Critically 3-frustrated signed graphs

Chiara Cappello[1], Reza Naserasr[2], Eckhard Steffen[1], and Zhouningxin Wang[3]

[1]Paderborn University, Department of Mathematics, Warburger Str. 100, 33098 Paderborn, Germany.
[2]Université Paris Cité, CNRS, IRIF, F-75013, Paris, France.
[3]School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China.
Emails: {ccappello, es}@mail.uni-paderborn.de, reza@irif.fr, wangzhou@nankai.edu.cn

Abstract

Extending the notion of maxcut, the study of the frustration index of signed graphs is one of the basic questions in the theory of signed graphs. Recently two of the authors initiated the study of critically frustrated signed graphs. That is a signed graph whose frustration index decreases with the removal of any edge. The main focus of this study is on critical signed graphs which are not edge-disjoint unions of critically frustrated signed graphs (namely indecomposable signed graphs) and which are not built from other critically frustrated signed graphs by subdivision. We conjecture that for any given k there are only finitely many critically k-frustrated signed graphs of this kind.

Providing support for this conjecture we show that there are only two of such critically 3-frustrated signed graphs where there is no pair of edge-disjoint negative cycles. Similarly, we show that there are exactly ten critically 3-frustrated signed planar graphs that are neither decomposable nor subdivisions of other critically frustrated signed graphs. We present a method for building indecomposable critically frustrated signed graphs based on two given such signed graphs. We also show that the condition of being indecomposable is necessary for our conjecture.

1 Introduction

In this paper, graphs are allowed to have multiedges and loops. For a graph G, let $E(G)$ and $V(G)$ denote the set of edges and the set of vertices of G, respectively. A *signed graph* (G, σ) is a graph G together with an assignment $\sigma : E(G) \to \{+, -\}$ called *signature*, where $\{+, -\}$ is viewed as a multiplicative group. An edge e of (G, σ) is called *negative* if $\sigma(e) = -$ and *positive* otherwise. The set of negative edges of (G, σ) is denoted by $E^-_\sigma(G)$. We may simply write E^-_σ if the underlying graph is clear from the context.

Furthermore, if $E^-_\sigma(G) = \emptyset$, then such a signature σ is called the *all-positive* signature, and the corresponding signed graph is denoted by $(G, +)$. Similarly, if $E^-_\sigma(G) = E(G)$, then σ is called the *all-negative* signature, and the corresponding signed graph is denoted by $(G, -)$. For a signed graph (G, σ) and a subgraph H of G, with rather an abuse of notation, we write (H, σ) to denote the signed graph $(H, \sigma|_{E(H)})$.
Let G be a graph. For a vertex $v \in V(G)$, the *degree* of v, denoted $d_G(v)$, or simply $d(v)$ when G is clear from the context, is the number of edges incident with v, where loops are counted twice. For $X \subseteq V$, we denote by $\partial_G(X)$ an *edge-cut* in G, which is a set of edges defined as follows: $\partial_G(X) := \{xy \in E(G) : x \in X, y \notin X\}$. The cardinality of an edge-cut $\partial_G(X)$ is denoted by $d_G(X)$, i.e., $d_G(X) = |\partial_G(X)|$. In particular, when we work with signed graphs without loops, if $X = \{v\}$, then in place of $d_G(\{v\})$ we write $d_G(v)$. Given a signature σ on G, a refinement of the notation $d_G(X)$ for (G, σ) is defined as follows:

$$d^-_{(G,\sigma)}(X) := |\partial_G(X) \cap E^-_\sigma| \text{ and } d^+_{(G,\sigma)}(X) := |\partial_G(X) \setminus E^-_\sigma|.$$

An edge-cut $\partial_G(X)$ is said to be *equilibrated* under σ if $d^+_{(G,\sigma)}(X) = d^-_{(G,\sigma)}(X)$. Whenever it is clear from the context, we may omit the index "(G, σ)" of the notations introduced above.
A *cycle* of G is a connected 2-regular subgraph. A cycle in (G, σ) is said to be *positive* (respectively, *negative*) if it contains an even (respectively, odd) number of negative edges. A signed graph (G, σ) is *balanced* if it contains no negative cycle and *unbalanced* otherwise.

For a signed graph (G, σ), *switching* at a vertex $v \in V(G)$ is to multiply the signs of all edges incident with v by $-$. For an edge-cut $\partial(X)$, switching at $\partial(X)$ is to switch at all vertices in X. Furthermore, two signed graphs (G, σ) and (G, σ') are *switching equivalent* if one can be obtained from the other by a series of switchings at some vertices. In such a case we may also say σ is switching equivalent to σ'. It has been proved by Zaslavsky in [9] that two signatures on the same graph are switching equivalent if and only if they induce the same set of negative cycles.
As being balanced is a desirable state, and since most signed graphs are not balanced, there has been many measures to decide how far a signed graph is from being balanced. See for example [2] and references therein. One of the basic parameters to measure this is the *frustration index* of a signed graph (G, σ), denoted by $\ell(G, \sigma)$, which is defined as follows:

$$\ell(G, \sigma) = \min_\pi \{|E^-_\pi| : (G, \pi) \text{ is switching equivalent to } (G, \sigma)\}.$$

If $\ell(G, \sigma) = k$, then (G, σ) is said to be *k-frustrated.* For a signed graph (G, σ), a signature π is said to be a *minimal equivalent signature* of (G, σ) if (G, π) is switching equivalent to (G, σ) and there is no equivalent signature π' of σ such that $E^-_{\pi'} \subset E^-_\pi$. In particular, a *minimum equivalent signature*, or simply a *minimum signature*, of (G, σ) is a signature π such that (G, π) is switching equivalent to (G, σ) and $|E^-_\pi| = \ell(G, \sigma)$.

A subset E' of the edges of (G, σ) is said to be *negative-cycle cover* if for each negative cycle C of (G, σ) we have $E(C) \cap E' \neq \emptyset$. A negative-cycle cover is minimal if no proper subset of it is a negative-cycle cover. The next lemma, which is well known, and to our knowledge first appears in [3], provides an equivalent definition of the frustration index of signed graphs. For completeness we provide a short proof of the lemma.

Lemma 1.1. *Given a signed graph (G, σ) a subset E' of the edges of G is a minimal negative cycle cover if and only if it is the set of negative edges of a minimal signature equivalent to σ.*

Proof. First, note that any signature, in particular a minimal signature, is a negative-cycle cover. Let E' be the set of edges of a minimal negative-cycle cover. We claim that E' is the set of negative edges of a minimal equivalent signature of (G, σ). To this end, observe that $(G - E', \sigma)$ is balanced and thus, it can be changed to $(G - E', +)$ by switching at a set X of vertices. Then, after switching at X, the set of negative edges of the resulting signed graph has to be E'. Otherwise, the set of negative edges is a proper subset of E' and also a negative-cycle cover, contradicting the minimality of E'. This also implies that a negative-cycle cover provided by a minimal equivalent signature is minimal, as otherwise, an included minimal negative-cycle cover would imply a smaller signature. □

1.1 Critically k-frustrated signed graphs

As computing the frustration index of a signed graph $(G, -)$ is equivalent to computing the size of a maximum cut of G, the problem of computing $\ell(G, \sigma)$ for an input signed graph (G, σ) is an NP-hard problem [1]. This motivates the study of the structure of signed graphs with high frustration index. For example, a basic observation is that the existence of k edge-disjoint negative cycle in (G, σ) implies $\ell(G, \sigma) \geq k$. To better understand the structural properties of signed graphs with the frustration index being at least k, the notion of critically k-frustrated signed graphs is introduced in [5]. This notion is formally defined as follows.

Definition 1.2. *A signed graph (G, σ) is* critically k-frustrated *if $\ell(G, \sigma) = k$ and for each edge $e \in E(G)$, we have $\ell(G - e, \sigma) = k - 1$.*

We note that critically k-frustrated signed graphs can be characterized in the following way.

Theorem 1.3. [5] *Let k be a positive integer and (G, σ) be a k-frustrated signed graph. The following statements are equivalent.*

(1) (G, σ) is critically k-frustrated.

(2) For each edge $e \in E(G)$, there exists a minimum signature σ' of (G, σ) such that $\sigma'(e) = -$.

(3) If $|E^-_\sigma| = \ell(G, \sigma)$, then every positive edge of (G, σ) is contained in an equilibrated edge-cut under σ.

Note that, given a critically k-frustrated signed graph (G, σ) with σ being a minimum signature, for each edge-cut $\partial(X)$ it holds that $d^-(X) \leq d^+(X)$.

