Homomorphisms of partial t-trees and edge-colorings of partial 3-trees

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Abstract

A reformulation of the four color theorem is to say that K_4 is the smallest graph to which every planar (loop-free) graph admits a homomorphism. Extending this theorem, the second author has proved (using the four color theorem) that the Clebsch graph (a well known triangle-free graph on 16 vertices) is a smallest graph to which every triangle-free planar graph admits a homomorphism. As a further generalization he has proposed that the projective cube of dimension $2k$, $PC(2k)$, (that is the Cayley graph $(\mathbb{Z}_2^{2k}, \{e_1, e_2, \dots, e_{2k}, J\}$, where the e_i's are the standard basis and $J = e_1 + e_2 + \cdots + e_{2k})$ is a smallest graph of odd-girth $2k+1$ to which every planar graph of odd-girth at least $2k+1$ admits a homomorphism. This conjecture is related to a conjecture of P. Seymour who claims that the fractional edge-chromatic number of a planar multigraph determines its edge-chromatic number (more precisely, Seymour conjectured that $\chi'(G) = \lceil \chi'_f(G) \rceil$ for any planar multigraph G). Note that the restriction of Seymour's conjecture to cubic (planar) graphs is Tait's reformulation of the four color theorem.

Both these conjectures are believed to be true for the larger class of K_5-minor-free graphs (which includes the class of planar graphs). Motivated by these conjectures and in extension of a recent work of L. Beaudou, F. Foucaud and the second author, which studies homomorphism bounds for the class of K_4-minor-free graphs, in this work we first give a necessary and sufficient condition for a graph B of odd-girth $2k+1$ to admit a homomorphism from any partial t-tree of odd-girth at least $2k+1$. Applying our results to the class of partial 3-trees, which is a rich subclass of K_5-minor-free graphs, we prove that $PC(2k)$ is in fact a smallest graph of odd-girth $2k+1$ to which every partial 3-tree of odd-girth at least $2k+1$ admits a homomorphism. We then apply this result to show that every planar $(2k+1)$-regular multigraph G whose dual is a partial 3-tree, and whose fractional edge-chromatic number is $2k+1$, is $(2k+1)$-edge-colorable. Both these results are the best known supports for the general cases of the above mentioned conjectures in extension of the four color theorem.

Key words: Planar graphs, treewidth, homomorphism, minor, edge-coloring.

1. Introduction

Given graphs G and H, a *homomorphism* of G to H is a mapping of the vertices of G to the vertices of H which preserves adjacency, that is to say, a mapping $h : V(G) \to V(H)$ such that if x and y are adjacent in G, then $h(x)$ and $h(y)$ are adjacent in H. When there exists a homomorphism of G to H we write $G \to H$ and we may say that G *maps* to H. It is easily verified that any mapping of G to K_k corresponds to a k-coloring of G, thus the notion of homomorphism generalizes the theory of coloring of graphs. Given a class $\mathcal{C}$ of graphs, we say that H *bounds* $\mathcal{C}$ if every member of $\mathcal{C}$ maps to H. Thus, in this terminology, the four color theorem is to say: K_4 bounds the class of planar graphs.

Given a graph G, the *odd-girth* of G is the length of a shortest odd cycle of G. It can be easily verified that an odd cycle of length $2l+1$ maps to an odd cycle of length $2k+1$ if and only if $l \geq k$. Thus we have the following no-homomorphism lemma:

Lemma 1.1. *If $G \to H$, then odd-girth$(G) \geq$ odd-girth(H).*

The following is a slightly stronger form of this lemma.

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Lemma 1.2. *Let G and B be graphs of odd-girth $2k+1$ and suppose that there is a homomorphism ϕ of G to B. Let u and v be two vertices of G which belong to a common $(2k+1)$-cycle of G. Then $\phi(u)$ and $\phi(v)$ are on a common $(2k+1)$-cycle of B and $d_G(u,v) = d_B(\phi(u), \phi(v))$.*

Observe that the odd-girth of a given graph G can be computed in time polynomial in the order of G. Given a vertex x of G compute, inductively, the set $N_i(x)$ consisting of vertices at distance i from x. The first i for which $N_i(x)$ induces an edge implies the existence of a closed walk of length $2i+1$ starting at x. The smallest such a closed walk is a shortest odd cycle of G. Thus, this classic no-homomorphism lemma is among the rare no-homomorphism lemmas which is based on a parameter computable in polynomial time (in contrast to other well known no-homomorphism lemmas which are based on parameters such as the clique number, the chromatic number, the fractional or circular chromatic numbers, all of which are NP-hard to compute).

The simplest use of this lemma concerns loops, that are odd cycles of length 1. It asserts that when H is a simple graph and $G \to H$, then G must have no loops. Hence, in this work we consider only graphs with no loops. However, multi-edges will be considered when dealing with applications to edge-coloring. When multi-edges are allowed, we will use the term *multigraph*, thus making sure the term graph refers solely to a simple graph.

In view of this no-homomorphism lemma, the four color theorem can be restated as follows:

Theorem 1.3 (4CT)**.** *The smallest graph with no loop which bounds the class of all planar graphs with no loop is K_4.*

A natural question then is to find a (or probably the unique) smallest graph of odd-girth $2k+1$ which bounds the class of all planar graphs of odd-girth at least $2k+1$. A conjecture of the second author, which implies, in particular, that the order of such a graph must be 2^{2k}, is discussed in Section 6.

Motivated by this question and the recent work of [2], we consider the question of bounding the class of partial t-trees of odd-girth at least $2k+1$ by a graph of odd-girth $2k+1$. A necessary and sufficient condition for a graph B of odd-girth $2k+1$ to admit a homomorphism from all K_4-minor-free graphs of odd-girth at least $2k+1$ is provided in [2]. Noting that the class K_4-minor-free graphs is the class of partial 2-trees, extending this work, we give a necessary and sufficient condition to test if a given graph B of odd-girth $2k+1$ bounds the class of partial t-trees of odd-girth at least $2k+1$. We will then apply our work to the class of partial 3-trees to obtain results which are the strongest known support for the general cases of some conjectures in generalization of the four color theorem.

The paper is organized as follows. In the next section we present our adaptation of various terminologies for the class of partial t-trees. In Section 2, extending the notion of odd-girth to weighted graphs, we build up the required concepts and provide some classifications. Then, in Section 4 we prove a necessary and sufficient condition for a graph B of odd-girth $2k+1$ to bound all partial t-trees of odd-girth at least $2k+1$. In Section 5 we discuss the algorithmic consequences of this necessary and sufficient condition. In Section 6 we discuss a conjecture in generalization of the four color theorem, and we provide support for this conjecture by finding an optimal bound of odd-girth $2k+1$ for partial 3-trees of odd-girth at least $2k+1$. In Section 7 we consider an edge-coloring conjecture of P. Seymour on the edge-chromatic number of planar graphs which extends Tait's reformulation of the four color theorem. We provide support for this conjecture by proving it for a subclass of multigraphs which are planar and whose duals are partial 3-trees.

2. Partial t-trees

The class of t-trees is a class of graphs built according to the following rules:

- K_{t+1} is a t-tree.
- Given a t-tree T and a t-clique C of T, the graph T' built from T by adding a vertex which is adjacent to all the vertices of C is also a t-tree.

After an arbitrary ordering of the vertices of the first K_{t+1}, this construction induces an ordering $v_1, v_2, \dots, v_n$ of the vertices which has the following properties: *i.* the subgraph induced by $v_1, v_2, \dots, v_t$ is a clique, *ii.* in the subgraph H_i induced by vertices $v_1, v_2, \dots, v_i$, $i > t$, the vertex v_i is of degree t and its neighbors induce a complete graph of order t. Such an ordering of the vertices of a t-tree T

is called a *t-tree ordering* of T. Given a t-tree T and a t-tree ordering $v_1, v_2, \ldots, v_n$, $n > t$, let X_i, $t+1 \leq i \leq n$, be the $(t+1)$-clique induced by v_i and all its neighbors in $\{v_1, v_2, \ldots, v_{i-1}\}$. The sequence $X_{t+1}, X_{t+2}, \ldots, X_n$, which is uniquely determined by the given t-tree ordering, is called a *clique-sequence* of T.

Observe that isomorphic copies of a t-tree might be associated with different t-tree orderings and that the class of 1-trees is exactly the class of trees, granting the name t-tree.

A *partial t-tree* is any subgraph of a t-tree. A partial t-tree G might be a subgraph of two or more non-isomorphic t-trees on the same set $V(G)$ of vertices. Given a partial t-tree G, any t-tree ordering of a t-tree T, which is built on $V(G)$ and contains G as a subgraph, is also a t-tree ordering of G. We will denote the class of all partial t-trees by $\mathcal{PT}_t$ and the subclass of partial t-trees of odd-girth at least $2k+1$ by $\mathcal{PT}_{t,2k+1}$.

The class $\mathcal{PT}_t$ is known under various equivalent definitions. It is most notably known as the class of graphs of treewidth at most t. It is easily verified that $\mathcal{PT}_t$ is a minor-closed class of graphs. For $t = 1$, we have the class of all forests which is the class of graphs with no K_3-minor. The class $\mathcal{PT}_2$ is the class of K_4-minor-free graphs, for a proof see for example [7]. The next class, $\mathcal{PT}_3$, is the class of all graphs having none of the four graphs of Figure 1 as a minor, this is proved in [1]. For $t \geq 4$ the full list of forbidden minors is not known, though this list is finite thanks to the Graph-Minor Theorem of Robertson and Seymour.

As K_{t+1} is a member of $\mathcal{PT}_t$ and K_{t+2} is not, K_{t+2} is a minor-minimal graph which does not belong to $\mathcal{PT}_t$, thus, $\mathcal{PT}_t$ is a subclass of the class of K_{t+2}-minor-free graphs. In particular $\mathcal{PT}_3$ forms a special subclass of K_5-minor-free graphs but it does include graphs that are not planar.

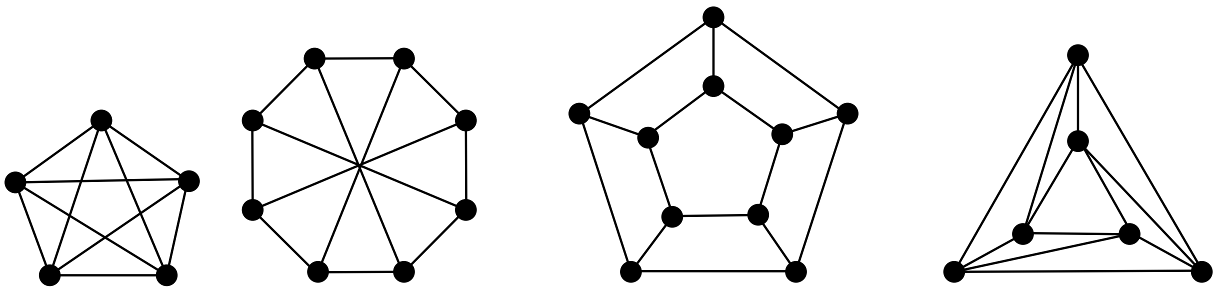


Figure 1: Forbidden minors of the class of partial 3-trees

3. Weighted graphs and $(2k+1)$-wideness

While previous definitions and notations are standard, here we introduce some notions we have developed to address our work. Throughout this section and also in the rest of the work, when it is clear from the context, k is a given positive integer such that $2k+1$ is a lower bound on the odd-girth of graphs and weighted graphs we are working with, the notion of odd-girth for weighted graphs being defined below.

A *weighted graph* (G, w) is a graph together with an assignment $w : E(G) \to \mathbb{Z}^+$ which assigns a positive integer $w(e)$ to each edge e of G.

A homomorphism of a weighted graph (G, w_1) to a weighted graph (H, w_2) is a homomorphism of G to H which preserves the weights, that is to say $f : V(G) \to V(H)$ is a homomorphism of (G, w_1) to (H, w_2) if $w_2(f(x)f(y)) = w_1(xy)$. We may denote existence of such a homomorphism by writing $(G, w_1) \to (H, w_2)$.

Definition 3.1. Given a weighted graph (G, w) satisfying $w(e) \leq 2k$ for each edge e, we define $\overline{\overline{(G, w)}}_{2k+1}$ to be a graph built from G as follows: for each edge uv of G, delete the edge uv and add two $u-v$ paths, one of length $w(e)$ and the other of length $2k+1-w(e)$ (all internal vertices of these paths being new and distinct). We then say that (G, w) is $(2k+1)$-*wide* if the graph $\overline{\overline{(G, w)}}_{2k+1}$ is of odd-girth $2k+1$.

