil \frac{2^{n+1}}{3} \rceil$ for odd values of n. For $n = 3$, we give a routing of AQ_3 whose edge-forwarding index is 4. However we prove that any HMS-routing of AQ_3 has an edge of load 6, i.e., the edge-forwarding index of any HMS-routing of AQ_3 is 6. For $n \geq 4$, we give an HMS-routing of AQ_n whose edge-forwarding index is 2^{n-1}.

2. HMS-routings in Cayley graphs

We would like to recall the HMS-routing defined in [3]. Let Γ be an additive group which is commutative. Let S be a symmetric subset of Γ and (Γ, S) be the corresponding Cayley graph. Let 0 denote the identity element of Γ. For each γ in Γ, the permutation ϕ_γ of Γ defined by $\phi_\gamma(h) = \gamma + h$ is an automorphism of (Γ, S) (i.e. a bijection that preserves adjacency). Furthermore, observe that if P is a path connecting vertices x and y, then the image of P under ϕ_γ is a path connecting $\phi_\gamma(x)$ and $\phi_\gamma(y)$.

Given a Cayley graph $G = (\Gamma, S)$, an HMS-routing R is a routing satisfying the following. For every vertex $v \neq 0$ in Γ, the route $R(0, v)$ is any shortest path from 0 to v. For an arbitrary pair of vertices x and y in Γ the route from x to y is defined by $R(x, y) = \phi_x(R(0, y - x))$.

We denote $00 \cdots 0$ by 0^n. Next, we want to prove that the minimal routing R_n defined in [1] is an HMS-routing.

Observation 3. *R_n is an HMS-routing of AQ_n for every $n \geq 1$.*

Proof. We know from [1] that R_n provides a shortest path from 0 to x for any $x \in \mathbb{Z}_2^n$. It remains to show that $R_n(x, y) = \phi_x(R_n(0, y - x))$. But this follows from the fact that vectors a and b of $\mathbb{Z}_2^n$ differs in the same coordinates as the vectors $\phi_\gamma(a)$ and $\phi_\gamma(b)$. It would then be enough to take $a = x$, $b = y$ and $\gamma = x$ (noting that $x = -x$ in $\mathbb{Z}_2^n$). $\square$

We recall some notations from [3]. Let (Γ, S) be a Cayley graph. If $v = u + s$ with $s \in S$, then assign the *type* s to the ordered pair (u, v), the type $-s$ to the ordered pair (v, u) and say that the edge $\{u, v\}$ is of type s or of type $-s$. A path $P = (u_0, u_1, \ldots, u_k)$ in (Γ, S) is uniquely determined by its initial vertex u_0 and the sequence $(s_1, s_2, \ldots, s_k)$ of the types of pairs of adjacent vertices. In other words, for $1 \leq i \leq k$, $u_i = u_{i-1} + s_i$. Denote by $t_s(P)$ the number of times the generator s occurs in the sequence $(s_1, s_2, \ldots, s_k)$. The following theorem was observed by Gauyacq [3].

Proposition 4. *[3] Let R be an HMS-routing in a Cayley graph (Γ, S). Let 0 be the identity and e be an edge of type s in (Γ, S). The load of e for the routing R is*

$$\pi(e) = \sum_{v \in \Gamma, v \neq 0} t_s(R(0, v)) + \sum_{v \in \Gamma, v \neq 0} t_{-s}(R(0, v)).$$

In the augmented cubes AQ_n, the identity is 0^n and $s = -s$ for any $s \in S$. We get the following corollary immediately.

Corollary 5. *Let R be an HMS-routing in AQ_n. The load of an edge e of type s for the routing R is*

$$\pi(e) = 2 \sum_{X \in V(AQ_n), X \neq 0^n} t_s(R(0^n, X)).$$

The corollary shows that, for an HMS-routing, the load of an edge depends only on its type. In a Cayley graph the problem of finding a shortest path from 0 to a vertex v is equivalent to finding a minimum length generating sequence for the element v. The problem of finding a minimum length generating sequence is known to be NP-hard [2]. More precisely, in [2] it is shown that given a set S of generators of the permutation group

S_n, a permutation P and an integer k, to determine if P is generated using at most k elements of S is NP-hard. Note that this is about finding a shortest path between two vertices of a Cayley graph, but the complexity of the question is considered based on the dimension of Γ rather than the number of elements of it. For the subject of our work, i.e., augmented cubes, it is shown in [1] that the greedy routing finds a shortest path between given two vertices in time polynomial of the dimension of the augmented cube.