Given positive integers $k, k_1, \ldots, k_t$ such that $k = \sum_{i=1}^{t} k_i$, a critically k-frustrated signed graph (G, σ) is said to be $(k_1, \ldots, k_t)$-*decomposable* if $E(G)$ can be partitioned into t parts $E_1, E_2, \ldots, E_t$ such that for each i, $i \in \{1, 2, \ldots, t\}$, the signed subgraph $(G[E_i], \sigma)$ is critically k_i-frustrated. If (G, σ) is $(k_1, \ldots, k_t)$-decomposable for some $t \geq 2$, then we simply say it is decomposable. A critically frustrated signed graph that is not decomposable is said to be *indecomposable*.

Observation 1.4. *Let (G, σ) be a critically k-frustrated signed graph. If for $k_1, \ldots, k_t$ with $k = \sum_{i=1}^{n} k_i$ we find edge-disjoint signed subgraphs (G_i, σ) of (G, σ) such that (G_i, σ) is k_i-frustrated for $i \in [t]$, then $E(G) = \bigcup_{i \in [t]} E(G_i)$ and thus (G, σ) is $(k_1, \ldots, k_t)$-decomposable.*
In particular, a critically k-frustrated signed graph containing k edge-disjoint negative cycles is the union of all these negative cycles.

Note also that, if a critically k-frustrated signed graph (G, σ) contains two parallel edges e_1 and e_2 having different signs, then each equilibrated cut of (G, σ) is also an equilibrated cut of $(G - \{e_1, e_2\}, \sigma)$. Hence, by Theorem 1.3, the following observation holds.

Observation 1.5. *Let (G, σ) be a critically k-frustrated signed graph. If (G, σ) contains a loop, then the loop is negative and (G, σ) is decomposable. If (G, σ) contains two parallel edges of different signs, then (G, σ) is decomposable.*

Since a decomposable signed graph relies on the structures of its critical subgraphs, in the following, we choose to focus on critically frustrated signed graphs which are not decomposable. In particular, from here on, the signed graphs that we consider have no loop and no parallel edges of different signs.
Another graph operation to preserve the property of being critically frustrated is the following (modified) notion of *subdivision* as introduced in [5]. For a signed graph (G, σ) and a positive integer t, a t-*multiedge* between two vertices x, y of G is a set of t edges connecting x and y, denoted by E_{xy}. As we have assumed above (and from here on) that there are no parallel edges of different signs, all the edges of a t-multiedge E_{xy} are of the same sign and, depending on this sign, it will be referred to as an *all-positive* or *all-negative* t-multiedge.
Given a signed graph (G, σ) and an all-positive (resp. all-negative) t-multiedge E_{xy} of (G, σ), let (G', σ') denote the signed graph obtained from $(G - E_{xy}, \sigma)$ by adding a new vertex v, and adding two t-multiedges E_{xv} and E_{vy}, so that both E_{xv} and E_{vy} are all-positive (resp. E_{xv} is all-negative and E_{vy} is all-positive). We say that (G', σ') is obtained from (G, σ) by *subdividing* at a multiedge E_{xy}. If a signed graph (H, π) (not isomorphic to (G, σ)) is obtained by subdividing at a series of multiedges of (G, σ), then we say that (H, π) is a (proper) *subdivision* of (G, σ). Equivalently we may say (H, π) is a (G, σ)-subdivision. Observe that, up to a switching, the signature in (H, π) is determined by the sequence of subdivisions and the signature of (G, σ).
If a signed graph (G, σ) is not a proper subdivision of any signed graph, then we say that (G, σ) is *irreducible*. Note that, if (G, σ) is decomposable, then all its subdivisions are also decomposable. See Figure 1 for examples.
The importance of this generalized notion of the subdivision in the study of critically frustrated signed graphs is highlighted in the following proposition.

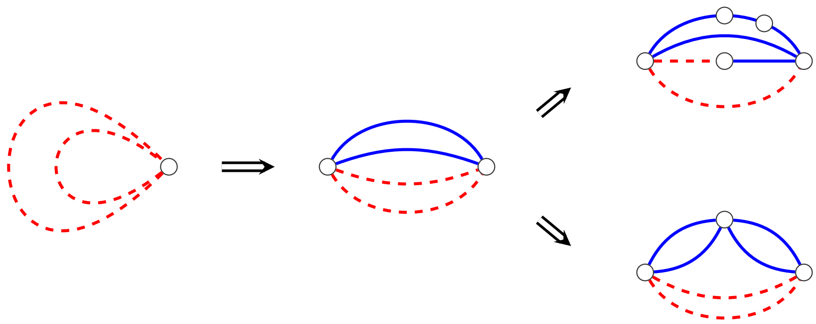


Figure 1: An example of possible subdivisions.

Proposition 1.6. [5] *For a signed graph (G, σ) and a subdivision (G', σ') of it, we have: (i). $\ell(G, \sigma) = \ell(G', \sigma')$; (ii). (G, σ) is critically frustrated if and only if (G', σ') is critically frustrated.*

Hence, without loss of generality, we can always limit our study to the class of irreducible signed graphs. It follows that for $k \geq 2$ every irreducible critically k-frustrated connected signed graph (G, σ) satisfies that $d(v) \geq 3$ for each $v \in V(G)$.

Let $\mathcal{L}(k)$ be the family of irreducible critically k-frustrated signed graphs and let $\mathcal{L}^*(k)$ be the family of irreducible indecomposable critically k-frustrated signed graphs.

Theorem 1.7. [5] *We have $\mathcal{L}(1) = \mathcal{L}^*(1) = \{C_{-1}\}$, $\mathcal{L}(2) = \{C_{-1} \cup C_{-1}, 2C_{-1}, (K_4, -)\}$ and $\mathcal{L}^*(2) = \{(K_4, -)\}$.*

Here C_{-1} is the signed graph on one vertex with a negative loop, $C_{-1} \cup C_{-1}$ is two disjoint copies of it, and $2C_{-1}$ is the signed graph on one vertex with two negative loops on it. Note that if (H, π) is a $(K_4, -)$-subdivision, then $\ell(H, \pi) = 2$. The following is one of the first structural results on k-frustrated signed graphs.

Theorem 1.8. [6] *If a k-frustrated signed graph contains no $(K_4, -)$-subdivision, then it contains k edge-disjoint negative cycles.*

We have seen that each of the families $\mathcal{L}^*(1)$ and $\mathcal{L}^*(2)$ is quite small and is precisely described. For $k \geq 3$, we conjecture the following.

Conjecture 1.9. *The set $\mathcal{L}^*(k)$ is finite for any positive integer k.*

A special subclass $\mathcal{S}^*(k)$ of $\mathcal{L}^*(k)$ consists of those elements (G, σ) in $\mathcal{L}^*(k)$ satisfying that for each integer $m \leq k$, every critically m-frustrated subgraph of (G, σ) is indecomposable. A relaxation of Conjecture 1.9 is that the set $\mathcal{S}^*(k)$ is finite.

Conjecture 1.10. *The set $\mathcal{S}^*(k)$ is finite for any positive integer k.*

A restriction of Conjecture 1.9 to signed plane graphs is also of special interest. More precisely, we ask:

Conjecture 1.11. *Every indecomposable critically k-frustrated signed plane graph has exactly $2k$ facial cycles each of which is a negative cycle.*

A similar conjecture on the structure of critically k-frustrated signed graphs is the following.

Conjecture 1.12. *Every critically k-frustrated signed graph (G, σ) satisfies $\Delta(G) \leq 2k$. Moreover, $\Delta(G) = 2k$ can only happen if G consists of k negative cycles pairwise edge-disjoint but all containing the same vertex.*

In this work, we verify Conjectures 1.10 and 1.11 for $k = 3$. The case $k = 3$ of Conjecture 1.9 will be addressed in a forthcoming paper. We show that the condition of being indecomposable is necessary in Conjectures 1.11 and 1.9. To support Conjecture 1.12, we prove it is true for a special class of signed graphs.

The rest of the paper is organized as follows. In Section 2 we show that the class $\mathcal{S}^*(3)$ contains exactly two elements. In Section 3 we show that, up to decomposition and subdivision, there exist ten critically 3-frustrated signed planar graphs. In Section 4 we present a construction of indecomposable critically frustrated signed graphs from two given such signed graphs. Moreover, we provide an infinite family of decomposable (plane) critically 3-frustrated signed graphs showing that in both Conjectures 1.9 and 1.11 the assumption of being indecomposable is necessary even for $k = 3$. In Section 5, we give a family of critically k-frustrated signed graphs having maximum degree at most $2k$.

2 Characterization of the family $\mathcal{S}^*(3)$

To characterize the elements of $\mathcal{S}^*(3)$, we use the following results.