Thus, whenever we claim that (G, w) is $(2k+1)$-wide, we implicitly imply that w satisfies $1 \leq w(e) \leq 2k$ for each edge e of G.

Given two weighted graphs (G, w) and (G, w') with $w(e), w'(e) \leq 2k$ for each edge e of G, if for each edge e we have $w'(e) = w(e)$ or $w'(e) = 2k+1-w(e)$, then $\overline{\overline{(G, w')}}_{2k+1}$ is isomorphic to $\overline{\overline{(G, w)}}_{2k+1}$.

Thus, we may freely assume that the weight of each edge is bounded from above by k. However, when dealing with the parity of a cycle in $\overline{\overline{(G,w)}}_{2k+1}$ going through three or more original vertices of G, we may need to consider some values of w which are greater than k.

From Lemma 1.2 we can derive the following lemmas:

Lemma 3.2. *Given two weighted graphs (G, w_1) and (H, w_2) such that w_1 and w_2 are both bounded from above by $2k$, if $(G, w_1) \to (H, w_2)$ and (H, w_2) is $(2k+1)$-wide, then so is (G, w_1).*

Lemma 3.3. *Given two weighted graphs (G, w_1) and (H, w_2) which are both $(2k+1)$-wide, if $(G, w_1) \to (H, w_2)$ then $\overline{\overline{(G,w_1)}}_{2k+1} \to \overline{\overline{(H,w_2)}}_{2k+1}$.*

Proof. Since homomorphisms must preserve the weights, an edge of weight say p in (G, w_1) is mapped to an edge of weight p in (H, w_2). Thus we just need, for each edge, to extend mapping of the endpoints of a path of length p to a path of length p and of a path of length $2k+1-p$ to a path of length $2k+1-p$. □

The converse of this lemma is true in the following sense:

Lemma 3.4. *Given two weighted graphs (G, w_1) and (H, w_2) which are both $(2k+1)$-wide, if ϕ is a homomorphism of $\overline{\overline{(G,w_1)}}_{2k+1}$ to $\overline{\overline{(H,w_2)}}_{2k+1}$ which maps vertices of G to vertices of H, then the restriction of ϕ to $V(G)$ is a homomorphism of (G, w_1) to (H, w_2).*

Proof. This is based on the fact that (G, w_1) and (H, w_2) are both $(2k+1)$-wide. This implies that both $\overline{\overline{(G,w_1)}}_{2k+1}$ and $\overline{\overline{(H,w_2)}}_{2k+1}$ are of odd-girth $2k+1$. The claim then follows from the fact that every mapping of C_{2k+1} to another cycle of the same length must preserve distances. □

Next, we would like to introduce procedures using which one can decide whether a given weighted complete graph is $(2k+1)$-wide. We first give two independent procedures for weighted triangles. We then show that for weighted complete graphs of larger order, it is enough to apply one of the procedures to all induced triangles. The first test, introduced in [2], applies when we assume all weights are bounded from above by k.

Proposition 3.5. *[2] Let (K_3, w) be a weighted triangle with edge weights a, b, c satisfying $1 \leq a \leq b \leq c \leq k$. Then it is $(2k+1)$-wide if and only if one of the following holds:*
(i) $a+b+c$ is odd and $a+b+c \geq 2k+1$,
(ii) $a+b+c$ is even and $a+b \geq c$.

The next procedure is when w is allowed to give also values between k and $2k$.

Proposition 3.6. *Let (K_3, w) be a weighted triangle with edge weights a, b, c satisfying $1 \leq a, b, c \leq 2k$ such that $a+b+c$ is odd. Let $f_{2k+1}(a,b,c) = \frac{1}{2}(a+b+c-(2k+1))$. Then (K_3, w) is $(2k+1)$-wide if and only if $0 \leq f_{2k+1}(a,b,c) \leq \min\{a,b,c\}$.*

Proof. Suppose (K_3, w) is $(2k+1)$-wide. By definition, the graph $\overline{\overline{(K_3,w)}}_{2k+1}$ is of odd-girth $2k+1$. Then each odd cycle of $\overline{\overline{(K_3,w)}}_{2k+1}$ has length at least $2k+1$. Let x, y, z be the three vertices of K_3. There are eight cycles in $\overline{\overline{(K_3,w)}}_{2k+1}$ containing all the three vertices x, y and z. Exactly four of these eight cycles are of odd length and they are of length $a+b+c$, $(2k+1-a)+(2k+1-b)+c$, $(2k+1-a)+b+(2k+1-c)$ and $a+(2k+1-b)+(2k+1-c)$. Since there is an odd cycle of length $a+b+c$, and since we have assumed that all odd cycles are of length at least $2k+1$, we get $a+b+c \geq 2k+1$, and thus $f_{2k+1}(a,b,c) \geq 0$. The odd cycle corresponding to the length $2k+1-a+2k+1-b+c$ implies that $2k+1-a+2k+1-b+c \geq 2k+1$, therefore $f_{2k+1}(a,b,c) \leq c$. Similarly, we have $f_{2k+1}(a,b,c) \leq b$ and $f_{2k+1}(a,b,c) \leq a$. So $0 \leq f_{2k+1}(a,b,c) \leq \min\{a,b,c\}$.

Conversely, assume $0 \leq f_{2k+1}(a,b,c) \leq \min\{a,b,c\}$. We want to show that (K_3, w) is $(2k+1)$-wide. By definition, we need to show that the graph $\overline{\overline{(K_3,w)}}_{2k+1}$ is of odd-girth $2k+1$. The odd cycles in $\overline{\overline{(K_3,w)}}_{2k+1}$ which contain exactly two vertices of K_3 are of length $2k+1$. There are four odd cycles which contain three vertices of K_3. Since $a+b+c$ is odd, these four odd cycles are of length $a+b+c$, $2k+1-a+2k+1-b+c$, $2k+1-a+b+2k+1-c$ and $a+2k+1-b+2k+1-c$. By the assumption that $f_{2k+1}(a,b,c) \geq 0$, we get $a+b+c \geq 2k+1$. Since $f_{2k+1}(a,b,c) \leq c$, we have $2k+1-a+2k+1-b+c \geq 2k+1$. Similarly, we get that the other two odd cycles have length at least $2k+1$. This completes the proof. □

Next we prove that, given a weighted complete graph (K_t, w), in order to decide whether (K_t, w) is $(2k+1)$-wide, it is enough to apply either one of the two previous propositions on all the induced triangles.

Theorem 3.7. *A complete weighted graph (K_t, w) is $(2k+1)$-wide if and only if each of its induced triangles is $(2k+1)$-wide.*

Proof. First we show that the condition is necessary. Assume that (K_t, w) is $(2k+1)$-wide. Then, by definition, $1 \leq w(e) \leq 2k$ for any edge $e \in E(K_t)$ and the graph $\overline{\overline{(K_t, w)}}_{2k+1}$ is of odd-girth $2k+1$. Let (K_3^*, w^*) be an induced triangle where w^* is induced by w over the edges of K_3^*. Thus $\overline{\overline{(K_3^*, w^*)}}_{2k+1}$ is a subgraph of $\overline{\overline{(K_t, w)}}_{2k+1}$ and, therefore, is also of odd-girth $2k+1$.

To prove that the condition is also sufficient, we assume that (K_t, w) is not $(2k+1)$-wide and we show that there exists a set of three vertices whose induced weighted triangle is not $(2k+1)$-wide. Since (K_t, w) is not $(2k+1)$-wide, there exists an odd cycle in $\overline{\overline{(K_t, w)}}_{2k+1}$ of length less than $2k+1$. Assume that C is such an odd cycle with minimum number of vertices from K_t. By construction, C must have at least three vertices of K_t. If it has three such vertices, then we have found our triangle. Thus, we assume that C has at least four vertices from K_t. Let x, y, z, u be four of these vertices in clockwise direction of C. Denote the two paths in C connecting x and z by P and Q. We know that the lengths of P and Q have different parity. There are two threads in $\overline{\overline{(K_t, w)}}_{2k+1}$ which connect x and z, denote them by R_1 and R_2. The sum of the lengths of R_1 and R_2 is $2k+1$, so the lengths of R_1 and R_2 have also different parity. Without loss of generality, assume that $P \cup R_1$ and $Q \cup R_2$ are odd cycles (if necessary, we relabel R_1 and R_2). Thus, each of $P \cup R_1$ and $Q \cup R_2$ induces a cycle of $\overline{\overline{(K_t, w)}}_{2k+1}$ which uses less vertices of K_t than C. Furthermore, the total length of these two cycles is $|C| + 2k + 1$, hence one of them is of length smaller than $2k+1$. This contradicts the choice of C, thus proving our claim. □

In the next theorem, we show how to use the previous results to test whether a given weighted t-tree is $(2k+1)$-wide.

Theorem 3.8. *Let G be a t-tree with a clique-sequence $X_{t+1}, X_{t+2}, \ldots, X_{t+l}$. Let w be a weighting of G such that each X_i, together with weights induced by w, is $(2k+1)$-wide. Then (G, w) is $(2k+1)$-wide.*

Proof. We prove our claim by induction on l. If $l = 1$, then $(G, w) = (X_{t+1}, w)$ and thus (G, w) is $(2k+1)$-wide. Suppose (G', w), which is obtained from the clique-sequence $X_{t+1}, X_{t+2}, \ldots, X_{t+l-1}$, is $(2k+1)$-wide and let (G, w) be the graph obtained by adding a vertex v joined to t vertices $u_1, u_2, \ldots, u_t$. Thus $u_1, u_2, \ldots, u_t$ together with v induces the $(t+1)$-clique X_{t+l}. Furthermore, suppose vu_i is of weight a_i $(i = 1, 2, \ldots, t)$. Since (G', w) is $(2k+1)$-wide, in order to prove that (G, w) is $(2k+1)$-wide, we only need to consider cycles of $\overline{\overline{(G, w)}}_{2k+1}$ which contain v and connect it to two vertices, say u_1, u_2, among $u_1, u_2, \ldots, u_t$, by paths of length a_i or $2k+1-a_i$, $i = 1, 2$. Let C be such a cycle, and assume, by contradiction, that C is an odd cycle of length smaller than $2k+1$. Let p be the length of the u_1vu_2 part of C and let p' be the length of the complementary part. Thus, p and p' are of different parity. Suppose that the $(2k+1)$-cycle connecting u_1 and u_2 in $\overline{\overline{(G, w)}}_{2k+1}$ is separated by u_1 and u_2 into two paths of length a and a', thus $a + a' = 2k+1$. By symmetry of a and a', assume $p + a$ is odd, then $p' + a'$ is also odd. Observe that each of $p + a$ and $p' + a'$ corresponds to the length of a cycle in $\overline{\overline{(G, w)}}_{2k+1}$. Since the sum of the lengths of these two cycles, (which is $|C| + 2k + 1$), is less than $2(2k+1)$, one of them is of length smaller than $2k+1$. But $p + a$ being smaller than $2k+1$ will contradict the assumption of the theorem that each X_i together with weights induced by w is $(2k+1)$-wide, and $p' + a'$ being smaller than $2k+1$ contradicts our inductive assumption that (G', w) is $(2k+1)$-wide. These contradictions complete our proof. □

Combining these results, we have the following criterion for a t-tree to be $(2k+1)$-wide.

Corollary 3.9. *A weighted t-tree (T, w), $t \geq 2$, is $(2k+1)$-wide if and only if all of its induced weighted triangles are $(2k+1)$-wide.*

3.1. Partial and k-partial distance graphs

In this work, as in [2], we will only consider weighted graphs for which the weight function is a specific metric function defined below.