Since R_n is an HMS-routing, we will determine the edge-forwarding index of R_n by calculating the number of times each generator appears in the set $\bigcup_X R(0^n, X)$, where X can be any nonzero vertex of AQ_n. In the following, we will denote the number $\sum_{X \in V(AQ_n)} t_s(R(0^n, X))$ by $|s|$ for short.

Theorem 6. *For any positive integer n, we have $\pi(AQ_n, R_n) = \frac{2^{n+1}}{3} - \frac{(-1)^n \times 2}{3}$.*

Proof. Since AQ_1, AQ_2 are both complete graphs, $\pi(AQ_1, R_1) = \pi(AQ_2, R_2) = 2$ and the formula holds.

If $n = 3$, by the definition of R_n, $R(000, 001) = (e_3^3), R(000, 010) = (e_2^3), R(000, 011) = (J_2^3), R(000, 100) = (e_1^3), R(000, 111) = (J_3^3), R(000, 101) = (e_1^3, e_3^3), R(000, 110) = (J_3^3, e_3^3)$. So $|e_1^3| = 2$, $|e_2^3| = 1$, $|e_3^3| = 3$, $|J_2^3| = 1$, $|J_3^3| = 2$. By the definition of the edge-forwarding index of a routing and by Corollary 5, we get $\pi(AQ_3, R_3) = 2|e_3^3| = 6 = \frac{2^4}{3} - \frac{(-1)^3 \times 2}{3}$. For example, the edge $(000, 001)$ is used in 6 routes of R_3 as depicted in Figure 1.

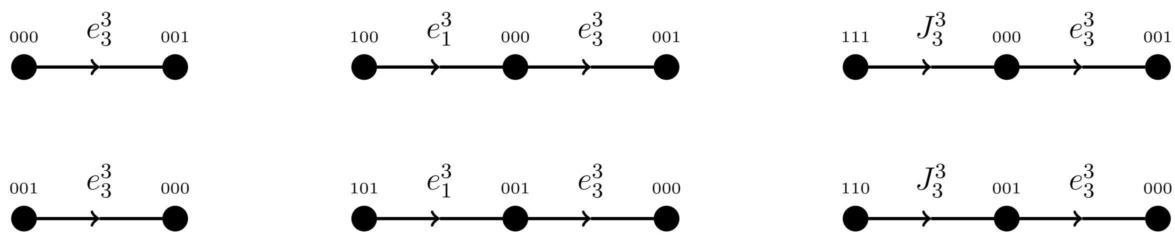

Figure 1: The 6 routes of R_3 using the edge (000,001).

If $n = 4$, by the definition of R_n, we have $R(0000, 0001) = (e_4^4)$, $R(0000, 0010) = (e_3^4)$, $R(0000, 0011) = (J_2^4)$, $R(0000, 0100) = (e_2^4)$, $R(0000, 0111) = (J_3^4)$, $R(0000, 1000) = (e_1^4)$, $R(0000, 1111) = (J_4^4)$, $R(0000, 0101) = (e_2^4, e_4^4)$, $R(0000, 0110) = (J_3^4, e_4^4)$, $R(0000, 1001) = (e_1^4, e_4^4)$, $R(0000, 1010) = (e_1^4, e_3^4)$, $R(0000, 1100) = (J_4^4, J_2^4)$, $R(0000, 1011) = (e_1^4, J_2^4)$, $R(0000, 1101) = (J_4^4, e_3^4)$, $R(0000, 1110) = (J_4^4, e_4^4)$. So $|e_1^4| = 4$, $|e_2^4| = 2$, $|e_3^4| = 3$, $|e_4^4| = 5$, $|J_2^4| = 3$, $|J_3^4| = 2$, $|J_4^4| = 4$. By the definition of the edge-forwarding index of a routing and by Corollary 5, we get $\pi(AQ_4, R_4) = 2|e_4^4| = 10 = \frac{2^5}{3} - \frac{(-1)^4 \times 2}{3}$.

In general, for $n \geq 5$ we show that in the routing R_n, the type e_n^n appears $\frac{2^n}{3} - \frac{(-1)^n}{3}$ times. This completes our proof by considering Corollary 5. To this end, we first describe the routing from 0 to X given by R_n based on the first two coordinates of X. For any $X \in V(AQ_n)$, either $X \in V(AQ_{n-1}^0)$ or $X \in V(AQ_{n-1}^1)$.