Proposition 2.1. [5] *Let (G, σ) be an irreducible critically k-frustrated signed graph for some positive integer k. Then $(G, \sigma) \in \mathcal{S}^*(k)$ if and only if (G, σ) contains no pair of edge-disjoint negative cycles.*

Using this fact and the characterization of signed graphs in which every two negative cycles intersect, provided in [7], the elements of $\mathcal{S}^*(k)$ can be characterized as follows. This characterization is based on embedding of (signed) graphs on the *projective plane*. Here the projective plane is viewed as a disk where the antipodal pairs of the boundary are identified where the boundary is referred to as *cross cap*. A graph embedded in this plane where two edges can only share a point on their end points is called *projective planar*, or a *projective plane* graph when the embedding is fixed. See Figure 6 for examples of projective plane graphs.

Theorem 2.2. [5] *Let (G, σ) be a signed graph in $\mathcal{S}^*(k)$ where σ is a minimum signature with $E^-_\sigma = \{x_1y_1, \ldots, x_ky_k\}$. Then we have the following.*

(1) Either $k = 1$, in which case (G, σ) is a negative loop.

(2) Or $k \geq 2$, G is a cubic projective plane graph where the edges passing through the cross cap are those of E^-_σ.

Let $(G,\sigma) \in \mathcal{S}^*(k)$ with σ being a minimum signature. An embedding of (G,σ) into the projective plane as described in Theorem 2.2 is called a *canonical projective-planar embedding* of (G,σ).
Let $(G,\sigma) \in \mathcal{S}^*(k)$ be a canonically projective-planar embedded signed graph. When the choice of the minimum signature σ is clear from the context, we will denote the subgraph $G - E_\sigma^-$ of G by G'. Moreover, G' will always be considered together with its planar embedding that is implied from Theorem 2.2. The facial cycle of the outer face of this plane graph G' will be denoted by C_O. One may observe that in G' the vertices $x_1, \dots, x_k, y_1, \dots, y_k$ are all of degree 2 and they appear on C_O in this cyclic order.
Given vertices u and v of C_O, by A_{uv} we denote the path on C_O connecting u to v which is in the clockwise direction starting at u and ending at v. When referring to a face of G' we do not consider the outer face. Thus a face F of G' is also a face of (G,σ) in the projective-planar embedding from which G' is defined. The boundary of this face F, which must be a cycle, will be denoted by C_F.
A face of G' is said to be *internal* if its boundary shares no edge with C_O. We note that, since G' is subcubic, the boundary of an internal face does not intersect C_O at a vertex either. In particular, the boundary of a face F which is not internal shares at least two vertices with C_O. We classify such faces depending on how many of those common vertices are in the set $\mathcal{R} = \{x_1, ..., x_k, y_1, ..., y_k\}$. More precisely, a face F is said to be an *i-face* of G' if C_F contains i elements from the set $\mathcal{R}$. Two faces F_1 and F_2 are said to be *adjacent on the boundary* if $V(C_{F_1} \cap C_{F_2} \cap C_O) \neq \emptyset$. A face F of G' is called a *bridge-face* if the subgraph induced by $C_F \cap C_O$ is disconnected. See Figure 2 for an example, noting that curves represent paths that might contain more vertices.

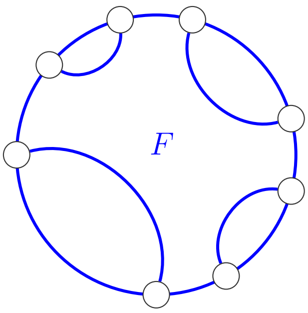


Figure 2: A bridge-face F in G'

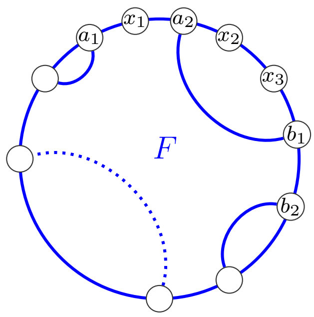


Figure 3: Proposition 2.5

Note that in (G,σ), each edge-cut with negative edges contains at least two edges of C_O. Furthermore, based on the cyclic order of the elements of $\mathcal{R}$ on C_O, we have the following observation.

Observation 2.3. *Let (G,σ) be a canonically projective-planar embedded signed graph in $\mathcal{S}^*(k)$ for $k \geq 2$. If an edge-cut $\partial_G(X)$ contains exactly two edges e_1 and e_2 of C_O, then $d^-_{(G,\sigma)}(X) = \min\{|V(A_1) \cap \mathcal{R}|, |V(A_2) \cap \mathcal{R}|\}$ where A_1 and A_2 are the two connected components of $C_O - \{e_1, e_2\}$.*

Lemma 2.4. *Let (G,σ) be a canonically projective-planar embedded signed graph in $\mathcal{S}^*(k)$ for $k \geq 2$. Assume that F is a bridge-face and let $A_{a_1a_2}$ and $A_{b_1b_2}$ be two connected* *components of $C_F \cap C_O$ such that $A_{a_2b_1}$ is a connected component in $C_O \setminus C_F$. Then $|V(A_{a_2b_1}) \cap \mathcal{R}| \in \{2, 2k-2\}$.*

Proof. Let e_1 (resp. e_2) be the edge in $A_{a_1a_2}$ (resp. $A_{b_1b_2}$) that has a_2 (resp. b_1) as an endpoint. Let G'' be the connected component of $G' - \{e_1, e_2\}$ containing a_2 (and b_1).
We first show that $|V(A_{a_2b_1}) \cap \mathcal{R}| \notin \{3, 4, \ldots, 2k-3\}$. Otherwise, by Observation 2.3 the edge-cut $\partial_G(V(G''))$ must contain at least three negative edges, but it has only two positive edges. This contradicts the fact that σ is a minimum signature.
Next we show that $|V(A_{a_2b_1}) \cap \mathcal{R}| \leq 1$ is not possible either. That $|V(A_{a_2b_1}) \cap \mathcal{R}| \geq 2k-1$ is not possible follows similarly. Suppose to the contrary that $|V(A_{a_2b_1}) \cap \mathcal{R}| \leq 1$. In $C_F - \{e_1, e_2\}$, there is a path connecting a_2 to b_1 and let e be an edge of this path. By criticality, there exists an equilibrated edge-cut $\partial(X)$ containing e. Since each equilibrated cut of (G, σ) contains at least two (positive) edges from C_O and noting that e is also a positive edge, we have $d^+(X) \geq 3$, and hence $d^-(X) \geq 3$. Moreover, by the choice of e, at least one of the edges of $A_{a_2b_1}$, say e', is in $\partial(X)$. Thus in total, at least two edges of G'' are in $\partial(X)$. We now consider the following two edge-cuts: $E_1 = \partial(X) \setminus \{e, e'\} \cup \{e_1\}$ and $E_2 = \partial(X) \setminus \{e, e'\} \cup \{e_2\}$. Since $|V(A_{a_2b_1}) \cap \mathcal{R}| \leq 1$, it follows that one of these two edge-cuts say E_1, has the same set of negative edges as $\partial(X)$. However, E_1 has fewer positive edges than $\partial(X)$, contradicting the minimality of σ. $\square$

Proposition 2.5. *Let (G, σ) be a canonically projective-planar embedded signed graph in $\mathcal{S}^*(k)$ for $k \geq 3$. Then we have the following:*

(i) Every bridge-face of G' is a 0-face.

(ii) For $i \geq 3$ there is no i-face in G'.

Proof. (i) Let F be a bridge-face of G' and assume that $C_F \cap C_O$ consists of t connected components (thus $t \geq 2$). Let $A_{a_1a_2}$ and $A_{b_1b_2}$ be two connected components of $C_F \cap C_O$ such that $A_{a_2b_1}$ is a connected component in $C_O \setminus C_F$. By Lemma 2.4, $|V(A_{a_2b_1}) \cap \mathcal{R}| \in \{2, 2k-2\}$. If a connected component of $C_O \setminus C_F$ contains $2k-2$ vertices from $\mathcal{R}$, then since $|\mathcal{R}| = 2k$, there is only one other component in $C_O \setminus C_F$. Furthermore, this component must contain the other two vertices of $\mathcal{R}$. This in turn implies that F is a 0-face. Thus we may assume that each connected component of $C_O \setminus C_F$ contains exactly two vertices from $\mathcal{R}$.
Toward a contradiction and without loss of generality, assume that $x_1 \in \mathcal{R} \cap V(A_{a_1a_2})$ and $x_2, x_3 \in \mathcal{R} \cap V(A_{a_2b_1})$, depicted in Figure 3. Let e_1 be the edge on $A_{a_1x_1}$ incident with x_1 and let e_2 be the edge on $A_{b_1b_2}$ incident with b_1. Then the set $\{e_1, e_2, x_1y_1, x_2y_2, x_3y_3\}$ is an edge-cut consisting of two positive edges and three negative edges, contradicting the fact that σ is a minimum signature.