Let G be a connected graph on n vertices. The *complete distance graph* of G is the weighted graph (K_n, d_G) where K_n is the complete graph on $V(G)$ and the weight $d_G(uv)$ of an edge uv is the distance in G between u and v. Thus, the edges of weight 1 induce G. Any spanning (weighted) subgraph of (K_n, d_G) will then be referred to as a *partial distance graph* of G or a *partial G-distance graph.* Thus, a partial G-distance graph (H, d_G) has $V(G)$ as its vertex set and each edge uv of H is given the weight $d_G(u, v)$, noting that H may not necessarily contain all edges of G. If, furthermore, we have $d_G(u, v) \leq k$ for each edge uv of H, then we say that (H, d_G) is a *k-partial distance graph* of G or a *k-partial G-distance graph.*

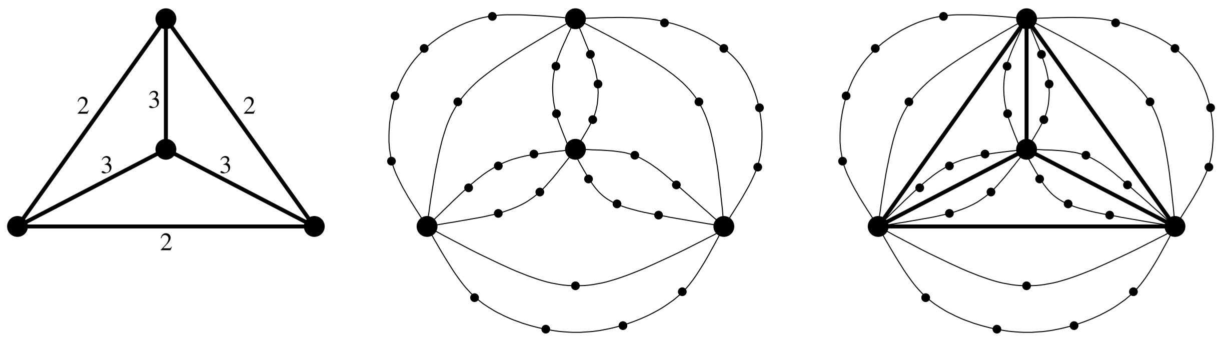
A special family of k-partial G-distance graphs is built as follows:

Definition 3.10. Let G be a t-tree and w be an edge-weighting of G with weights from $\{1, 2, \ldots, k\}$ such that (G, w) is $(2k+1)$-wide. Recall that we have associated with (G, w) a graph $\overline{\overline{(G, w)}}_{2k+1}$ whose vertices contain vertices of G and, furthermore, an edge of G is also an edge of $\overline{\overline{(G, w)}}_{2k+1}$ if and only if it is of weight 1. Let $G^*_{(w,2k+1)}$ be the weighted graph obtained from $\overline{\overline{(G, w)}}_{2k+1}$ by adding all missing edges of G and assigning a weight as follows: edges that are also in $\overline{\overline{(G, w)}}_{2k+1}$ are assigned weight 1 and each edge e which is also an edge of G is assigned the weight $w(e)$. As common edges of G and $\overline{\overline{(G, w)}}_{2k+1}$ are of weight 1, this works fine.

Observe that for $t \geq 2$, given a partial t-tree and an edge e of it, adding an edge parallel to e and subdividing it, the result is still a partial t-tree. Thus we may claim the following:

Observation 3.11. *Given a weighted t-tree (G, w) which is $(2k+1)$-wide, the underlying graph of $G^*_{(w,2k+1)}$ is a partial t-tree.*

An example is given in Figure 2. The figure on the right is a weighted graph built from the weighted K_4 presented on the left. The weights of bold edges are the same as the corresponding edges of (K_4, w) on the left, and all other edges are of weight 1.



(K_4, w) $\qquad$ $\overline{\overline{(K_4, w)}}_7$ $\qquad$ $(K_4)^*_{(w,7)}$

Figure 2: Examples of $\overline{\overline{(G, w)}}_{2k+1}$ and $G^*_{(w,2k+1)}$

As claimed in the following proposition, the condition that (G, w) is $(2k+1)$-wide implies that the extension of (G, w) into $G^*_{(w,2k+1)}$ transforms the general weight function w into a k-partial distance function (of $\overline{\overline{(G, w)}}_{2k+1}$). We leave the verification of this claim to the reader.

Proposition 3.12. *Given a t-tree G and an edge weighting w with weights from $\{1, 2, \ldots, k\}$ such that (G, w) is $(2k+1)$-wide, the weighted graph $G^*_{(w,2k+1)}$ is a k-partial $\overline{\overline{(G, w)}}_{2k+1}$-distance graph.*

3.2. Odd-girth and wideness

Here we will prove that for a complete distance graph of G to be $(2k+1)$-wide is equivalent with G being of odd-girth at least $2k+1$. One direction of this claim follows from the definitions. We prove the other direction in the following general setting.

Lemma 3.13. *If B is of odd-girth $2k+1$, then any partial B-distance graph $\hat{B}$ is $(2k+1)$-wide.*

Proof. By the definition of a weighted graph to be $(2k+1)$-wide, we only need to prove that the complete B-weighted graph $\tilde{B}$ is $(2k+1)$-wide. To do this, we need to show that every odd cycle $\tilde{C}$ of $\overline{\overline{(\tilde{B},w)}}_{2k+1}$ has length at least $2k+1$. If $\tilde{C}$ has exactly two vertices from B, then the length of $\tilde{C}$ is exactly $2k+1$ by construction. Assume $\tilde{C}$ has at least three vertices from B and denote them by $x_1, x_2, \ldots, x_t$ in the clockwise orientation of $\tilde{C}$. Denote the distance from x_i to x_{i+1} in $\tilde{C}$ by l_i. If there exist l_i and l_j such that $l_i, l_j \geq k$, then $\sum_{s=1}^{t} l_s \geq 2k+1$. So the length of $\tilde{C}$ is at least $2k+1$. Suppose that $l_i < k$ for every $i = 1, 2, \ldots, t$. Then we know that $l_i = d_B(x_i, x_{i+1})$, so there is a closed walk of odd length going through $x_1, x_2, \ldots, x_t$ in B whose length is $\sum_{s=1}^{t} l_s$. Since B is of odd-girth $2k+1$, we have $\sum_{s=1}^{t} l_s \geq 2k+1$. So, the length of $\tilde{C}$ is at least $2k+1$. Finally, consider the case where there is exactly one l_i with $l_i \geq k$. By symmetry, we may assume $l_1 \geq k$. Hence, $l_i = d_B(x_i, x_{i+1})$ for $i = 2, \ldots, t$. We consider two subcases. If $l_1 = k$, then we have $l_1 = w(x_1, x_2) = k \leq d_B(x_1, x_2)$. By the triangle inequality, we get $l_2 + \cdots + l_t \geq d_B(x_1, x_2)$. So, $\sum_{s=1}^{t} l_s \geq k + d_B(x_1, x_2) \geq 2k$. Since $\sum_{s=1}^{t} l_s$ is odd, we have $\sum_{s=1}^{t} l_s \geq 2k+1$. If $l_1 \geq k+1$, then $l_1 = 2k+1 - w(x_1, x_2)$. Hence, $\sum_{s=1}^{t} l_s = 2k+1 - w(x_1, x_2) + l_2 + \cdots + l_t$. Since $w(x_1, x_2) \leq d_B(x_1, x_2)$, by the triangular inequality, we have $\sum_{s=1}^{t} l_s \geq 2k+1$. We conclude that $\tilde{C}$ has length at least $2k+1$ in each case. □

The following theorem is a key tool of our work. It claims, basically, that a homomorphism of a graph G to a graph H where both G and H are of odd-girth $2k+1$ can be viewed as a mapping which preserves more than just adjacency. It is only a restatement of Lemma 1.2 using the terminology we have just developed.

Theorem 3.14. *Let G and H be two graphs of odd-girth $2k+1$. Let (G', w_1) be the partial G-distance graph consisting of all edges xy where x and y belong to a common $(2k+1)$-cycle of G. Similarly, define (H', w_2) to be the partial H-distance graph consisting of all edges uv where u and v belong to a common $(2k+1)$-cycle of H. Then each homomorphism of G to H is also a homomorphism of (G', w_1) to (H', w_2).*

Proof. This is also an application of the fact that a mapping of C_{2k+1} to another cycle of the same length is rather an isomorphism and distances are preserved. □

3.3. List of all $(2k+1)$-wide complete graphs on $t+1$ vertices

Given positive integers k and t, and towards working with graphs in $\mathcal{PT}_{t,2k+1}$, we will need the list of all weightings w of the complete graphs on $t+1$ vertices for which (K_{t+1}, w) is $(2k+1)$-wide. As k and t are fixed numbers, this list can be easily computed. The computation of this list is further simplified by Theorem 3.7. We denote the list of such weightings by $L(t+1, 2k+1)$. The total number of weightings on a K_{t+1} whose vertices are labeled, with the weight of each edge being an integer between 1 and k, is $k^{\binom{t+1}{2}}$. This provides an upper bound on the number of elements of $L(t+1, 2k+1)$. This upper bound is not too far from the exact number of elements by the following observation:

Proposition 3.15. *If w is a weighting of K_{t+1} satisfying $\frac{2k}{3} \leq w(e) \leq k$ for each edge e, then (K_{t+1}, w) is $(2k+1)$-wide.*

Proof. By Theorem 3.7 it is enough to check that for each triangle the induced weighted triangle is $(2k+1)$-wide. Let a, b, c be the weights of a triangle in (K_{t+1}, w). We use Proposition 3.5 to test if this triangle is $(2k+1)$-wide. If $a+b+c$ is odd, then it is at least $2k+1$. If it is even, then $a+b \geq \frac{4k}{3}$ but $c \leq k$, which implies $a+b \geq c$. □

Thus, given k and t, there are at least $\frac{1}{\binom{t+1}{2}!}(\frac{k}{3})^{\binom{t+1}{2}}$ non isomorphic elements in $L(t+1, 2k+1)$. We leave finding the exact number of elements of $L(t+1, 2k+1)$ as an open question. To help with a better understanding of the notion, in Figure 3, we have provided the full list of 7 non-isomorphic elements of $L(4, 5)$.

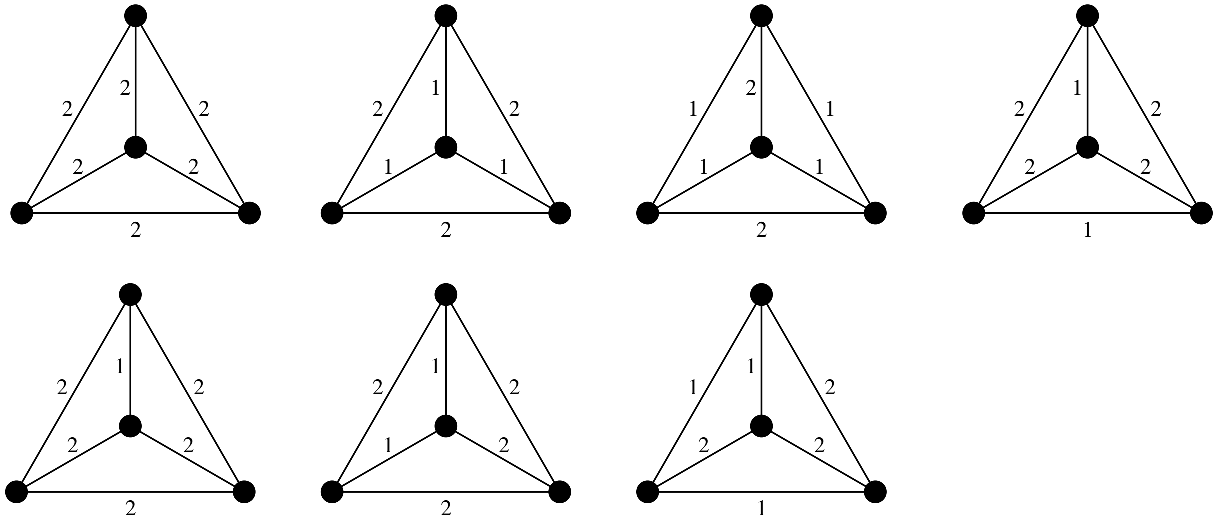


Figure 3: The list $L(4, 5)$ of all 5-wide K_4's

In the next lemma, we show that our list $L(t+1, 2k+1)$ maintains a weak notion of monotonicity. That is to say, if in a given $(2k+1)$-wide weighting, all entries corresponding to a vertex u are raised to the maximum possible value (which is k), then the result will still be a member of our list.

Lemma 3.16. *Let w be a weighting of K_{t+1} such that (K_{t+1}, w) is $(2k+1)$-wide. If we change the weight of each of the t edges incident to a same vertex to k, then the resulting weighted graph (K_{t+1}, w') is still $(2k+1)$-wide.*

Proof. By the assumption, we have $w'(e) \leq k$ for every edge $e \in E(K_{t+1})$. In order to prove that (K_{t+1}, w') is $(2k+1)$-wide, by the definition of $\overline{\overline{(K_{t+1}, w')}}_{2k+1}$, we only need to prove that every odd cycle C which contains at least three vertices of K_{t+1} has length at least $2k+1$. Let v_0 be the vertex for which we change the weights of its adjacent edges. If $v_0 \in V(C)$, then the length of C is at least $2k+1$. Otherwise, C is an odd cycle of $\overline{\overline{(K_{t+1}, w)}}_{2k+1}$. Hence, the length of C is at least $2k+1$. □

4. Necessary and sufficient conditions

Here we would like to give a necessary and sufficient condition for a graph B of odd-girth $2k+1$ to bound the class of partial t-trees of odd girth $2k+1$. Our condition is expressed in the form of the existence of a family of cliques in a partially B-weighted graph. We start with a description of the main property that this set of cliques must satisfy. We point out that deciding if there is a weighting on B for which such a set of cliques exists, and finding one such a set of cliques, when it exists, can be done in polynomial time (in terms of the order of B, k and t). Such an algorithm will be given in Section 5.