Case 1. $X \in V(AQ_{n-1}^0)$, then $X = 0x_2x_3 \cdots x_n$, where $x_2x_3 \cdots x_n \in V(AQ_{n-1})$. Suppose that $R(0^{n-1}, x_2x_3 \cdots x_n) = (s_{i_1}, s_{i_2}, \ldots, s_{i_j})$, then by the definition of R_n, we get $R(0^n, X) = (0s_{i_1}, 0s_{i_2}, \ldots, 0s_{i_j})$. (For any $p \in \{1, 2, \ldots, j\}$, if $s_{i_p} = e_k^{n-1}$ then $0s_{i_p} = e_{k+1}^n$; if $s_{i_p} = J_k^{n-1}$ then $0s_{i_p} = J_k^n$.)

Case 2. $X \in V(AQ^1_{n-1})$, there are two subcases.

Case 2.1. $X \in V(AQ^{10}_{n-2})$, then $X = 10x_3 \cdots x_n$, where $x_3 \cdots x_n \in V(AQ_{n-2})$. Suppose that $R(0^{n-2}, x_3 \cdots x_n) = (s_{i_1}, s_{i_2}, \dots, s_{i_l})$, then by the definition of R_n, we get $R(0^n, X) = (e^n_1, 00s_{i_1}, 00s_{i_2}, \dots, 00s_{i_l})$. (For any $q \in \{1, 2, \dots, l\}$, if $s_{i_q} = e^{n-2}_k$ then $00s_{i_q} = e^n_{k+2}$; if $s_{i_q} = J^{n-2}_k$ then $00s_{i_q} = J^n_k$.)

Case 2.2. $X \in V(AQ^{11}_{n-2})$, then $X = 11x_3 \cdots x_n$, where $x_3 \cdots x_n \in V(AQ_{n-2})$. Then we know $\bar{x}_3 \cdots \bar{x}_n \in V(AQ_{n-2})$. Suppose that $R(0^{n-2}, \bar{x}_3 \cdots \bar{x}_n) = (s_{i_1}, s_{i_2}, \dots, s_{i_l})$, then by the definition of R_n, we get $R(0^n, X) = (J^n_n, 00s_{i_1}, 00s_{i_2}, \dots, 00s_{i_l})$.

We know there are 2^{n-2} vertices in AQ_{n-2}, so $|e^n_1| = 2^{n-2}$. Similarly, $|J^n_n| = 2^{n-2}$. Since e^n_2 only appears in the set $\bigcup\limits_X R(0^n, X)$, where $X \in V(AQ^0_{n-1})$, we obtain $|e^n_2| = |e^{n-1}_1| = 2^{n-3}$. Similarly, $|J^n_{n-1}| = 2^{n-3}$.

For $3 \le i \le n$, $e^n_i = 0e^{n-1}_{i-1}$ and $e^n_i = 00e^{n-2}_{i-2}$, so $|e^n_i| = |e^{n-1}_{i-1}| + 2 \times |e^{n-2}_{i-2}|$. Similarly, for $2 \le i \le n-2$, we get $|J^n_i| = |J^{n-1}_i| + 2 \times |J^{n-2}_i|$.

Claim. *For any $i \in \{1, \dots, n-1\}$ and $l \in \{2, \dots, n\}$, we have $|e^n_i| \le 2^{n-2}$, $|J^n_l| \le 2^{n-2}$. But $|e^n_n| > 2^{n-2}$.*

We prove the claim by induction. We already saw that for $n = 3, 4$, the claim holds. Suppose the claim holds for $n = k$, where $k \ge 5$. We want to prove that it also holds for $n = k + 1$.

Clearly, both $|e^{k+1}_1|$ and $|e^{k+1}_2|$ satisfy the inequality. For $i \in \{3, \dots, k\}$, by the induction hypothesis, $|e^{k+1}_i| = |e^k_{i-1}| + 2 \times |e^{k-1}_{i-2}| \le 2^{k-2} + 2 \times 2^{k-3} = 2^{k-1}$. The proof for $|J^{k+1}_l|$ is similar, where $l \in \{2, \dots, k+1\}$. By the induction hypothesis, $|e^{k+1}_{k+1}| = |e^k_k| + 2|e^{k-1}_{k-1}| > 2^{k-2} + 2 \times 2^{k-3} = 2^{k-1}$.