(ii) Suppose to the contrary that F is an i-face of G' for $i \geq 3$. By Claim (i), we know that F is not a bridge-face. Therefore, by the symmetry of labeling, we assume that

$x_1, x_2, x_3 \in V(C_F) \cap \mathcal{R}$. Let $e_1 = vx_1, e_2 = x_3u \in E(C_F \cap C_O)$ such that $v \notin V(A_{x_1x_2})$ and $u \notin V(A_{x_2x_3})$. Then the edge set $\{e_1, e_2, x_1y_1, x_2y_2, x_3y_3\}$ is an edge-cut that contains three negative edges but only two positive edges, a contradiction. □

From now on, we focus on the family $\mathcal{S}^*(3)$. We give some structural properties of signed graphs in $\mathcal{S}^*(3)$ in the following lemmas.

Lemma 2.6. *Let (G, σ) be a canonically projective-planar embedded signed graph in $\mathcal{S}^*(3)$. Then each face of G' is either a bridge-face or an i-face for $i \in \{1, 2\}$.*

Proof. By Proposition 2.5 (ii), if F is an i-face of G', then $i \in \{0, 1, 2\}$. It remains to show that there are no internal faces and that every 0-face is a bridge-face.

For the first claim, assume to the contrary that there exists an internal face F of G'. Note that each equilibrated cut containing one edge of C_F must have at least two (positive) edges from C_F and two (positive) edges from C_O. However, there are only three negative edges in (G, σ), contradicting the fact that each equilibrated cut has the same number of positive and negative edges.

For the second claim, assume that F is a 0-face of G' which is not a bridge-face. As C_F shares at least one edge with the outer facial cycle C_O of G', there is a face F' such that $C_{F'}$ shares a common vertex with both C_F and C_O. Assume that F' is an i-face for $i \in \{0, 1, 2\}$. Let e_0 be a (positive) edge in the path $C_F \cap C_{F'}$. Let $\partial(X)$ be an equilibrated cut containing e_0. Recall that any equilibrated cut must contain at least two edges of C_O. As $(G, \sigma) \in \mathcal{S}^*(3)$, $\partial(X)$ contains exactly two edges of C_O. Furthermore, one of these two edges belongs to $E(C_O \cap C_F)$ while the other is in $E(C_O \cap C_{F'})$. To complete the proof, it suffices to show that $|X \cap \mathcal{R}| \neq 3$, which would contradict the fact that $\partial(X)$ is an equilibrated cut. If F' is not a bridge-face, then by Proposition 2.5 there are at most two elements of $\mathcal{R}$ in $C_{F'}$, and consequently at most two elements of $\mathcal{R}$ in X (i.e., $|X \cap \mathcal{R}| \leq 2$). If F' is a bridge-face, then by Lemma 2.4 the number of elements of $\mathcal{R}$ in each connected component of $C_O \setminus C_{F'}$ is either 2 or 4. As either all of the vertices of a connected component of $C_O \setminus C_{F'}$ are contained in X or none of them is in X, $|X \cap \mathcal{R}|$ has to be an even number and clearly $|X \cap \mathcal{R}| \neq 3$. □

Lemma 2.7. *Let (G, σ) be a canonically projective-planar embedded signed graph in $\mathcal{S}^*(3)$. If F is a bridge-face of G', then $C_F \cap C_O$ has exactly three connected components. In particular, there is at most one bridge-face.*

Proof. As $(G, \sigma) \in \mathcal{S}^*(3)$, we have $|\mathcal{R}| = 6$. As F is a bridge-face, $C_F \cap C_O$ has at least two components, and, by Lemma 2.4, has at most three components. It remains to show that $C_F \cap C_O$ does not have exactly two components. Assume to the contrary that $C_F \cap C_O$ has exactly two components, say $A_{a_1a_2}$ and $A_{b_1b_2}$. Then one of $A_{a_2b_1}$ or $A_{b_2a_1}$, say $A_{a_2b_1}$

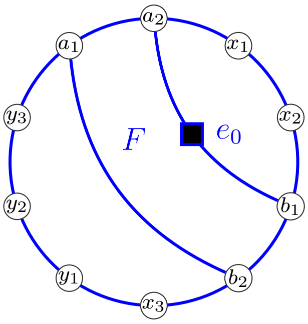


Figure 4: Case in Lemma 2.7

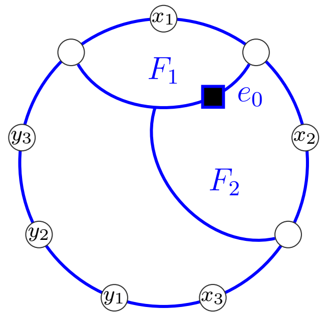


Figure 5: Case in Lemma 2.8

without loss of generality, has two elements from $\mathcal{R}$, and the other, $A_{b_2a_1}$ in this case, has four elements from $\mathcal{R}$. See Figure 4 for a depiction.

Let e_0 be an edge on the a_2b_1-path of C_F which is internally vertex-disjoint from C_O. Let $\partial(X)$ be an equilibrated cut containing e_0. As $\partial(X)$ must contain two (positive) edges, say e_1 and e_2, of C_O, it has to be an edge-cut of size 6 and hence e_0, e_1, and e_2 are the only positive edges of it. Thus one of e_1 or e_2 is on $A_{a_2b_1}$ and the other is on $A_{a_1a_2} \cup A_{b_1b_2}$. Noting that each bridge-face is a 0-face by Proposition 2.5 (i) and $A_{a_2b_1}$ contains two elements from $\mathcal{R}$, X has at most two vertices of $\mathcal{R}$ and, therefore, $\partial(X)$ contains at most two negative edges, contradicting the fact that it is an equilibrated cut.

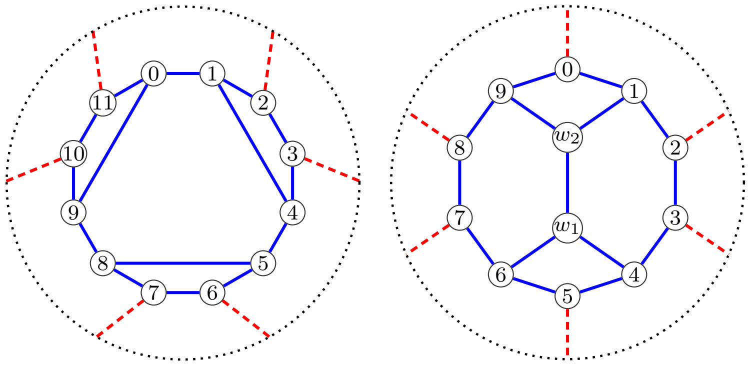
Finally, by Lemma 2.4, as each of the connected components of $C_O \setminus C_F$ must contain either two or four elements of $\mathcal{R}$, and since there are three connected components, each of them contains exactly two elements of $\mathcal{R}$ and thus there is no other bridge-face. □

Lemma 2.8. *Let (G, σ) be a canonically projective-planar embedded signed graph in $\mathcal{S}^*(3)$. Let F_1 and F_2 be an i_1-face and an i_2-face of G', respectively. If F_1 is adjacent to F_2 on the boundary, then either (i) $i_1 + i_2 \geq 3$ or (ii) one of F_1 and F_2 is a bridge-face.*

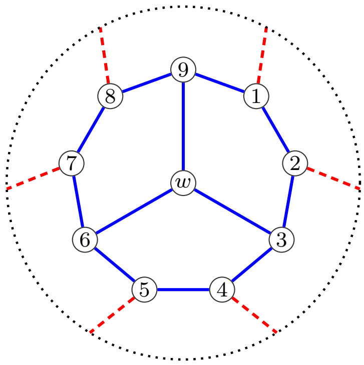
Proof. Assume that neither of F_1 and F_2 is a bridge-face. By Lemma 2.6 $i_1 + i_2 \geq 2$, and it remains to prove that $i_1 + i_2 \neq 2$. Assume to the contrary that $i_1 + i_2 = 2$. Let e_0 be an edge on the path $C_{F_1} \cap C_{F_2}$. See Figure 5. Each equilibrated cut containing the edge e_0 must have two more (positive) edges of C_O say e_1 and e_2. It follows as before that e_1 is on $C_{F_1} \cap C_O$ and e_2 is on $C_{F_2} \cap C_O$. Since $i_1 + i_2 = 2$, a similar argument implies that X can contain at most two vertices from $\mathcal{R}$, leading to a contradiction with $\partial(X)$ being an equilibrated cut. □

We are now ready to give the full description of $\mathcal{S}^*(3)$.