4.1. $(t, 2k+1)$-closed sets of cliques

The following definition is a key property of a set of cliques in a partial B-weighted graph which we need for B, which is of odd-girth $2k+1$, to bound the class of all partial t-trees of odd-girth at least $2k+1$. We will provide an example to explain it better.

Definition 4.1. Let (G, w) be a weighted $(2k+1)$-wide graph. Let $\mathcal{W}$ be a collection of (weighted) $(t+1)$-cliques (G, w). Observe that each member of $\mathcal{W}$, without labeling of its vertices, is a member of $L(t+1, 2k+1)$. We say $\mathcal{W}$ is $(t, 2k+1)$-*closed* if the following is satisfied for each member W of $\mathcal{W}$: Let v be a vertex of W and let $a_1, a_2, \ldots, a_t$ be the weights of the edges of W incident to v ($a_i = w(vv_i)$). Suppose that switching the weights $a_1, a_2, \ldots, a_t$ to $a'_1, a'_2, \ldots, a'_t$ while keeping all other weights the same results in another element of $L(t+1, 2k+1)$. Then, there must exist a clique W' in $\mathcal{W}$ which is obtained from W by removing the vertex v and adding a vertex v' where $w(v'v_i) = a'_i$ for $i = 1, 2, \ldots, t$.

The definition is inspired by the construction of t-trees. Given a $(t+1)$-clique W' of a weighted t-tree G, if a new vertex is added which is joined to t vertices of W', and weighted in such a way that the new graph G' is also $(2k+1)$-wide, we want to be able to extend any mapping of G to the weighted graph B to a mapping of G' to B. A $(t, 2k+1)$-closed set of $(t+1)$-cliques in the weighted B will allow us to do exactly this. And as we will prove, the existence of at least one $(t, 2k+1)$-closed set of $(t+1)$-cliques in the weighted B is also a necessary condition for B to bound all partial t-trees of odd-girth at least $2k+1$.

For further clarification, we would like to point out a main difference between this definition and the list $L(t+1, 2k+1)$: a $(t, 2k+1)$-closed set is a set of (weighted) cliques in a weighted version of a given graph, thus in particular its vertices are labeled. In contrast, for the list $L(t+1, 2k+1)$ the labeling of vertices is of no importance. The key here is the edge-weights of graphs in the list which helps with keeping us within the condition of odd-girth at least $2k+1$.

For a better understanding of this definition we use the list of Figure 3. Suppose (B, w) is a weighted graph which is 5-wide (in a sense it is triangle-free), and let $\mathcal{W}$ be a collection of its 4-cliques which is $(3, 5)$-closed. Let W_1 be a member of $\mathcal{W}$ on vertices x, y, z, t and suppose that all edges are of weight 2, i.e., W_1 is isomorphic to the first weighted graph in the list of Figure 3. Consider the triangle induced by xyz, all whose edges are of weight 2. There are three other weighted K_4's in the list $L(4, 5)$ which contain triangles all whose edges are of weight 2. Those are the second, fifth and sixth graphs in Figure 3. Considering the first, the condition of Definition 4.1 is the existence of a vertex u_{111} which is adjacent to x, y and z (each edge having weight 1). The fifth element implies the existence of three other vertices: a vertex u_{122} which is adjacent to x with an edge of weight 1 and to y and z both with edges of weight 2. Vertices u_{212} and u_{221} are defined similarly. Note that, by our definition, each of the three cliques induced by $\{x, y, z, u_{122}\}$, $\{x, y, z, u_{212}\}$ and $\{x, y, z, u_{221}\}$ must be in $\mathcal{W}$. Similarly, there must be three other vertices u_{112}, u_{121} and u_{211} giving cliques isomorphic to the sixth element of the list. We note that each of these vertices might be used again to satisfy the condition for other triangles of W_1 or for elements of $\mathcal{W}$. Altogether, as we will see later, the smallest 5-wide weighted graph with a nontrivial $(3, 5)$-closed set of 4-cliques has 16 vertices.

A $(t, 2k+1)$-closed set of cliques of a weighted graph must then have a large number of $(t+1)$-cliques. To find a smallest such a set or a smallest weighted graph with such a set of cliques seems to be a challenging question. An easy lower bound on the number of elements of such a set is the number of elements of $L(t+1, 2k+1)$. More precisely, given a $(t, 2k+1)$-closed set of cliques, if it is nonempty, then it must have one $(t+1)$-clique for each element of $L(t+1, 2k+1)$ as we prove in the following theorem.

Theorem 4.2. *Given a $(2k+1)$-wide weighted graph (G, w), if a nonempty collection $\mathcal{W}$ of its $(t+1)$-cliques is $(t, 2k+1)$-closed, then for any weighting w of K_{t+1} such that $(K_{t+1}, w) \in L(t+1, 2k+1)$ there is an isomorphic copy of (K_{t+1}, w) in $\mathcal{W}$.*

Proof. Since $\mathcal{W}$ is nonempty, there exists an element (K_{t+1}, w_0) in $\mathcal{W}$. Let $V(K_{t+1}) = \{v_0, v_1, \ldots, v_t\}$. If we change the weights of the edges incident with v_0 to k, then, by Lemma 3.16, the resulting $(t+1)$-clique is $(2k+1)$-wide. As $\mathcal{W}$ is $(t, 2k+1)$-closed, there exists a vertex u_0 which is adjacent to each of $v_1, v_2, \ldots, v_t$ with an edge of weight k and such that the $(t+1)$-clique induced by $u_0, v_1, v_2, \ldots, v_t$ is in $\mathcal{W}$. By repeating this process, we may replace each v_i with a vertex u_i which is adjacent to the vertices of the current clique we are working on with edges of weight k. Thus, at the final step, we have a clique in $\mathcal{W}$ all whose edges are of weight k.

In summary, starting with any $(2k+1)$-wide $(t+1)$-clique in $\mathcal{W}$, we get that there is a clique in $\mathcal{W}$ all whose edges are of weight k. However, this is a reversible construction, and starting with one such clique (where all edges are of weight k) and using the property of being $(t, 2k+1)$-closed, we can get an isomorphic copy of any other member of $L(t+1, 2k+1)$ in $\mathcal{W}$.

□

4.2. A necessary and sufficient condition

Recall that $\mathcal{PT}_t$ is the class of partial t-trees, and $\mathcal{PT}_{t,2k+1}$ is the subclass of partial t-trees of odd-girth at least $2k+1$. We are now ready to state and prove the following theorem, which provides a necessary and sufficient condition for a graph B of odd-girth $2k+1$ to bound $\mathcal{PT}_{t,2k+1}$. We will give an algorithm to check this in Section 5.

Theorem 4.3. *A graph B of odd-girth $2k+1$ admits a homomorphism from every partial t-tree of odd-girth at least $2k+1$ if and only if there exists a partial B-distance graph $(\hat{B}, w)$ with a nonempty set $\mathcal{W}$ of $(t+1)$-cliques of $(\hat{B}, w)$ which is $(t, 2k+1)$-closed.*

Proof. First, we prove the sufficient part. Assume such a weighted graph $\hat{B}$ and a nonempty set $\mathcal{W}$ of $(t+1)$-cliques exist. Suppose G is a partial t-tree of odd-girth at least $2k+1$. Let $\tilde{G}$ be a t-tree containing G as a subgraph. We form a weighted graph on $\tilde{G}$ by defining the weight function $\varphi : E(\tilde{G}) \to \{1, 2, \dots, k\}$ by $\varphi(xy) = \min\{d_G(x, y), k\}$ for every $xy \in E(\tilde{G})$. By Lemma 3.13, we know that $(\tilde{G}, \varphi)$ is a $(2k+1)$-wide weighted graph. Let $X_1, X_2, \dots, X_s$ be a $(t+1)$-clique-sequence from which $\tilde{G}$ is built. Since X_1 is a subgraph of $\tilde{G}$, we know that (X_1, φ) is $(2k+1)$-wide. By Theorem 4.2, there is an isomorphic copy of X_1 in $\mathcal{W}$, let C_1 be this copy. Let h be an isomorphism of X_1 to C_1. Observe that, in particular, h is a homomorphism of X_1 to C_1. Our goal is to extend h to a homomorphism of $\tilde{G}$ to $\hat{B}$. Let $V(X_2) = \{x_1, x_2, \dots, x_t, x_{t+1}\}$. By the definition of partial t-trees, and without loss of generality, assume $x_1, x_2, \dots, x_t \in V(X_1)$. Since $\mathcal{W}$ is $(t, 2k+1)$-closed and X_2 is $(2k+1)$-wide, there exists $y \in V(B)$ such that $\{y, h(x_1), h(x_2), \dots, h(x_t)\}$ induces a member of $\mathcal{W}$ and the weight of $yh(x_i)$ is the same as the weight of $\varphi(x_{t+1}x_i)$ for each $i \in \{1, 2, \dots, t\}$. Let $h(x_{t+1}) = y$, hence, this extended mapping h is a homomorphism of $(X_1 \cup X_2, \varphi)$ to $\hat{B}$. By continuing this process, we eventually extend h to a homomorphism of $(\tilde{G}, \varphi)$ to $(\hat{B}, w)$. Thus, for each edge xy of $\tilde{G}$ of weight 1, $h(x)h(y)$ is an edge of $\hat{B}$ of weight 1. That is to say, $h(x)h(y)$ is an edge of B. Therefore, h is also a homomorphism of G to B.

Next, we want to prove that the condition is necessary.

We first define $\hat{B}$ as follows: the vertices of $\hat{B}$ are the same as the vertices of B. A pair x, y of vertices is an edge of $\hat{B}$ if x and y belong to a common $(2k+1)$-cycle of B. Thus, $\hat{B}$ does not necessarily contain all edges of B, but it does contain each pair that belongs to a common $(2k+1)$-cycle and thus we may apply Theorem 3.14. We then define the weight φ of an edge xy of $\hat{B}$ to be the distance between x and y in B. That is the same as their distance in any $(2k+1)$-cycle containing them since B is of odd-girth $2k+1$. Our aim is to show that $(\hat{B}, \varphi)$ admits a nonempty set $\mathcal{W}$ of $(t+1)$-cliques which is $(t, 2k+1)$-closed.

Recall that given a t-tree G and a weighting w $(1 \le w(e) \le k)$ of its edges, we have associated with (G, w) a weighted graph $G^*_{(w,2k+1)}$ which, by Proposition 3.12, is a partial distance graph of $\overline{\overline{(G, w)}}_{2k+1}$.

By Observation 3.11, for $t \ge 2$, the underlying graph of $G^*_{(w,2k+1)}$ is a partial t-tree. Since $\overline{\overline{(G, w)}}_{2k+1}$ is a partial t-tree of odd-girth $2k+1$, by our assumption, it admits a homomorphism to B. Thus, by Theorem 3.14, $G^*_{(w,2k+1)}$ maps to $(\hat{B}, \varphi)$.

Intuitively speaking, the set $\mathcal{W}$ of $(t+1)$-cliques we are looking for is a minimal set of $(2k+1)$-cliques in a weighted version of B which are the images of cliques when mapping $(2k+1)$-wide weighted graphs to the weighted version of B. More details are as follows.