For $|e^n_n|$, as $|e^3_3| = 3$, $|e^4_4| = 5$ and considering $|e^n_n| = |e^{n-1}_{n-1}| + 2 \times |e^{n-2}_{n-2}|$ $(n \ge 5)$, we get $|e^n_n| = \frac{2^n}{3} - \frac{(-1)^n}{3}$. By the definition of the edge-forwarding index of a routing and by the claim and Corollary 5, we conclude that $\pi(AQ_n, R_n) = 2|e^n_n| = \frac{2^{n+1}}{3} - \frac{(-1)^n \times 2}{3} > 2^{n-1}$. □

3. The optimal routing in augmented cubes

In this section, we will provide a minimal and optimal routing of AQ_n. First we prove that no HMS-routing of AQ_3 is optimal.

Proposition 7. *Given an HMS-routing R of AQ_3, we have $\pi(AQ_3, R) = 6$.*

Proof. Denote the minimal length generating sequence of X by $D\{X\}$. We present the vertex X in AQ_3 by the minimal length generating sequence: $001 = D\{e^3_3\}, 010 = D\{e^3_2\}, 011 = D\{J^3_2\}, 100 = D\{e^3_1\}, 111 = D\{J^3_3\}, 101 = D\{e^3_1, e^3_3\} = D\{J^3_3, e^3_2\}, 110 = D\{e^3_1, e^3_2\} = D\{J^3_3, e^3_3\}$. Each generator appears once in the set $\bigcup_{X_1} R(000, X_1)$, where X_1 is the vertex with distance one to 000 in AQ_3. But at least one generator from the set $\{e^3_1, e^3_2, e^3_3, J^3_3\}$ appears twice in the set $R(000, 101) \cup R(000, 110)$. Thus, by Corollary 5, the edge-forwarding index of any HMS-routing of AQ_3 is 6. □

Next, we will provide a minimal routing for AQ_3 whose edge-forwarding index attains the lower bound of $\pi(AQ_3)$. But it is not an HMS-routing. However for $n \ge 4$, the optimal routing that we provide for AQ_n is also an HMS-routing.

Theorem 8. *For $n \ge 2$, there is a minimal routing $\mathcal{R}_n$ of AQ_n with $\pi(AQ_n, \mathcal{R}_n) = 2^{n-1}$. Furthermore, for $n \ge 4$ such a routing can be chosen to be also an HMS-routing.*

Proof. For $n = 2$, since AQ_2 is the complete graph on four vertices, the unique minimal routing uses each edge twice.

For $n = 3$ we know, by Proposition 7, that we cannot have an HMS routing with edge-forwarding index equal to four. Thus we provide a minimal but not an HMS-routing for AQ_3 as follows: for any two adjacent vertices X, Y in AQ_3, let $R(X, Y) = (X, Y)$. Each edge has thus already been used twice. It remains to provide a routing for pairs of non adjacent vertices using each edge at most twice. Observe that the set of non adjacent pairs are the edges of the complement of AQ_3. This graph is disjoint union of two cycles of length four. After choosing an orientation on these four cycles, we provide a routing for this directed pairs which uses each edge of AQ_3 at most once. Then given a routing for the directed edge XY, we consider the reverse route for the directed pair YX. Then in this routing of non adjacent pairs each edge is either used twice or not used. In total then each edge is used either two times or four times.

A choice of direction for non edges of AQ_3 together with routing for these these pairs is giving in Figure 2. In this figure, a gray directed edge shows a non edge of AQ_3. A black path of similar pattern then shows a routing for each of these directed non edges. One can check that each edge of AQ_3 connecting a vertex of one 4-cycle to a vertex of the other 4-cycle is used exactly once and diagonal edges of the 4-cycles are not used. For clarification the routings of this Figure are described below.
$R(000, 101) = (000, 001, 101)$; $R(101, 011) = (101, 111, 011)$;
$R(011, 110) = (011, 010, 110)$; $R(110, 000) = (110, 100, 000)$;
$R(100, 001) = (100, 011, 001)$; $R(001, 111) = (001, 110, 111)$;
$R(111, 010) = (111, 000, 010)$; $R(010, 100) = (010, 101, 100)$;

We note that there are more than one routing of AQ_3 with edge-forwarding index 4, we only provided an example.