Theorem 2.9. *The class $\mathcal{S}^*(3)$ consists of two signed graphs, depicted in Figure 6.*



(a) $\hat{G}_1$ with two embeddings



(b) $\hat{G}_2$

Figure 6: $\mathcal{S}^*(3)$

Proof. We consider the following three cases:

- G' has a bridge-face F. By Lemma 2.7, F is the only bridge-face of G' and $C_O \setminus C_F$ consists of three components each of which has exactly two elements from $\mathcal{R}$. Furthermore, it follows from Lemma 2.8 that the vertices of $\mathcal{R}$ are the only vertices on each of these components, as otherwise an additional vertex (not in $\mathcal{R}$) would result in an i_1-face and an i_2-face with $i_1 + i_2 \leq 2$ which are adjacent to each other, a contradiction. This leads to the projective planar graph of Figure 6a (left).

- G' has at least one 1-face (and no bridge-face). Let F_1 be a 1-face of G'. As G' has no bridge-face, by Lemma 2.6, it has no 0-face. Furthermore, by Lemma 2.8, each of the two faces adjacent to F_1 on the boundary are 2-faces. As there are only six vertices in $\mathcal{R}$, and as there is no internal face by Lemma 2.6, there is only one remaining face. Furthermore, this face is a 1-face. Let G'' be the graph obtained from G' by suppressing all vertices of $\mathcal{R}$ and note that G'' is cubic and planar.

  It then follows from Euler's formula that $|V(G'')| - \frac{3}{2}|V(G'')| + 5 = 2$, i.e., $|V(G'')| = 6$. But there are only two cubic graphs on 6 vertices: $K_{3,3}$ and the 3-prism. As $K_{3,3}$ is not planar, G'' is the 3-prism. As each 1-face of G' is adjacent to two 2-faces of G', both of the triangles of G'' correspond to faces of the same type in G'. More precisely, either each corresponds to a 1-face or each corresponds to a 2-face. The former case leads to the projective planar graph of Figure 6a (right). In the latter case, we consider the middle edge of the 3-path and we observe that this edge cannot be in an equilibrated cut.

- Each face of G' is a 2-face. Hence, G' has exactly three 2-faces. Similar to the previous case we consider the graph G'' obtained from G' by suppressing all vertices of $\mathcal{R}$. It follows from Euler's formula that G'' has four vertices and noting that G'' is cubic, hence, it must be K_4. Thus (G, σ) is the signed graph in Figure 6b.

We note that the two signed graphs in Figure 6a are switching-isomorphic and thus up to switching isomorphism $\mathcal{S}^*(3)$ consists of two signed graphs. $\square$

Note that the signed graph $\hat{G}_2$ of Figure 6b is a signed Petersen graph.

3 Critically 3-frustrated signed planar graphs

Let $\mathcal{P}^*(3)$ denote the class of irreducible indecomposable critically 3-frustrated signed planar graphs. In this section, we show that each signed plane graph in the class $\mathcal{P}^*(3)$ has exactly six negative facial cycles and no positive facial cycles. Using this we conclude that there are ten non-isomorphic signed graphs (with respect to switching isomorphism) in $\mathcal{P}^*(3)$. They are depicted in Figure 10.
We will need the next lemma that follows from the description of $\mathcal{L}^*(2)$.

Lemma 3.1. *Let C_1, C_2, and C_3 be three negative cycles of a signed graph (G, σ). If $E(C_1) \cap E(C_2) \cap E(C_3) = \emptyset$, then the signed subgraph induced by C_1, C_2, and C_3 contains either a $(K_4, -)$-subdivision or two edge-disjoint negative cycles.*

Proof. Since $E(C_1) \cap E(C_2) \cap E(C_3) = \emptyset$, the frustration index of the signed subgraph induced by $C_1 \cup C_2 \cup C_3$ is at least 2. Hence, it contains a critically 2-frustrated subgraph. The statement then follows from Theorem 1.7. $\square$

Noting that each edge of a plane graph belongs to exactly two facial cycles, we have the following observation, which implies that any element of $\mathcal{P}^*(3)$ has at most six negative facial cycles.

Observation 3.2. *Every critically k-frustrated signed plane graph has at most $2k$ negative facial cycles. Moreover, if there are $2k$ negative facial cycles, then they are the only facial cycles.*

Next, we show that each signed plane graph in $\mathcal{P}^*(3)$ has exactly six negative facial cycles. In fact, we prove this for a larger class of critically 3-frustrated signed graphs which are not necessarily irreducible.

Theorem 3.3. *Let (G, σ) be an indecomposable critically 3-frustrated signed plane graph. Then (G, σ) consists of six negative facial cycles.*

Proof. Since (G, σ) is not decomposable, by Theorem 1.8 (G, σ) contains a $(K_4, -)$-subdivision (H, σ) as a subgraph. Let e_1 be an edge of $E(G \setminus H)$, noting that it is not an empty set because $\ell(G, \sigma) = 3$. Without loss of generality, we assume that σ is a minimum signature where e_1 is assigned to be negative. We observe that the other two negative edges of E^-_σ is on the $(K_4, -)$-subdivision (H, σ).
To prove the theorem it suffices to show that each facial cycle of (G, σ) contains at most one negative edge. That is because, this together with the fact that $\ell(G, \sigma) = 3$ would imply the existence of six negative facial cycles. The claim then follows from Observation 3.2.

As there are only two negative edges in (H, σ), say e_2 and e_3, no facial cycle of (H, σ) contains two negative edges. Thus in (G, σ) no facial cycle contains three negative edges. It remains to show that no facial cycle of (G, σ) contains two negative edges. Assume to the contrary that C_{F_2} is such a facial cycle. As the negative edges cannot be e_2 and e_3, and by the symmetry between these two labels, we may assume that e_1 and e_2 are the negative edges of C_{F_2}. Let C_{F_1} and C_{F_3} be the other facial cycles incident with e_1 and e_2, respectively. Observe that e_3 neither belongs to C_{F_1} nor to C_{F_3}, as otherwise (G, σ) would have only two negative faces, contradicting the fact that it contains a $(K_4, -)$-subdivision. See Figure 7 for an illustration where a blue (or solid) $x_i x_j$-connection presents an all-positive path some of which could be of length 0, red (or dashed) connections each shows a negative path (i.e., a signed path with a single negative edge), thus each of length at least 1. We first claim that C_{F_1} and C_{F_3} have no common edge. Otherwise, a common edge e' together with e_1 and e_2 forms an edge-cut, and by switching at this edge-cut we have a signature with only 2 negative edges.

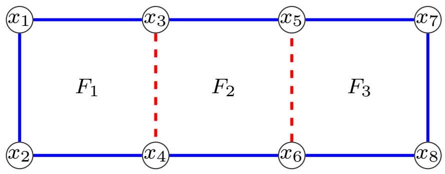


Figure 7: F_1, F_2 and F_3

Let C_{F_4} and C_{F_5} be the two negative facial cycles of (G, σ) such that $e_3 \in E(C_{F_4} \cap C_{F_5})$. Observe that each of C_{F_4} and C_{F_5} must share at least one edge with either C_{F_1} or C_{F_3}. Otherwise, we would have a set of three edge-disjoint negative cycles (for example, C_{F_1}, C_{F_3}, and C_{F_4}), by Observation 1.4 contradicting the assumption that (G, σ) is indecomposable. We now consider the following two cases.

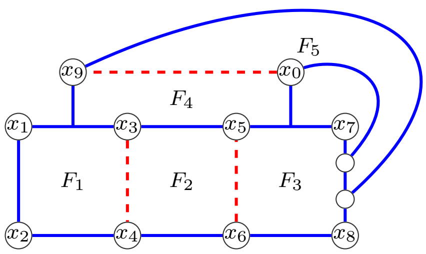


Figure 8: Case (1)

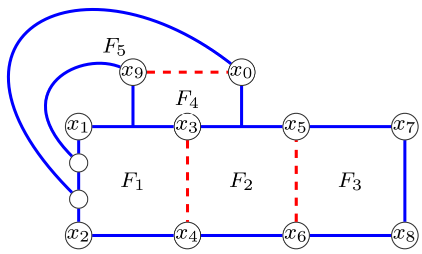


Figure 9: Case (2)

Case (1): C_{F_4} shares a common edge with (at least) one of C_{F_1} and C_{F_3}, and C_{F_5} shares a common edge with the other.

By symmetry, we assume that C_{F_4} shares a common edge with C_{F_1}, and hence C_{F_5} shares a common edge with C_{F_3}. See Figure 8. Then there is an edge-cut crossing the faces

F_4, F_1, F_2, F_3, F_5, and F_4 in this order containing two positive edges and three negative edges, a contradiction with $\ell(G, \sigma) = 3$.