Let $\mathcal{PT}^*_{t,2k+1}$ be the set of weighted graphs $G^*_{(w,2k+1)}$, where G is a t-tree such that, together with edge-weighting w, (G, w) is $(2k+1)$-wide. As mentioned above, each member of $\mathcal{PT}^*_{t,2k+1}$ admits a homomorphism to $(\hat{B}, \varphi)$. Observe that, in a mapping of any such member $G^*_{(w,2k+1)}$ to $(\hat{B}, \varphi)$, the image of any $(t+1)$-clique of $G^*_{(w,2k+1)}$ is a weighted $(t+1)$-clique in $(\hat{B}, \varphi)$. Let $\mathcal{W}$ be a minimal possible set of $(t+1)$-cliques in $(\hat{B}, \varphi)$ as the homomorphic images of such a mapping. More precisely, $\mathcal{W}$ is a set of $(t+1)$-cliques in $(\hat{B}, \varphi)$ such that (i) for every $G^*_{(w,2k+1)}$ in $\mathcal{PT}^*_{t,2k+1}$ there exists a homomorphism of $G^*_{(w,2k+1)}$ to $(\hat{B}, \varphi)$ such that each $(t+1)$-clique of $G^*_{(w,2k+1)}$ is mapped to an element of $\mathcal{W}$, and (ii) for each clique $W \in \mathcal{W}$, $\mathcal{W} - W$ is not such a set. We will denote the existence of a homomorphism mentioned in (i) by $G^*_{(w,2k+1)} \to (\hat{B}, \varphi, \mathcal{W})$. A restatement of (ii) is that for each clique $W \in \mathcal{W}$, there exists a weighted graph $G^*_{(w,2k+1)}$ in $\mathcal{PT}^*_{t,2k+1}$ such that, in any mapping of $G^*_{(w,2k+1)}$ to $(\hat{B}, \varphi, \mathcal{W})$, at least one of its $(t+1)$-cliques is mapped to W.

We claim that $\mathcal{W}$ satisfies the condition of our theorem, i.e., $\mathcal{W}$ is not empty and it is $(t, 2k+1)$-closed.

To see that $\mathcal{W}$ is not empty it is enough to take any $(2k+1)$-wide K_{t+1} as (G, w) and consider the corresponding $G^*_{(w,2k+1)}$.

To prove that $\mathcal{W}$ is $(t, 2k+1)$-closed, we take a $(t+1)$-clique, say C_1, then we take t vertices $v_1, v_2, \ldots, v_t$ from C_1 and we assume that adding a vertex x with $\varphi(xv_i) = a_i$ to these t vertices produces a $(t+1)$-clique which is also $(2k+1)$-wide. Then, we need to show that there exists a vertex $v \in V(B)$ such that $v_1, \ldots, v_t, v$ induce a $(t+1)$-clique in $\mathcal{W}$ and such that $\varphi(vv_i) = a_i$ for $i = 1, \ldots, t$. Recall that, by the minimality of $\mathcal{W}$, there exists an element $\hat{G}_{C_1}$ which admits a $(\hat{B}, \varphi, \mathcal{W})$-mapping, but in any such mapping, at least one $(t+1)$-clique is mapped to C_1.

Consider all isomorphic copies of C_1 in $\hat{G}_{C_1}$ (these are the $(t+1)$-cliques that could potentially map to C_1). For each such a $(t+1)$-clique W, do as follows: let $u_1, u_2, \ldots, u_t$ be the vertices of W corresponding to $v_1, v_2, \ldots, v_t$. Let W' be the induced t-clique by $u_1, u_2, \ldots, u_t$. Suppose that adding a vertex v to W' and joining v to each v_i with an edge of weight a_i results in a $(t+1)$-clique which is $(2k+1)$-wide, i.e., the weighted clique is in $L(t+1, 2k+1)$. Then, for each isomorphic copy of C_1 in $\hat{G}_{C_1}$, add a new vertex which is joined to vertices in the isomorphic copy with edges of corresponding weight from $a_1, a_2, \ldots, a_t$. Furthermore, for each newly added edge of weight a_i, add two paths of respective length a_i and $2k+1-a_i$ connecting the two ends of this edge. Each edge of these two paths is of weight 1. Let G^* be the new weighted graph. By construction and by Proposition 3.12, the weighted graph G^* is a member of $\mathcal{PT}^*_{t,2k+1}$, and, therefore, there exists a homomorphism ρ of G^* to $(\hat{B}, \varphi, \mathcal{W})$.

Recall that to complete our proof, given a clique C_1 in $\mathcal{W}$, we need to show that, if for a vertex v of C_1, the weights of the edges incident to v are changed from $a_1, a_2, \ldots, a_t$ to $a'_1, a'_2, \ldots, a'_t$, then there is a clique $C'_1 \in \mathcal{W}$ realizing this new set of weights.

Observe that $\hat{G}_{C_1} \subseteq G^*$, thus ρ induces a homomorphism of $\hat{G}_{C_1}$ to $(\hat{B}, \varphi, \mathcal{W})$. By the choice of $\hat{G}_{C_1}$, some $(t+1)$-clique of $\hat{G}_{C_1}$, say K, is mapped to C_1. However, in G^* we have built a clique K' such that t of its vertices are from K and a new vertex x is added, which is adjacent to the common vertices of K and K' with weights $a'_1, a'_2, \ldots, a'_t$. The image of x is then the vertex we are looking for and this completes our proof. □

5. Algorithmic implications

Here we discuss how to apply the necessary and sufficient condition of Theorem 4.3 to decide whether a given graph B of odd-girth $2k+1$ bounds $\mathcal{PT}_{t,2k+1}$. Since t and k are fixed integers, we can assume that the set $L(t+1, 2k+1)$ is already computed. This is a list of order at most $k^{\binom{t+1}{2}}$ and, therefore, of a constant size when t and k are fixed integers. We first form a weighted graph $\hat{B}$ whose vertices are vertices of B and edges are pairs uv such that u and v belong to a common $(2k+1)$-cycle of B, the weight of each such edge being the distance between u and v in the graph B. By the condition on the odd-girth of B, this distance is determined by the distance between u and v in one of the odd cycles they both belong to.

Observe that determining if vertices u and v are in a common $(2k+1)$-cycle of B is simple: starting at u, and iteratively, we compute the set $N_i(u)$ which consists of all vertices at distance i from u and, at each step, we check if v is in $N_i(u)$. If $v \notin N_i(u)$ for $i \leq k$, then we conclude that the pair is in no common $(2k+1)$-cycle. Otherwise, we may assume l is the first i such that $v \in N_i(u)$ (thus $l = d_B(u, v)$). We then compute the set $N_{2k+1-2l}(v)$. If this set has no intersection with $N_l(u)$, then u and v do not belong to a $(2k+1)$-cycle. Otherwise we have found such a cycle.

Once we have our weighted graph $\hat{B}$, we may list, in time at most $\binom{|V(B)|}{t+1}$, the set of all weighted $(t+1)$-cliques of $\hat{B}$. As t is a fixed number, with respect to the order of B, this list is provided in polynomial time. Let $\mathcal{W}$ be this ordered set of cliques whose elements are labeled $W_1, W_2, \ldots, W_r$. Next, we would like to figure out if any subset of this list is $(t, 2k+1)$-closed, and if so, to output such a subset. To this end, given an element W_i of $\mathcal{W}$, we first check whether W passes the test of being $(t, 2k+1)$-closed with respect to W_i. This is done in the following loop.

Let $v_1, v_2, \ldots, v_{t+1}$ be the vertices of W_i, and let W'_i be a t-clique induced by W_i after deleting a vertex v_j. For simplicity, we assume that $j = t+1$, but we must do this next inner loop for all v_j, $j = 1, 2, \ldots, t+1$. In this inner loop, we look for all elements of $L(t+1, 2k+1)$ which can be regarded as an extension of the weights of W'_i by adding one more vertex and, for each such element of $L(t+1, 2k+1)$, we consider all possible extensions. The number of such possibilities is a function of t and k, so it is constant with respect to our parameter which is the order of B. Consider one such extension φ. Thus, φ is regarded as an edge weighting of a $(t+1)$-clique, t of whose vertices are labeled $v_1, v_2, \ldots, v_t$ and the last vertex is labeled v' (which is not a vertex of B). Moreover, for each edge v_iv_j, $i, j \leq t$, we have

$\varphi(v_iv_j) = d_B(v_i, v_j)$. Assume $\varphi(v_iv') = a_i$, $i = 1, 2, \ldots, t$. What we need to find now is a vertex v of B such that $d_B(v_i, v) = a_i$ for all $i = 1, 2, \ldots, t$. The existence of such a vertex v then can be checked easily by trying each vertex of B that is not also a vertex of W_i. If for some vertex $v \in V(B) \setminus V(W_i)$ we have $d_B(v_i, v) = a_i$, and the $(t+1)$-clique obtained for W'_i by adding the vertex v is in $\mathcal{W}$, then we consider this step of the inner loop verified and check the next embedding of W'_i among the members of $L(t+1, 2k+1)$. If we cannot find any such vertex, then we conclude that W_i cannot be in a set of $(t, 2k+1)$-closed cliques. In that case, we remove W_i from our list $\mathcal{W}$ and we start over. If at some point, for any choice of an element W of $\mathcal{W}$, any choice of an induced t-clique W' of W, and any embedding of W' in elements of $L(t+1, 2k+1)$, we find our required vertex v, then this list W of $(t+1)$-cliques is $(t, 2k+1)$-closed by definition and, therefore, it provides a certificate that B (or rather a subgraph of B induced by the edges of weights 1 of cliques in W) bounds the class $\mathcal{PT}_{t,2k+1}$. If, by repeating our loops and inner loops, we eventually delete all considered $(t+1)$-cliques, i.e., when we arrive at the case $\mathcal{W} = \emptyset$, we claim that our weighted graph $\hat{B}$ has no nonempty $(t, 2k+1)$-closed set of $(t+1)$-cliques. That is because if there were such a set $\mathcal{W}_1$, since at the start we considered all $(t+1)$-cliques, we would have $\mathcal{W}_1 \subseteq \mathcal{W}$. But then each element of $\mathcal{W}_1$ passes all our tests and thus no element of $\mathcal{W}_1$ will ever be deleted.

Overall, in the discussion above, we have proved the following facts:

Theorem 5.1. *Let t and k be given (fixed) positive integers and let B be a graph of odd-girth $2k+1$. Then*

- *we can decide in a finite number of steps if B bounds $\mathcal{PT}_{t,2k+1}$,*
- *this number of steps is bounded by a polynomial function of $|V(B)|$ whose degree and coefficients are dependent on t and k,*
- *if the above steps certify B as a bound for $\mathcal{PT}_{t,2k+1}$, then it also outputs a $(t, 2k+1)$-closed list of $(t+1)$-cliques that we can use to find a mapping of any member of $\mathcal{PT}_{t,2k+1}$ to B.*

There are a few important remarks to make here. First, in order not to underestimate the power of Theorem 4.3 and its algorithmic application, we would like to mention that, for the same general graph B of odd-girth $2k+1$, we do not know of any finite algorithm to decide whether B bounds the class of all planar graphs of odd-girth $2k+1$. While for certain cases, such as when B contains a K_4, we have the trivial YES answer, for a general choice of B, finding such an algorithm may help to deal with some of the difficult conjectures which we have mentioned in this work. Another note is that, in the discussion before Theorem 5.1, we only cared to show that the algorithm we provide runs in a time polynomial in the order of B when t and k are fixed. To actually implement the algorithm, one may also apply further optimization. For example, having in hand a lower bound for the order of a $(t, 2k+1)$-closed set of $(t+1)$-cliques, if the order of $\mathcal{W}$ goes below such a threshold, we may stop with a negative answer. An example of such a lower bound is the number of elements of $L(t+1, 2k+1)$. Recall that we must already have a list of them for our algorithm. The last note worthy comment here is about a weakness of our algorithm: when B is a bound, our algorithm provides a $(t, 2k+1)$-closed set $\mathcal{W}$ of $(t+1)$-cliques that we can use to map each member of $\mathcal{PT}_{t,2k+1}$ to B. However, the set $\mathcal{W}$ created here is the largest of all such sets while, in practice, and towards optimization, we are interested in a smallest of such sets.

6. Projective cubes and bounding partial 3-trees

Theorem 4.3 is an extension of a necessary and sufficient condition provided in [2] to test whether a given graph of odd-girth $2k+1$ bounds the class of K_4-minor-free graphs of odd-girth at least $2k+1$. The question of finding a smallest such a bound is discussed in that paper, where it is shown that the order of a smallest such graph must be $\Theta(k^2)$. The exact answer remains an open problem for $k \geq 4$. One naturally expects that the analogous question would become more difficult for $\mathcal{PT}_{t,2k+1}$ with $t \geq 3$. While generally this might be the case, in this section we show that the case $t = 3$ is rather special by providing the optimal answer. Here, we exhibit a graph of odd-girth $2k+1$ on 2^{2k} vertices which bounds the class $\mathcal{PT}_{3,2k+1}$, and we point out that there can be no such graph with smaller order. Our results here can be viewed as the strongest support provided so far for the general case of a conjecture in extension of the four color theorem. This conjecture will be restated after we introduce the necessary definitions.