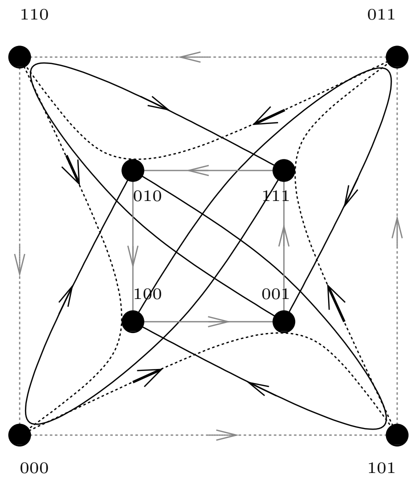

Figure 2: Routing of a direction of non edges of AQ_3

If $n \geq 4$, then we will prove the theorem by the following method: first we find a minimum length generating sequence for each element $X \in AQ_n, X \neq 0^n$ such that each generator appears at most 2^{n-2} times in the generating sequences. Then given generating sequence $D(X) = \{s_{j_1}, s_{j_2}, \ldots, s_{j_l}\}$, we define $R(0^n, X) = (s_{j_1}, s_{j_2}, \ldots, s_{j_l})$. Finally, if we define $\mathcal{R}_n$ to be the corresponding HMS-routing, then by Corollary 5, $\pi(AQ_n, \mathcal{R}_n) \leq 2^{n-1}$.

For $n = 4$ the minimum length generating sequence of the vertices of AQ_4 are as follows: $D(0001) = \{e_4^4\}$, $D(0010) = \{e_3^4\}$, $D(0011) = \{J_2^4\}$, $D(0100) = \{e_2^4\}$, $D(0111) = \{J_3^4\}$, $D(1000) = \{e_1^4\}$, $D(1111) = \{J_4^4\}$, $D(0101) = \{e_2^4,\ e_4^4\}$, $D(0110) = \{e_2^4, e_3^4\}$, $D(1001) = \{e_1^4, e_4^4\}$, $D(1010) = \{e_1^4, e_3^4\}$, $D(1100) = \{J_4^4, J_2^4\}$, $D(1011) = \{e_1^4, J_2^4\}$, $D(1101) = \{J_4^4, e_3^4\}$, $D(1110) = \{J_4^4, e_4^4\}$. Some of the routings are depicted in Figure 3. We can check that each generator appears at most four times in the generating sequences of vertices of AQ_4.

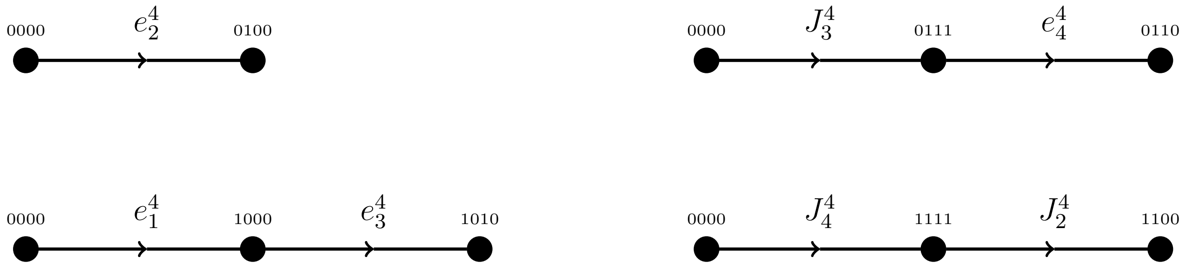

Figure 3: Some routings of AQ_4 starting from 0000.