Case (2): Each of C_{F_4} and C_{F_5} shares a common edge with the same C_{F_i} for $i \in \{1, 3\}$. By symmetry, assume that each of C_{F_4} and C_{F_5} shares a common edge with C_{F_1} but none with C_{F_3}. See Figure 9. Therefore, C_{F_3} is edge-disjoint from the negative facial cycles C_{F_1}, C_{F_4}, and C_{F_5}. Noting that each edge is in two faces, we have $E(C_{F_1}) \cap E(C_{F_4}) \cap E(C_{F_5}) = \emptyset$. Thus by Lemma 3.1, $C_{F_1} \cup C_{F_4} \cup C_{F_5}$ contains a critically 2-frustrated signed graph. Note that such a critically 2-frustrated signed graph is edge-disjoint from C_{F_3}. Since C_{F_3} is a negative facial cycle (i.e., a critically 1-frustrated signed graph), (G, σ) is decomposable, a contradiction. □

Corollary 3.4. *If $(G, \sigma) \in \mathcal{P}^*(3)$, then (G, σ) is simple. Moreover, for each minimum signature σ, every facial cycle contains exactly one negative edge.*

Proof. By Observation 1.5, there is no loop in (G, σ) and no two parallel edges of different signs. If there exist two parallel edges with the same sign, then in some planar embedding of (G, σ) they induce a positive facial cycle, contradicting Theorem 3.3. The moreover part is immediate from the fact that there are six facial cycles. □

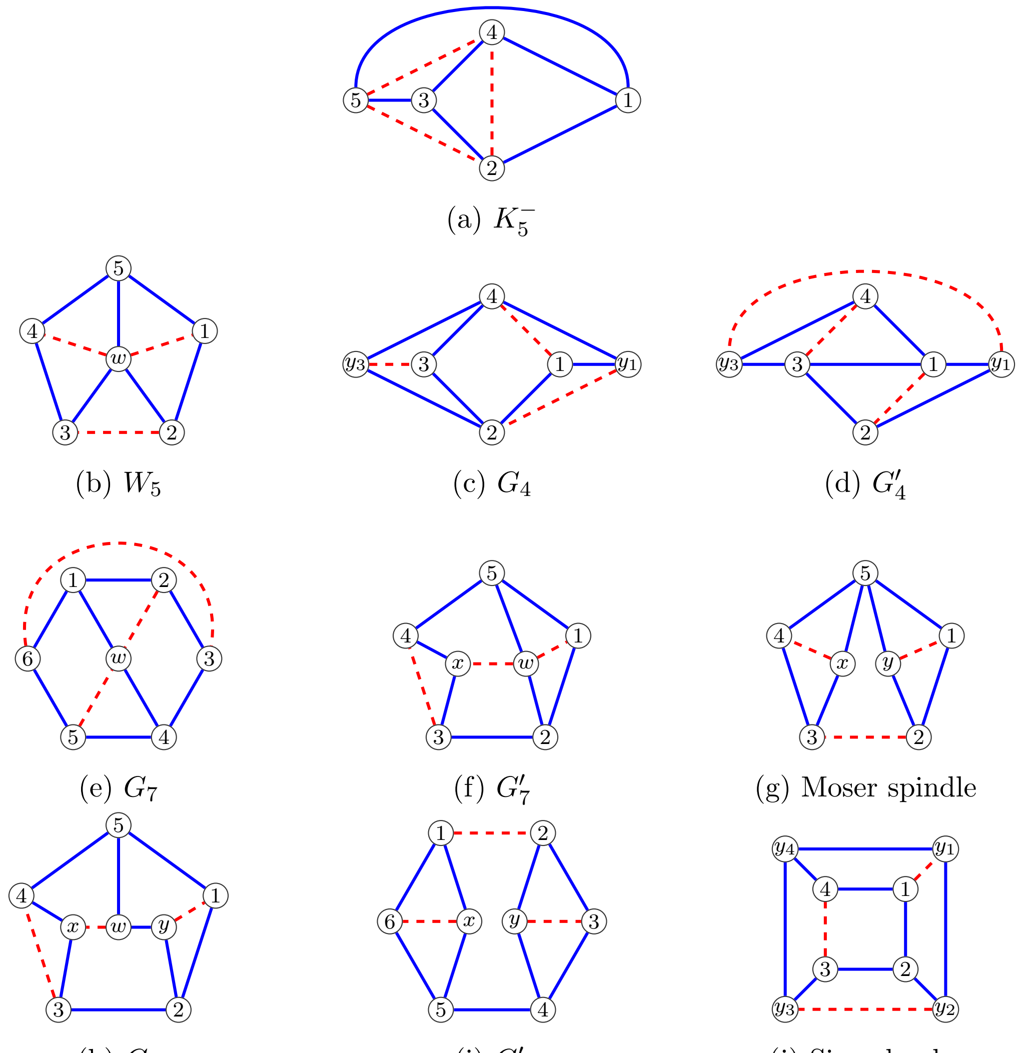
Now we are ready to describe the elements of the class $\mathcal{P}^*(3)$.

Theorem 3.5. *The class $\mathcal{P}^*(3)$ consists of ten signed graphs, depicted in Figure 10.*

Proof. Let $(G, \sigma) \in \mathcal{P}^*(3)$ with a planar embedding. By Theorem 3.3, in (G, σ) there are six facial cycles all of which are negative. This determines the signature up to a switching. So it remains to classify the underlying graphs G. Let $n = |V(G)|$, $m = |E(G)|$, and $f = |F(G)|$ where $F(G)$ is the set of facial cycles of G. Note that $f = 6$ by Theorem 3.3. By Euler's formula and the fact that $\delta(G) \geq 3$, we have that $n - \frac{3}{2}n + 6 \geq 2$. Hence, every irreducible indecomposable critically 3-frustrated signed planar graph contains at most 8 vertices. Note that any simple signed graph on at most four vertices has its frustration index at most 2, thus $n \geq 5$. Depending on the values of n we consider four cases. Noting that in each case G has 6 faces, the number of edges is determined by Euler's formula.

- $n = 5,\ m = 9$: The underlying graph is K_5^- as it has only one edge less than K_5. This graph has a unique planar embedding and in (G, σ) all facial cycles must be negative. In Figure 10a one such signature is presented.

- $n = 6,\ m = 10$: Either G consists of one 5-vertex and four 3-vertices or it consists of four 3-vertices and two 4-vertices. In the first case, G is isomorphic to W_5, see Figure 10b. In the second case, we consider two subcases: (1) The two 4-vertices are not adjacent. In this case, these two 4-vertices are both adjacent to all the remaining vertices, moreover, there are only two edges induced by the four



(a) K_5^-

(b) W_5 (c) G_4 (d) G_4'

(e) G_7 (f) G_7' (g) Moser spindle

(h) G_8 (i) G_8' (j) Signed cube

Figure 10: The class $\mathcal{P}^*(3)$

3-vertices. See Figure 10c. (2) The two 4-vertices are adjacent. In this case, the two 4-vertices share at most two common neighbors. Otherwise, a $K_{3,3}$ is forced by just counting degrees, contradicting planarity. The degree conditions then lead to the unique example of Figure 10d.

- $n = 7,\ m = 11$: G consists of one 4-vertex and six 3-vertices. We consider the graph G_1 obtained from G by removing the 4-vertex. Note that G_1 consists of two 3-vertices and four 2-vertices, and moreover, G is planar and there is a planar embedding such that the four 2-vertices are in a facial cycle. Then one of the following must be the case for G_1: (1) It consists of two 4-cycles sharing one edge, see Figure 10e; (2) It consists of one 5-cycle sharing one edge with a triangle, see Figure 10f; (3) It consists of two triangles connected by an edge, see Figure 10g.

- $n = 8,\ m = 12$: There is a total of five cubic 2-connected graphs, see for example [4]. Of these, we have one Wagner graph which is not planar, and one obtained from $K_{3,3}$ by blowing up a vertex to a triangle. The other three form the full list of cubic 2-connected simple planar graphs on 8 vertices. They are depicted in Figures 10h, 10i, and 10j.

To complete the proof, we need to verify that each signed graph in the list is critically 3-frustrated. That is to say, removing any edge in any of these signed graphs the remaining subgraph has its frustration index being at most 2. To see this, we note that each of these ten graphs is 2-edge-connected and each has only six facial cycles all of which are negative. Thus once an edge is removed, we have five facial cycles, one of which (the new facial cycle) is positive and the other four are negative. It can then be readily verified that in each case these four negative facial cycles can be covered with 2 edges. By Lemma 1.1, we are done. □

4 Constructions of critically k-frustrated signed graphs

In this section, we first introduce a method to build critically frustrated signed graphs from two given critically frustrated signed graphs, and show that it preserves the property of being indecomposable and irreducible. Secondly, we build an infinite family of decomposable irreducible critically 3-frustrated signed graphs. In particular, it implies that the condition of being indecomposable in Conjectures 1.9 and 1.11 is necessary.