We first recall the definition of a Cayley graph: let Γ be an additive group and let S be a symmetric subset of Γ (i.e., for each $x \in S$ we have $-x \in S$), furthermore, suppose that 0 is not a member of S. Then the Cayley graph (Γ, S) is the graph whose vertices are elements of Γ, and where two vertices x and y are adjacent if and only if $x - y \in S$. The fact that S is symmetric implies that G is a graph and not a digraph. As we will consider only binary groups here, we have $-x = x$ for each x, and thus all sets are symmetric here. The important advantage of considering Cayley graphs on binary groups is the naturally associated edge-coloring. Given a Cayley graph on a binary group, to each edge uv we assign as a color $u + v = u - v = v - u \in S$.

The main objects of study in this section are the following Cayley graphs on the binary group $\mathbb{Z}_2^k$.

Definition 6.1. The *projective cube* of dimension k is the Cayley graph $(\mathbb{Z}_2^k, \{e_1, e_2, \ldots, e_k, J\})$ where the e_i's are the vectors of the standard basis and J is the all-1 vector.

Recall that the Cayley graph $(\mathbb{Z}_2^k, \{e_1, e_2, \ldots, e_k\})$ is the well-known hypercube $H(k)$, where graph distances are the same as Hamming distances. Therefore, $PC(k)$ is built from $H(k)$ by adding an edge between each antipodal pair of vertices. It is not hard to see that this graph is also built from $H(k+1)$ by identifying antipodal pairs. This projection has given the choice of name "projective cube" for these graphs. However, they are more often referred to as "folded cube". It can be easily checked that $PC(2k-1)$ is bipartite. On the other hand, $PC(2k)$ is a 4-chromatic graph of odd-girth $2k+1$ (see [21] and [3]). It is easily verified that $PC(2)$ is isomorphic to K_4, thus the following conjecture of the second author from [15] is an extension of the four color theorem:

Conjecture 6.2. *Every planar graph of odd-girth at least $2k + 1$ admits a homomorphism to $PC(2k)$.*

The conjecture is believed to be true for the larger class of K_5-minor-free graphs (see [17]). Even a further extension using the notion of signed graphs is considered, we refer to [17] for more details. The conjecture is about finding an optimal bound in the following sense: it is proved in [19] that if the conjecture is true, then $PC(2k)$ is a smallest graph (both in terms of the number of edges and of the number of vertices) of odd-girth $2k + 1$ which bounds the class of all planar graphs of odd-girth at least $2k+1$. That is to say, if B is a graph of odd-girth $2k+1$ to which every planar graph of odd-girth at least $2k + 1$ admits a homomorphism, then B has at least as many vertices and as many edges as $PC(2k)$.

The case $k = 1$ of the conjecture is just the four color theorem, and its extension to the K_5-minor-free graphs is obtained by Wagner's theorem on the decomposition of K_5-minor-free graphs. The case $k = 2$ is proved in [15] (using the four color theorem) and its extension to K_5-minor-free graphs is given in [20]. The case $k = 3$ follows from recent results on edge-coloring of planar multigraphs which are also based on the four color theorem (see [17]).

For larger values of k the conjecture remains open, and the best known support was a verification for the class of K_4-minor-free graphs proved in [2]. Here, as an application of our work, we show that the conjecture holds for the larger class of partial 3-trees of odd-girth at least $2k + 1$. We note that this is a subclass of K_5-minor-free graphs, but it does contain some non-planar graphs. Thus, our result here provides the strongest evidence so far in support of the conjecture. To proceed, we need to establish some notations and prove some properties of $PC(2k)$. To this end, we will have to provide proofs for some already known facts about projective cubes and then, using our techniques, strengthen these results.

We start by recalling a general fact about Cayley graphs on binary groups. Let G be a Cayley graph on a binary group and let ϕ be the associated edge-coloring. Suppose $v_1, v_2, \ldots, v_l, v_1$ is a closed walk of G (with this order of vertices). Then $\phi(v_1v_2) + \phi(v_2v_3) + \cdots + \phi(v_lv_1) = 2(v_1 + v_2 + \cdots + v_l) = 0$. In particular, this is the case for all cycles.

Consider the set $S_{2k} = \{e_1, e_2, \ldots, e_{2k}, J\}$. The only linear relations among elements of S_{2k} are $2x = 0$ for every x and $\sum_{x \in S_{2k}} x = 0$. This fact, together with the previous observation, partitions the set of cycles of $PC(2k)$ into two types: (i) cycles C where $\phi(v_1v_2) + \phi(v_2v_3) + \cdots + \phi(v_lv_1) = 0$ because every element of S_{2k} appears an even number of times as a color on the edges of C (0 as an even number is also allowed), and (ii) cycles C where $\phi(v_1v_2) + \phi(v_2v_3) + \cdots + \phi(v_lv_1) = \sum_{x \in S_{2k}} x = 0$, that is to say every element of S_{2k} appears as color on edges of C an odd number of times. As S_{2k} has an odd number of elements, this second type of cycles then corresponds to the class of odd cycles of $PC(2k)$ while the first type corresponds exactly to the class of even-cycles. As a consequence, we get that $PC(2k)$ has odd-girth $2k + 1$. But in fact a stronger statement follows, based on the following notation:

Let u and v be two vertices of $PC(2k)$, P be a shortest $u-v$ path and S_{uv} be the set of colors of the edges of P (thus $S_{uv} \subset S_{2k}$).

Theorem 6.3. *Given $PC(2k)$ and any two vertices u and v, the set S_{uv} corresponding to the colors of the edges of a shortest $u-v$ path P is independent of the choice of P and thus well-defined. Furthermore, given a $u-v$ path P' of length $2k+1-d(u,v)$, the set of colors of the edges of P' is $\bar{S}_{uv} = S_{2k} \setminus S_{uv}$ and is thus independent of the choice of P'.*

Proof. Let P be a $u-v$ path of length $d(u,v)$ and let S_{uv} be the set of colors of the edges of P. Our first claim is that $|S_{uv}| = d(u,v)$. Let $s_1, s_2, \dots, s_l$ be the elements of S_{uv} that appear an odd number of times on the edges of P. If $l = d(u,v)$, then each s_i must appear exactly once and we are done. Otherwise, consider the path $ux_1x_2\dots x_l$, with $x_1 = u + s_1$ and $x_i = x_{i-1} + s_i$ for $i = 2, \dots, l$. It follows that $x_l = v$ and that this is a shorter $u-v$ path.

Next, we claim that for $S'_{uv} = S_{2k} \setminus S_{uv}$ there is a $u-v$ path P' of length $2k+1-d(u,v)$ whose set of colors is S'_{uv}. Labeling elements of S'_{uv} as $s'_1, s'_2, \dots, s'_r$, one such a path P' is $uy_1y_2\dots y_r$, with $y_1 = u + s'_1$ and $y_i = y_{i-1} + s'_i$ for $i = 2, \dots, r$, noting that y_r must be v.

Observe that the color sets corresponding to the two paths P and P' form a partition of S_{2k} into S_{uv} (of size $d(u,v)$) and its complement S'_{uv} (of size $2k+1-d(u,v)$). Now, as every $u-v$ path Q of length d together with P' forms a closed walk of length $2k+1$, and since $PC(2k)$ has odd-girth $2k+1$, the set of colors corresponding to the edges of Q must be the complement of S'_{uv}, that is S_{uv}. Similarly, every $u-v$ path Q' of length $2k+1-d$ must have the complement of S_{uv} as its color set. □

This leads to the following labeling of the edges of the complete graph on 2^{2k} vertices using $PC(2k)$.

Definition 6.4. The $PC(2k)$-*edge-labeled complete graph* is the complete graph on the vertices of $PC(2k)$, where each edge uv is labeled by the partition $\{S_{uv}, \bar{S}_{uv}\}$.

Observe that by replacing each such label with the order of S_{uv} (that is the smallest of the two orders), we obtain the complete $PC(2k)$-distance graph, which is $(2k+1)$-wide since $PC(2k)$ is of odd-girth $2k+1$. Before we use this labeling to apply Theorem 4.3 on the complete $PC(2k)$-distance graph, we use it to prove a high level of symmetry in $PC(2k)$. To this end, we will need the following property of the complete $PC(2k)$-distance graph.

Proposition 6.5. *Let x, y and z be three vertices of the $PC(2k)$-edge-labeled complete graph. Let $A \in \{S_{xy}, \bar{S}_{xy}\}$, $B \in \{S_{xz}, \bar{S}_{xz}\}$ and $C \in \{S_{yz}, \bar{S}_{yz}\}$. Depending on the parity of $|A|+|B|+|C|$, one of the following holds:*

- *Each element of S_{2k} appears in an even number of the sets A, B or C, that is to say, either it appears in none of them, or in exactly two of them (this is the case when $|A|+|B|+|C|$ is even).*
- *Each element of S_{2k} appears in an odd number of the sets A, B or C, that is to say, it appears either in exactly one of them, or in all three of them (this is the case when $|A|+|B|+|C|$ is odd).*

Proof. Observe that the choice of A corresponds to the colors of the edges of an $x-y$ path P_A. Similarly, the choice of B and C corresponds to $x-z$ path P_B and $y-z$ path P_C, respectively. Thus $|A|+|B|+|C|$ corresponds to the length of the closed walk starting at x and traversing P_A, then P_C and then returning to x through P_B. Thus, the sum of the elements of A and B and C is 0. This means that if a color comes an odd number of times in the closed walk, then so do all other colors, and thus $|A|+|B|+|C|$ is an odd number. Otherwise, all colors appear an even number of times, which means $|A|+|B|+|C|$ is also an even number. □

Corollary 6.6. *Given subsets A, B and C of the previous proposition such that $|A|+|B|+|C|$ is odd, the number of elements that appear in all three of them is $\frac{1}{2}(|A|+|B|+|C|-(2k+1))$.*

This corollary is of importance for two main reasons: The first reason is that $|A|$, $|B|$ and $|C|$ are the edge-weights of a weighted triangle in the complete $PC(2k)$-distance graph induced by x, y and z. Thus, this triangle is $(2k+1)$-wide and the value $\frac{1}{2}(|A|+|B|+|C|-(2k+1))$ of this corollary corresponds to the value $f_{2k+1}(|A|,|B|,|C|)$ of Proposition 3.6. The second reason is that the number of elements appearing in all three sets A, B and C is a function of the order of A, B and C and does not depend on the choice of A, B, C. This implies a very high level of symmetry, as we will discuss below.

A graph G is said to be *distance-transitive* if for every two pairs (u, v) and (x, y) of vertices, if $d_G(u, v) = d_G(x, y)$, then there is an automorphism of G which maps u to x and v to y. Furthermore, G is said to be *triple transitive* if for every two triples (u, v, w) and (x, y, z), if $d_G(u, v) = d_G(x, y)$, $d_G(u, w) = d_G(x, z)$, $d_G(v, w) = d_G(y, z)$, then there is an automorphism ζ of G such that $\zeta(u) = x$, $\zeta(v) = y$ and $\zeta(w) = z$. In the next theorem, we give a proof of a result of [14] (see also [4]) using the terminology we have developed here which claims that $PC(2k)$ is triple transitive. The ideas developed in this proof are essential for the proof of Theorem 6.8.

Theorem 6.7. *The graph $PC(2k)$ is triple-transitive.*

Proof. In order to prove this we need to introduce the automorphisms of this graph. To this end, similar to the labeling of the edges of the complete $PC(2k)$-labeled graph on 2^{2k} vertices, and using this labeling of edges, we label the vertices of $PC(2k)$ also by partitions of S_{2k}. To start with, we label the vertex 0 by the trivial partition $\{\emptyset, S_{2k}\}$. Then, the vertex u receives the label of the edge $0u$, that is $\{S_{0u}, \bar{S}_{0u}\}$. Using this labeling, one easily observes that every permutation of S_{2k} is an automorphism of the complete $PC(2k)$-labeled graph and, therefore, also an automorphism of $PC(2k)$. In fact, this is the full list of automorphisms that has 0 as its fixed point, but we do not need to prove this fact here. Given a set $A \subset \{e_1, e_2, \ldots, e_{2k}\}$, one may get an automorphism of the complete $PC(2k)$-labeled graph (and thus of $PC(2k)$) by mapping the vertex labeled $\{X, \bar{X}\}$ to the vertex labeled $\{X \oplus A, \bar{X} \oplus A\}$ (if a is the vector in $\mathbb{Z}_2^{2k}$ whose support is A, then this automorphism corresponds to the mapping $u \to u + a$). Unless A is the empty set, this automorphism never fixes the vertex 0. By composing two of the automorphisms described above, one clearly gets an automorphism of the complete $PC(2k)$-labeled graph and, thus, of $PC(2k)$. This is indeed the full list of automorphisms of this graph, but we do not need to prove this stronger statement here either.