For $n = 5$, the minimum length generating sequence of the vertices of AQ_5 are as follows: $D(00001) = \{e_5^5\}$, $D(00010) = \{e_4^5\}$, $D(00011) = \{J_2^5\}$, $D(00100) = \{e_3^5\}$, $D(00111) = \{J_3^5\}$, $D(01000) = \{e_2^5\}$, $D(01111) = \{J_4^5\}$, $D(00101) = \{e_3^5, e_5^5\}$, $D(00110) = \{e_3^5, e_4^5\}$, $D(01001) = \{e_2^5, e_5^5\}$, $D(01010) = \{e_2^5, e_4^5\}$, $D(01100) = \{J_4^5, J_2^5\}$, $D(01011) = \{e_2^5, J_2^5\}$, $D(01101) = \{J_4^5, e_4^5\}$, $D(01110) = \{J_4^5, e_5^5\}$, $D(10000) = \{e_1^5\}$, $D(10001) = \{e_1^5, e_5^5\}$, $D(10010) = \{e_1^5, e_4^5\}$, $D(10011) = \{e_1^5, J_2^5\}$,$D(10100) = \{e_1^5, e_3^5\}$, $D(10111) = \{e_1^5, J_3^5\}$, $D(11000) = \{J_5^5, J_3^5\}$, $D(11111) = \{J_5^5\}$, $D(10101) = \{e_1^5, e_3^5, e_5^5\}$, $D(10110) = \{e_1^5, e_3^5, e_4^5\}$, $D(11001) = \{J_5^5, J_3^5, e_5^5\}$, $D(11010) = \{J_5^5, J_3^5, e_4^5\}$, $D(11100) = \{J_5^5, J_2^5\}$, $D(11011) = \{J_5^5, e_3^5\}$, $D(11101) = \{J_5^5, e_4^5\}$, $D(11110) = \{J_5^5, e_5^5\}$. Again it can be readily checked that each generator appears at most 8 times.

For $n \geq 6$, we recursively build minimum length generating sequence for the vertices of AQ_n based on the minimum length generating sequence of vertices of AQ_{n-1} and AQ_{n-2}. Let $X \in V(AQ_n)$, either $X \in V(AQ_{n-1}^0)$ or $X \in V(AQ_{n-1}^1)$. If $X \in V(AQ_{n-1}^0)$, then $X = 0x_2x_3 \cdots x_n$, where $x_2x_3 \cdots x_n \in V(AQ_{n-1})$. Assume that $D(x_2x_3 \cdots x_n) = (s_{i_1}, s_{i_2}, \ldots, s_{i_j})$, then let $D(X) = (0s_{i_1}, 0s_{i_2}, \ldots, 0s_{i_j})$. When $X \in V(AQ_{n-1}^1)$, we consider two subcases. Either $X = 10x_3 \cdots x_n$, or $X = 11x_3 \cdots x_n$. If $X = 10x_3 \cdots x_n$, assuming that $D(x_3 \cdots x_n) = (s_{i_1}, s_{i_2}, \ldots, s_{i_l})$, we set $D(X) = (e_1^n, 00s_{i_1}, 00s_{i_2}, \ldots, 00s_{i_l})$. If $X = 11x_3 \cdots x_n$, then assuming that $D(\bar{x}_3 \cdots \bar{x}_n) = (s_{i_1}, s_{i_2}, \ldots, s_{i_l})$, we set $D(X) = (J_n^n, 00s_{i_1}, 00s_{i_2}, \ldots, 00s_{i_l})$.

It remains to show that each generator appears at most 2^{n-2} times. It is easy to see that both e_1^n and J_n^n appear exactly 2^{n-2} times. Similarly, both e_2^n and J_{n-1}^n appear exactly 2^{n-3} times. For $3 \leq i \leq n$, $|e_i^n| = |e_{i-1}^{n-1}| + 2 \times |e_{i-2}^{n-2}|$. By induction we have $|e_{i-1}^{n-1}| \leq 2^{n-3}$ and $|e_{i-2}^{n-2}| \leq 2^{n-4}$, thus $|e_i^n| \leq 2^{n-2}$. Similarly, for $2 \leq i \leq n-2$, we have $|J_i^n| = |J_i^{n-1}| + 2 \times |J_i^{n-2}|$ and the inequality follows from similar induction. □

Corollary 9. *The edge-forwarding index of the augmented cube of dimension n ($n \geq 2$) is 2^{n-1}.*

Proof. The lower bound $\pi(AQ_n) \geq 2^{n-1}$ is proved in [5]. We sketch the proof for the sake of completeness. Consider the inductive definition of AQ_n. There are two vertex disjoint copies of AQ_{n-1}, connected by 2^n edges. The number of paths connecting vertices from one copy to vertices in the other copy is 2^{2n-1}, each such path must use at least one edge connecting the two parts, thus at least one of such edges being used at least 2^{n-1} times. The upper bound follows from Theorem 8. □

Remark The proof of Theorem 6 of [5] has the correct ideas but it misses the simple fact that AQ_1 doesn't satisfy the formula. More precisely, $\pi(AQ_1) = 2$ but $2^{1-1} = 1$. However, the wrong formula "$\pi(AQ_1) = 2^{1-1} = 1$" is used in their inductive proof.

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