4.1 Construction of indecomposable critically frustrated signed graphs

In this subsection, we build signed graphs in $\mathcal{L}^*(k)$ from two given indecomposable critically frustrated signed graphs, one being k_1-frustrated and the other being k_2-frustrated such that $k = k_1 + k_2 - 1$.

Definition 4.1. *Let (G_1, σ_1) and (G_2, σ_2) be two signed graphs, and let xy be a negative edge of (G_1, σ_1) and uv be a negative edge of (G_2, σ_2). We define $H[(G_1, \sigma_1)_{xy},\ (G_2, \sigma_2)_{uv}]$ to be the signed graph obtained from disjoint union of (G_1, σ_1) and (G_2, σ_2) by deleting edges xy and uv, and then adding a negative edge xu and a positive edge yv.*

Proposition 4.2. *Given integers $k_1, k_2 \geq 2$, let $(G_1, \sigma_1) \in \mathcal{L}^*(k_1)$ and $(G_2, \sigma_2) \in \mathcal{L}^*(k_2)$ be two signed graphs such that $|E^-_{\sigma_1}| = k_1$ and $|E^-_{\sigma_2}| = k_2$. Let xy be a negative edge of (G_1, σ_1) and uv be a negative edge of (G_2, σ_2). Then $H[(G_1, \sigma_1)_{xy},\ (G_2, \sigma_2)_{uv}] \in \mathcal{L}^*(k_1 + k_2 - 1)$.*

Proof. Let σ be the signature of $H[(G_1, \sigma_1)_{xy},\ (G_2, \sigma_2)_{uv}]$ and note that it has $k_1 + k_2 - 1$ negative edges. We first verify that σ is a minimum signature by showing that there is no edge-cut with more negative edges than positive ones. Suppose to the contrary that there exists an edge-cut of $H[(G_1, \sigma_1)_{xy},\ (G_2, \sigma_2)_{uv}]$ with more negative edges than positive ones. As σ_1 (resp. σ_2) is a minimum signature of (G_1, σ_1) (resp. (G_2, σ_2)), such an edge-cut, say $\partial(X)$, must contain the new negative edge xu. The vertices x and y are not separated by $\partial(X)$ because otherwise in the restriction of $\partial(X)$ to (G_1, σ_1) we will find a contradiction. Similarly, u and v are not separated by $\partial(X)$. Then yv is also an edge of $\partial(X)$. However, in this case in one of the restrictions of $\partial(X)$ to (G_1, σ_1) and (G_2, σ_2) we find a contradiction.

Next we show that $H[(G_1, \sigma_1)_{xy},\ (G_2, \sigma_2)_{uv}]$ is critically frustrated. By Theorem 1.3, it suffices to prove that each positive edge of $H[(G_1, \sigma_1)_{xy},\ (G_2, \sigma_2)_{uv}]$ belongs to an equilibrated cut. For any positive edge e of $E(G_1, \sigma_1)$, an equilibrated cut of (G_1, σ_1) containing e is also an equilibrated cut of $H[(G_1, \sigma_1)_{xy},\ (G_2, \sigma_2)_{uv}]$ by replacing xy with xu if needed. The same argument holds for positive edges of (G_2, σ_2). For the new positive edge yv, $\partial(V(G_1))$ is the required equilibrated cut. Note that $H[(G_1, \sigma_1)_{xy},\ (G_2, \sigma_2)_{uv}]$ is irreducible because it has no vertex with exactly two neighbors.

It remains to show that $H[(G_1, \sigma_1)_{xy},\ (G_2, \sigma_2)_{uv}]$ is not decomposable. Assume to the contrary that it is and suppose there is a $(r_1, \dots, r_t)$-decomposition ($r_1 + \cdots + r_t = k_1 + k_2 - 1$) into signed subgraphs $\hat{H}'_1, \dots, \hat{H}'_t$. We may furthermore assume that each $\hat{H}'_i$ is connected. Then they must be 2-connected because a critically frustrated signed graph cannot have a bridge. Thus one of the $\hat{H}'_i$'s, say $\hat{H}'_1$, should contain both xu and yv. Each of the others then should be a subgraph of either (G_1, σ_1) or (G_2, σ_2). Without loss of generality, we assume $\hat{H}'_2$ is a subgraph of (G_2, σ_2). Let $(H_2, \sigma) = \hat{H}'_2$, and let (H_1, σ) be the signed subgraph obtained from putting together all other $\hat{H}'_i$'s (that is $H[(G_1, \sigma_1)_{xy},\ (G_2, \sigma_2)_{uv}] - (H_2, \sigma)$). This gives us an (l_1, l_2)-decomposition where $l_1 = k_1 + k_2 - 1 - r_2$ and $l_2 = r_2$.

Observe that $l_2 \leq k_2 - 1$, because uv is not an edge of the critically l_2-frustrated signed graph (H_2, σ) which is a subgraph of the critically k_2-frustrated signed graph (G_2, σ_2). Let (H', σ_2) be the signed subgraph of (G_2, σ_2) by removing all edges of H_2 (recall that uv is a negative edge of this signed subgraph). Observe that $\ell(H', \sigma_2) \le k_2 - l_2$, but moreover if $\ell(H', \sigma_2) = k_2 - l_2$ then by Observation 1.4 (G_2, σ_2) is $(l_2, k_2 - l_2)$-decomposable, a contradiction. Thus $\ell(H', \sigma_2) \le k_2 - l_2 - 1$. Thus there exists a switching-equivalent signature π of σ_2 such that $|E^-_\pi(H')| = k_2 - l_2 - 1$. Assume π is obtained by switching at a set X of vertices of G_2.

We consider two cases based on whether $uv \in E^-_\pi$. If $uv \in E^-_\pi$, then X contains either both of u and v or none of them. We now consider a switching at the subset $X \cap V(H_1)$ of the vertices of (H_1, σ). This switching does not change the signs of the edges in (G_1, σ_1) part, thus there remain $k_1 - 1$ negative edges in this part, noting that xy is not an edge in $E(H_1 \cap G_1)$. On $\{xu, yv\}$ there would remain one negative edge. And on $(H' - uv, \pi)$ we have $k_2 - l_2 - 1$ negative edges. Altogether we have $k_1 + k_2 - l_2 - 2$ negative edges in this switching of (H_1, σ), contradicting the fact that its frustration index is $k_1 + k_2 - l_2 - 1$. If $uv \notin E^-_\pi$, then X contains exactly one of u or v, by symmetry of switching on X or X^c, we may assume $u \in X$. As in the previous case we consider a switching at the subset X of the vertices of (H_1, σ). Since $u \in X$ and $v \notin X$, both xu and yv are positive edges after this switching. A similar calculation as before then counts the number of negative edges in this switched signed graph to be $k_1 + k_2 - l_2 - 2$, which leads to the same contradiction. □

4.2 An infinite family of critically 3-frustrated signed graphs

As mentioned before, the family of critically 2-frustrated signed graphs consists of $(K_4, -)$-subdivisions and edge-disjoint union of two negative cycles. If we furthermore require that they are irreducible, then there are only three such signed graphs: $(K_4, -)$, two disjoint negative loops, and two negative loops on the same vertex. In other words, the set $\mathcal{L}(2)$ of irreducible critically 2-frustrated signed graphs consists of only three elements even without the added assumption of being indecomposable. However, that is not the case for critically k-frustrated signed graphs for $k \ge 3$. In this subsection, we show the next result.

Theorem 4.3. *The set $\mathcal{L}(3)$ contains infinitely many irreducible critically 3-frustrated signed graphs.*

By adding a number of negative loops to the signed graphs of $\mathcal{L}(3)$, one gets examples for any k as long as $k \ge 3$.

Corollary 4.4. *The set $\mathcal{L}(k)$ is infinite for any positive integer $k \ge 3$.*

In order to prove our statements, we first define a sequence of signed graphs as follows: Let $\hat{G}_0$ be the signed graph obtained from K_4 on vertices x, y, z, w by first assigning negative signs to xw and yz, positive signs to the remaining four edges, and secondly adding a positive edge xw and a negative edge yz. See Figure 11. Observe that $\hat{G}_0$ can be decomposed into three negative cycles: xwx (2-cycle), $xyzx$ (3-cycle), and $wyzw$ (3-cycle).