To prove the claim of the theorem, let $\{x, y, z\}$ and $\{u, v, w\}$ be two triples of vertices of $PC(2k)$ such that $d_{PC(2k)}(x, y) = d_{PC(2k)}(u, v)$, $d_{PC(2k)}(x, z) = d_{PC(2k)}(u, w)$ and $d_{PC(2k)}(y, z) = d_{PC(2k)}(v, w)$. Note that by taking a symmetric difference (more precisely, by the automorphism $t \to t + x - u$) one may map the vertex u to the vertex x. If v and w are mapped to v' and w' by this automorphism, then it is enough to prove that the triple $\{x, v', w'\}$ can be mapped to $\{x, y, z\}$ by an automorphism of $PC(2k)$. Hence, we may assume $u = x$ in the rest of this proof.

The assumption of $d_{PC(2k)}(x, y) = d_{PC(2k)}(x, v)$, $d_{PC(2k)}(x, z) = d_{PC(2k)}(x, w)$ and $d_{PC(2k)}(y, z) = d_{PC(2k)}(v, w)$ implies that $|S_{xy}| = |S_{xv}|$, $|S_{xz}| = |S_{xw}|$ and $|S_{yz}| = |S_{vw}|$. Consider the triangle induced by the vertices x, y, z in the complete $PC(2k)$-labeled graph. The edges of this triangle are labeled by $\{S_{xy}, \bar{S}_{xy}\}$, $\{S_{xz}, \bar{S}_{xz}\}$ and $\{S_{yz}, \bar{S}_{yz}\}$. Let $A \in \{S_{xy}, \bar{S}_{xy}\}$, $B \in \{S_{xz}, \bar{S}_{xz}\}$, and $C \in \{S_{yz}, \bar{S}_{yz}\}$ be such that $|A| + |B| + |C|$ is odd. Noting that $|S_{xy}| \leq k < \bar{S}_{xy}$, let A' be the element of $\{S_{xv}, \bar{S}_{xv}\}$ which is of the same order as A and define similarly B' and C'. To complete the proof, it is enough to find a permutation of S_{2k} which maps A to A', B to B' and C to C'.

Thanks to Corollary 6.6, we know that A, B, C have $\frac{1}{2}(|A| + |B| + |C| - (2k + 1))$ common elements (each appearing in all three of them) and every other element of S_{2k} appears in exactly one of them. The same holds for A', B' and C'. As $|A| = |A'|$, $|B| = |B'|$ and $|C| = |C'|$, the three sets A, B and C have as many common elements as A', B' and C'. Thus it is enough (and necessary) to choose a permutation of S_{2k} which maps the common elements of A, B, C to the common elements of A', B', C'. □

We are now ready to prove that the projective cube of dimension $2k$ satisfies the conditions of Theorem 4.3 for $t = 3$ and odd-girth $2k + 1$.

Theorem 6.8. *The set $\mathcal{W}$ of all 4-cliques of the complete $PC(2k)$-distance graph is $(3, 2k+1)$-closed.*

Proof. Our proof is based on Theorem 6.7, and on the terminology developed in this section and in the proof of Theorem 6.7. Since the projective cube $PC(2k)$ is triple-transitive, we only need to prove that, for each $(2k + 1)$-wide weighted graph (K_4, w), there is an isomorphic copy of (K_4, w) in $\mathcal{W}$.

Consider a weighted complete graph (K_4, w) on vertices x, y, z, t and assume it is $(2k + 1)$-wide. Suppose that $w(xy) = a', w(xz) = b', w(xt) = c', w(zt) = a, w(yt) = b$ and $w(yz) = c$. To show that there exists at least one isomorphic copy of this graph in the $PC(2k)$-distance graph, using the development in the proof of the previous theorem, it is enough to find subsets S_x, S_y, S_z and S_t of S_{2k} such that the symmetric difference of any two of them, or the complement of this set, has the same order as the weight on the corresponding edge. For example, we want the symmetric difference $S_x \oplus S_y$ to be of order either a' or $2k + 1 - a'$.

We recall that, by Proposition 6.5, with such a choice of the three subsets S_x, S_y and S_z, the three symmetric differences corresponding to the edges of the xyz-triangle, i.e., $S_x \oplus S_y$, $S_y \oplus S_z$ and $S_z \oplus S_x$, have the property that each element of S_{2k} is in an odd number of them, that is to say, each element is either exactly in one or in all three of them. Thus, to prove our theorem, in what follows, we show how to make this property hold for each triangle of the K_4.

If necessary, we change the weight of an edge to its complementary weight $2k+1-w$ to make sure that the elements of each pair $\{a,a'\},\{b,b'\},\{c,c'\}$ are of the same parity and the sum $a+b+c$ is odd. Therefore, the weight sums of the four triangles of (K_4,w) are $a+b+c$, $a+b'+c'$, $a'+b+c'$ and $a'+b'+c$, each being an odd number. Let $f_x = \frac{1}{2}(a+b+c-2k-1)$, $f_y = \frac{1}{2}(a+b'+c'-2k-1)$, $f_z = \frac{1}{2}(a'+b+c'-2k-1)$, and $f_t = \frac{1}{2}(a'+b'+c-2k-1)$. Since (K_4,w) is $(2k+1)$-wide, by Theorem 3.7, each of its four triangles is $(2k+1)$-wide. Then, by Proposition 3.6, we conclude that each of the four values f_v, $v \in \{x,y,z,t\}$, is a nonnegative integer not larger than the minimum of the three elements defining it.

Next, we will assign a subset $A_{vv'}$ of $S_{2k} = \{e_1, e_2, \ldots, e_{2k}, J\}$ to each edge vv' of K_4 such that $|A_{vv'}|$ (the order of $A_{vv'}$) is either $w(vv')$ or $2k+1-w(vv')$ and such that, for each triangle uvw of K_4, each element of $\{e_1, e_2, \ldots, e_{2k}, J\}$ appears either in exactly one of A_{uv}, A_{uw}, A_{vw} or in all of them, in order to satisfy Proposition 6.5 (here, u, v and w can be any triple among x,y,z,t; similarly v,v' can be any pair of vertices chosen from these four vertices).

Without loss of generality, assume that $f_x = \min\{f_x,f_y,f_z,f_t\}$. Our idea is to choose the six subsets $A_{vv'}$ of S_{2k} (v and v' being two distinct vertices of (K_4,w)) in such a way that an f_x number of the elements of S_{2k} belong to all six of them (unlike the other conditions, this is not the only way, depending on the weight function w of (K_4,w), there could be other types of solutions). Then, for the three edges not incident to x (i.e., the edges of the triangle yzt), we will partition the remaining elements of S_{2k} according to the weight of the corresponding edges. Next, for each of the three triangles, namely xyz, xyt and xzt, we need to pick, respectively, $f_t - f_x$, $f_z - f_x$ and $f_y - f_x$ more common elements. The details of the whole process is as follows.

Let S_x be a subset of $S_{2k} = \{e_1, e_2, \ldots, e_{2k}, J\}$ of order f_x. We partition the set $S_{2k} \setminus S_x$ into three disjoint subsets S_{yz}, S_{yt} and S_{zt} of order $|S_{yz}| = c - f_x$, $|S_{yt}| = b - f_x$, and $|S_{zt}| = a - f_x$. Observe that one of these sets might be empty, but we cannot have two of them empty. For example, if $c - f_x = b - f_x = 0$, then $c = \dfrac{1}{2}(a+b+c-(2k+1))$ and $b = \dfrac{1}{2}(a+b+c-(2k+1))$. Adding up these two identities, we conclude that $a = 2k+1$, but we are restricted to the weights from 1 to $2k$. Define $A_{yz} = S_{yz} \cup S_x$, $A_{yt} = S_{yt} \cup S_x$ and $A_{zt} = S_{zt} \cup S_x$.

Let S'_y, S'_z and S'_t be subsets of S_{2k} such that $S'_y \subset S_{zt}$ and $|S'_y| = f_y - f_x$, $S'_z \subset S_{yt}$ and $|S'_z| = f_z - f_x$, $S'_t \subset S_{yz}$ and $|S'_t| = f_t - f_x$. The set $S_y = S'_y \cup S_x$ is of order f_y, $S_z = S'_z \cup S_x$ is of order f_z and $S_t = S'_t \cup S_x$ is of order f_t. Let $A_{tx} = (S_{yz} - S'_t) \cup S_y \cup S_z$, $A_{zx} = (S_{yt} - S'_z) \cup S_y \cup S_t$ and $A_{yx} = (S_{zt} - S'_y) \cup S_z \cup S_t$.

In summary, we have built subsets A_{zt}, A_{yt}, A_{yz}, A_{tx}, A_{zx} and A_{yx} such that: (1) $|A_{yz}| = c$, $|A_{yt}| = b$, $|A_{tz}| = a$, $|A_{tx}| = c'$, $|A_{zx}| = b'$, $|A_{yx}| = a'$, (2) for each triangle induced by three vertices among $\{x,y,z,t\}$, in the three sets corresponding to the three edges, each element of S_{2k} appears either once or three times.

With these choices of $A_{vv'}$, we may choose the following set of four vertices in the $PC(2k)$-distance graph. The first vertex, say x, can be any vertex. The vertices y, z and t are defined as follows: $y = x + \sum\limits_{s_i \in A_{xy}} s_i$, $z = x + \sum\limits_{s_i \in A_{xz}} s_i$, $t = x + \sum\limits_{s_i \in A_{xt}} s_i$, where the addition is done in Z_2^{2k}. The fact that every element of S_{2k} is in either all three of A_{xy}, A_{xz} and A_{yz} or just in one of them, implies that $z = y + \sum\limits_{s_i \in A_{yz}} s_i$. For the same reason, similar relations hold between any two of the four vertices, that is to say, their binary difference is equal to the sum of the elements of the set associated with the corresponding edge.

Recall that the size of A_{uv} is either $d(u,v)$ or $2k+1-d(u,v)$. Thus, the four vertices x,y,z,t we have defined above will have distances in $PC(2k)$ equal to the corresponding weights in (K_4,w) that we have started with. This completes the proof. □

Remark. In the proof of this theorem, we chose S_x to be a subset of all the sets A_{uv}. Depending on the weight function w, other solutions may exist. This shows that $PC(2k)$ being triple transitive is the limit of the symmetries of this graph and that $PC(2k)$ is not 4-tuple transitive.

In support of Conjecture 6.2, we have the following theorem which is an immediate corollary of Theorem 4.3 and Theorem 6.8.

Theorem 6.9. *The projective cube of dimension* $2k$*,* $PC(2k)$*, is a graph of odd-girth* $2k+1$ *which bounds the class of partial 3-trees of odd-girth at least* $2k+1$*.*

Next, we claim that $PC(2k)$ is a smallest graph of odd-girth $2k+1$ which may satisfy the statement of the previous theorem. This claim is implicitly proved in [19] based on the following notation.

Definition 6.10. Given a graph G of odd-girth $2k+1$ and an integer l, $l<k$, the *walk-power* $G^{(2l+1)}$ of G is defined to be the graph on $V(G)$ where vertices u and v are adjacent if and only if there is a $(u-v)$-walk of length $2l+1$.

Given an odd number $2l+1$, traversing an edge uv an odd number of times is a $(u-v)$-walk of length $2l+1$. Thus, G is a subgraph of $G^{(2l+1)}$ for any choice of l. This is one of the main reasons because of which we only considered walks of odd length in this definition. The other reason is that the choice of an odd number $2l+1$ together with the condition of G being of odd-girth at least $2l+3$ ensure that $G^{(2l+1)}$ has no loop. The following easy lemma is the main interest of this definition for us:

Lemma 6.11. *Let* G *and* H *be two graphs of odd-girth at least* $2k+1$ *and let* $\phi : V(G) \to V(H)$ *be a homomorphism of* G *to* H*. Then,* ϕ *is also a homomorphism of* $G^{(2k-1)}$ *to* $H^{(2k-1)}$*.*

Thus, if a graph B of odd-girth $2k+1$ bounds a class $\mathcal{C}$ of graphs, then $|V(B)|$ is an upper bound on the order of the clique number of graphs in $\{G^{(2k-1)}|G \in \mathcal{C}\}$. To prove that $PC(2k)$ has the smallest order among graphs of odd-girth $2k+1$ bounding $\mathcal{PT}_{3,2k+1}$, we prove that this class of graphs has some weak sense of perfectness. That is to say, there is a graph $G \in \mathcal{PT}_{3,2k+1}$ for which the clique number of $G^{(2k-1)}$ is 2^{2k}. This approach was introduced in [19], where a planar graph G of odd-girth $2k+1$ satisfying $\omega(G^{(2k-1)}) = 2^{2k}$ was built. Our observation here is that the construction given there is also a partial 3-tree. Thus, we only describe the construction to verify that the result is a partial 3-tree, and refer the reader to [19] for the verification of $\omega(G^{(2k-1)}) = 2^{2k}$.

Theorem 6.12. *Given a positive integer* k *there exists a partial 3-tree* G *of odd-girth* $2k+1$ *for which the graph* $G^{(2k-1)}$ *has clique number* 2^{2k}*.*

Sketch of proof: We consider the following construction of a planar graph G of odd-girth $2k+1$ whose walk power $2k-1$ has a clique of order 2^{2k}. The graph G is built iteratively, starting with a particular subdivision G_0 of K_4 such that, in a planar embedding, all four faces are odd cycles of order $2k+1$. It is shown that for any such subdivision, the walk power $2k-1$ is a complete graph on $4k+2$ vertices. Given an already constructed graph G_i, if there is a $u-v$ path P all of whose internal vertices are of degree 2, then first add a new $u-v$ path P' of the same length as P, and parallel to P so P and P' make a new face of the plane graph, and then an internal vertex of P' is connected to a particular internal vertex of P in such a way that the PP' face is split into two faces, each being a $(2k+1)$-cycle, the result is the graph G_{i+1}. If $G_i^{(2k-1)}$ has a clique that uses l internal vertices of P, then we will find an extension of it in $G_{i+1}^{(2k-1)}$ which uses l more vertices. With a specific choice of a subdivision of K_4 and a specific rule for connecting P and P', at the end of this process we will have a graph G for which $G^{(2k-1)}$ has a 2^{2k}-clique.

What we observe furthermore are a few easy facts whose proofs we leave to the reader:

* K_4 is the first 3-tree.
* If G is a partial 3-tree and e is an edge of G, then any subdivision of e results in a partial 3-tree.
* If G is a partial 3-tree and P is a path whose internal vertices are of degree 2, then adding a vertex which is joined to (only) three internal vertices of P is also a partial 3-tree.

One can view the construction of [19], sketched above, as repeated applications of these three steps. Thus, the resulting graph at each step, and in particular at the final step, besides being a planar graph, is also a partial 3-tree. We have thus constructed a partial 3-tree G of odd-girth $2k+1$ for which $G^{(2k-1)}$ has clique number at least 2^{2k}. That the clique number of $G^{(2k-1)}$ is indeed 2^{2k} and not larger follows from the fact that G maps to $PC(2k)$ by Theorem 6.9 (this fact was not verified in [19]).

In summary we have proved:

Theorem 6.13. *The projective cube of dimension* $2k$*,* $PC(2k)$*, is a smallest graph, in terms of the number of vertices, of odd-girth* $2k+1$ *which bounds the class* $\mathcal{PT}_{3,2k+1}$*.*

7. Application to edge-coloring

The *edge-chromatic number* of a multigraph G, denoted $\chi'(G)$, is the smallest number of matchings into which $E(G)$ can be partitioned. The *fractional edge-chromatic number* of G, denoted $\chi'_f(G)$, is the smallest total sum of nonnegative weights assigned to the matchings of G such that the total weight of the matchings containing each given edge is at least 1. Given a subset S of an odd number of vertices, it can be verified that $\frac{2|E(G[S])|}{|S|-1}$ is a lower bound for $\chi'_f(G)$. Let $\Lambda(G) = \max_{|S| odd}\{\frac{2|E(G[S])|}{|S|-1}\}$. We refer to the textbook [22] for more on fractional coloring and for a proof and references to the following theorem:

Theorem 7.1. *Given a multigraph G we have $\chi'_f(G) = \max\{\Delta(G), \Lambda(G)\}$.*

Using this theorem, it can easily be verified that a cubic multigraph has fractional edge-chromatic number exactly 3 if and only if it is bridgeless. Thus Tait's classic reformulation of the four color theorem ([26], then a conjecture) is to say that the edge-chromatic number of a cubic bridgeless planar multigraph is equal to its fractional edge-chromatic number. With such a view, P. Seymour proposed the following strong generalization:

Conjecture 7.2. *[23] For every planar multigraph we have $\chi'(G) = \lceil \chi'_f(G) \rceil$.*

The conjecture is proved for K_4-minor-free graphs in [25] (a different proof is provided in [10]), and using this result it is proved in [13] for the larger class of K_5^--minor-free graphs, where K_5^- is the graph obtained from K_5 by removing one edge. As a direct generalization of Tait's statement, the case of k-regular planar multigraphs satisfying $\chi'_f(G) = k$ has got more attention. After verifying a condition under which a k-regular multigraph satisfies $\chi'_f(G) = k$, this special case of the conjecture can be restated as:

Conjecture 7.3. *S79 If G is a planar k-regular multigraph in which any set X of an odd number of vertices is connected to $V(G) \setminus X$ by at least k edges, then G is k-edge-colorable.*

The cases $k = 4, 5$ of this conjecture are proved in [11], the case $k = 6$ in [8], the case $k = 7$ in [9] (see also [5]) and the case $k = 8$ in [6]. Proofs are based on induction on k, thus dependent on the case $k = 3$ which is equivalent to the four color theorem. The case $k = 4$ is known to imply the four color theorem and one cannot expect an independent proof, however, it is not known whether the cases $k \geq 5$ are stronger than the four color theorem, though we expect them to be.

In this section, providing support for the general case of Conjecture 7.3, and as an application of our work, we prove that Conjecture 7.3 holds on the subclass of $(2k + 1)$-regular planar multigraphs whose duals are partial 3-trees. This subclass of planar graphs can be characterized as a minor-closed family of graphs with four forbidden minors given in Figure 4. Observe that two of these forbidden minors are K_5 and $K_{3,3}$, whose absence as a minor implies planarity, and that the other two are duals of the two planar graphs of Figure 1.

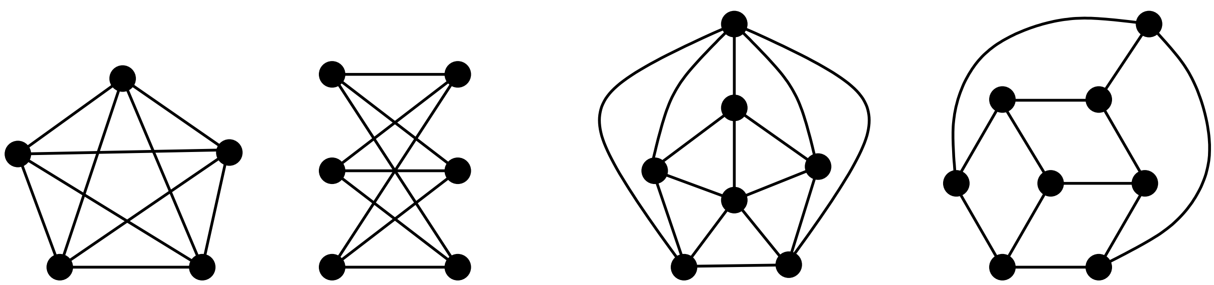


Figure 4: Forbidden minors of the class of planar graphs whose duals are partial 3-trees

Theorem 7.4. *Let G be a planar $(2k+1)$-regular multigraph whose dual is a partial 3-tree. Furthermore, assume that for each subset X of an odd number of vertices, the number of edges connecting X to $V(G)\setminus X$ is at least $2k + 1$. Then, $\chi'(G) = 2k + 1$.*

Proof. Let G be a planar $(2k+1)$-regular multigraph, together with a planar embedding, and G^D be its dual together with the corresponding planar drawing. Let X be a set of vertices of G and let $\delta(X)$ be the set of edges with one end in X and another end in $V(G) \setminus X$. Each vertex of G corresponds to a facial cycle of G^D, which is a cycle of length $2k+1$. More generally, given a set X of vertices of G, $\delta(X)$ corresponds to a cycle C_X of length $|\delta(X)|$ which bounds $|X|$ faces of G^D. As all the facial cycles are of odd-length, the parity of C_X is determined by the parity of $|X|$. In other words, C_X is an odd cycle if and only if X contains an odd number of vertices of G. Thus, the condition on G that for every set X of an odd number of vertices $\delta(X)$ has at least $2k+1$ edges, is equivalent to the statement that G^D has odd-girth $2k+1$.

Since G^D has odd-girth $2k+1$, and since we have assumed it is a partial 3-tree, by Theorem 6.9 it admits a homomorphism to $PC(2k)$. Under any such mapping, the image of each $(2k+1)$-cycle of G^D is a $(2k+1)$-cycle of $PC(2k)$. Recall that the edges of $PC(2k)$ are colored by the elements of the $(2k+1)$-set $S_{2k} = \{e_1, e_2, \ldots, e_{2k}, J\}$, where each $(2k+1)$-cycle receives each color exactly once. Let ϕ be a mapping of G^D to $PC(2k)$; we color the edges of G^D using S_{2k} as the color set, assigning to each edge e the color of $\phi(e)$ in $PC(2k)$. Consider a facial cycle C_v corresponding to a vertex v of G. As v is of degree $2k+1$, C_v is a $(2k+1)$-cycle which maps to a $(2k+1)$-cycle of $PC(2k)$. By Theorem 6.3, the set of colors assigned to C_v is the whole set S_{2k} having each color used exactly once. We may now color the edges of G by assigning to each edge the color of its corresponding edge in G^D. As the edges incident to a vertex v correspond to the edges of a $(2k+1)$-cycle of G^D, they receive distinct colors. □

We restate the theorem also using the set of four forbidden minors:

Theorem 7.5. *Let G be a $(2k+1)$-regular multigraph which does not contain any of the four graphs of Figure 4 as a minor. Furthermore, assume that for each subset X of an odd number of vertices, the number of edges connecting X to $V(G) \setminus X$ is at least $2k+1$. Then $\chi'(G) = 2k+1$.*

8. Remarks and discussion

At the end we have a few remarks:

1. We gave an independent proof that $PC(2k)$ is triple-transitive. The proof is based on the fact that given triples x, y, z and subsets A, B and C of S_{2k} corresponding to three edges of the triangle induced by xyz in the $PC(2k)$-distance complete graph, if $|A|+|B|+|C|$ is odd, then each element of S_{2k} is either in all three of them, or exactly in one of them. Furthermore, the number of elements in all three of them is only a function of the distances between x, y and z. But if a set of four vertices is selected, then the common elements among six sets corresponding to the six edges of K_4 is no longer determined uniquely by the distances between the six pairs. Thus, we can easily show that $PC(2k)$ is not 4-tuple-transitive. When viewed as signed graphs, these properties extend to $PC(2k-1)$ as well. This will be addressed in forthcoming work.

2. We find it rather surprising that a sort of perfectness holds for the class of partial 3-trees in the following sense. The order of a smallest graph of odd-girth $2k+1$ to which every partial 3-tree of odd-girth at least $2k+1$ admits a homomorphism, the largest chromatic number of $G^{(2k-1)}$ when G is a partial 3-tree of odd-girth at least $2k+1$ and the largest clique number of $G^{(2k-1)}$ when G is a partial 3-tree of odd-girth at least $2k+1$ are all 2^{2k}. This is not necessarily the case for the class of partial t-trees, $t \neq 3$. For example, for the class of partial 2-trees, which is the same as the class of K_4-minor-free graphs, the smallest triangle-free graph to which every triangle-free graph with no K_4-minor admits a homomorphism is of order 8, but the largest clique one can find in $G^{(3)}$ where G is a triangle-free graph with no K_4-minor is 6 (ongoing work of W. He, the second author and Q. Sun, [12]).

3. We showed that $PC(2k)$ has the smallest number of vertices among all graphs of odd-girth $2k+1$ which bounds all partial 3-trees of odd-girth at least $2k+1$. It follows from a simple construction in [16] that any such minimal bound must be of minimum degree at least $2k+1$. Thus, $PC(2k)$ is also optimal in terms of the number of edges.

4. Using terminology and discussions of [17] and [18], the question of bounding signed bipartite partial t-trees will be addressed in a forthcoming work. Using such results, we will be able to present a $2k$-regular analogue of Theorem 7.4.

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