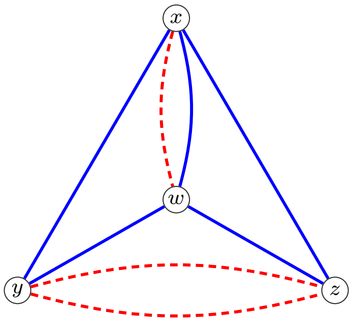


Figure 11: $\hat{G}_0$.

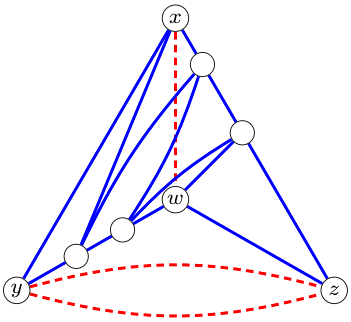


Figure 12: $\hat{G}_2$

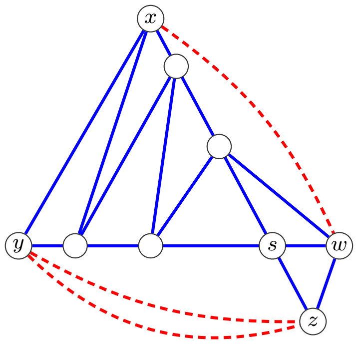


Figure 13: $\hat{G}'_2$

The signed graph $\hat{G}_t$ of the sequence is built from $\hat{G}_0$ as follows. We first introduce $2t$ points by subdividing the positive edge connecting x and w, and two sets of t points by subdividing each of xz and yw. Then we identify the $2t$ points of the xw-path with the $2t$ points, alternating between the points from xz and wy. See Figure 12 for the case of $t = 2$.

Proof of Theorem 4.3. We shall prove this claim by showing that $\hat{G}_t \in \mathcal{L}(3)$. Observe that subdivisions of each of the three cycles given in decomposition of $\hat{G}_0$ gives a decomposition of $\hat{G}_t$. It implies that $\ell(\hat{G}_t) = 3$. What remains is to show that $\hat{G}_t$ is irreducible and critically 3-frustrated.
That $\hat{G}_t$ is irreducible follows from the fact that in a subdivision of a graph, there is always a vertex that has only two distinct neighbors. But there is no such vertex in $\hat{G}_t$. Now we provide a sketch of the proof of $\hat{G}_t$ being critically 3-frustrated. First, observe that each edge incident with y (or z) is in an equilibrated cut $\partial(y)$ (respectively, $\partial(z)$). All other edges are the results of subdivisions (and then identifying some vertices). For an edge uv where u is a vertex on the subdivision of xz and v is a vertex on the subdivision of yw, the following six edges form an equilibrated cut: uv, the edge on the xz-path that forms a triangle with uv, the edge on the yw-path that forms a triangle with uv and the three negative edges. $\square$

In fact, we can modify these signed graphs to get an infinite family of irreducible critically 3-frustrated signed planar graphs. For each $\hat{G}_t$, we apply the following modification to get $\hat{G}'_t$. First, by modifying the embedding of Figure 12 and putting w on the outside of the xyz-triangle, we may have an embedding with one cross which is the crossing of the edge of the yw-path incident with w and the edge of the xz-path incident with z. Then introduce a new vertex, say s at this crossing point to get the planar signed graph $\hat{G}'_t$. See Figure 13 for a depiction of $\hat{G}'_2$. The only remaining point to verify is that each of the new edges is in an equilibrated cut. Such two cuts are $\partial(\{w, z\})$ and $\partial(\{w, z, s\})$. Therefore, we obtain the following result for planar graphs.

Theorem 4.5. *There exist infinitely many irreducible critically k-frustrated planar signed graphs for $k \geq 3$.*

We remark that even though the classes $\mathcal{S}^*(3)$ and $\mathcal{P}^*(3)$ are fully described in this work, the full description of the class $\mathcal{L}^*(3)$ is far from clear. In particular, $\mathcal{L}^*(3) \setminus (\mathcal{S}^*(3) \cup$

$\mathcal{P}^*(3)) \neq \emptyset$. Two examples of such signed graphs are given in Figure 14. The class $\mathcal{L}^*(3)$ is shown to contain finitely many elements in forthcoming work.

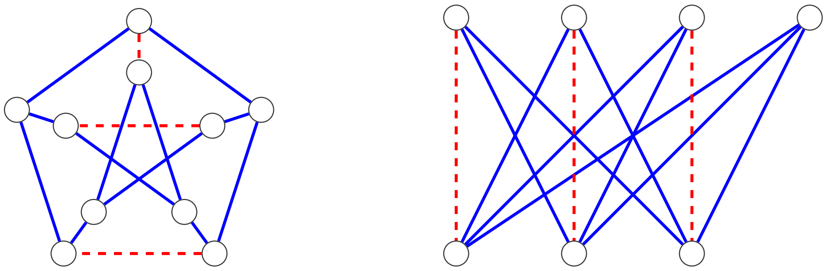


Figure 14: Examples in $\mathcal{L}^*(3)$ neither in $\mathcal{S}^*(3)$ nor in $\mathcal{P}^*(3)$

5 A support for Conjecture 1.12

In this section, we show that Conjecture 1.12 holds if we add the extra condition that (G, σ) has no $(K_5, -)$-minor. A signed graph (H, π) is a *minor* of (G, σ) if it is obtained from (G, σ) by a sequence of the following operations: Deleting vertices or edges, contraction of positive edges, and switching.
Our claim concludes from some known results on the frustration index of $(K_5, -)$-minor-free signed graphs.

Theorem 5.1. [8] *Let (G, σ) be an Eulerian signed graph without $(K_5, -)$-minor. Then the maximum number of edge-disjoint negative cycles of (G, σ) is equal to its frustration index.*

Let (G, σ) be a $(K_5, -)$-minor-free signed graph. Note that, by doubling each edge with the respective sign, we obtain a new $(K_5, -)$-minor-free signed graph which is Eulerian and whose frustration index equals $2\ell(G, \sigma)$. By applying Theorem 5.1 to this new signed graph, we obtain the following result.

Theorem 5.2. *Given a $(K_5, -)$-minor-free signed graph (G, σ) there exists a set $\mathcal{C}$ of negative cycles in (G, σ) such that each edge is in at most two cycles of $\mathcal{C}$, and $|\mathcal{C}| = 2\ell(G, \sigma)$.*

Observe that if $\ell(G, , \sigma) = k$, and if a set $\mathcal{C}$ of negative cycles in (G, σ) is such that each edge is in at most two cycles of $\mathcal{C}$, then size of $\mathcal{C}$ is at most $2k$, that is because each negative edge in a minimum signature can cover at most two of these cycles.
In the next theorem we show that when working with critically frustrated signed graphs we can ask for each edge to be covered precisely twice.

Theorem 5.3. *Let (G, σ) be a $(K_5, -)$-minor-free signed graph which is critically k-frustrated. Then there exists a set $\mathcal{C}$ of negative cycles in (G, σ) such that each edge of G is in exactly two cycles of $\mathcal{C}$, and $|\mathcal{C}| = 2k$.*

Proof. Let (G, σ) be a $(K_5, -)$-minor-free signed graph and assume it is critically k-frustrated. By Theorem 5.2 there exists a set $\mathcal{C}$ of $2k$ negative cycles of (G, σ) such that each edge is in at most two of them. What remains to prove is that each edge of G is in exactly two cycles of $\mathcal{C}$. Assume to the contrary that an edge e is not in two cycles of $\mathcal{C}$, thus it is either in none of them or only in one of them.

First, consider the case when e does not belong to any cycle of $\mathcal{C}$. Then $\ell(G-e, \sigma) \geq k = \frac{1}{2}|\mathcal{C}|$, contradicting the criticality of (G, σ).

Next, suppose that e belongs to exactly one cycle C_e of $\mathcal{C}$. Since (G, σ) is critically k-frustrated, $\ell(G-e, \sigma) = k-1$. Thus any set of negative cycles of $(G-e, \sigma)$, where each edge belongs to as most two cycles in it, is of order at most $2k-2$. However, with e being an edge of C_e but no other cycle in $\mathcal{C}$, the set $\mathcal{C} \setminus C_e$ is a set of $2k-1$ cycles of $(G-e, \sigma)$ which contains each edge at most twice, this contradiction completes our proof. $\square$

As the edges incident with each vertex v belong to at most $2k$ cycles, Conjecture 1.12 is implied when (G, σ) has no $(K_5, -)$-minor.

Corollary 5.4. *Every $(K_5, -)$-minor-free critically k-frustrated signed graph (G, σ) satisfies $\Delta(G) \leq 2k$.*

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