When an optimal dominating set with given constraints exists

Omid Etesami[a], Narges Ghareghani[*,b,a], Michel Habib[c], Mohammadreza Hooshmandasl[d], Reza Naserasr[c], Pouyeh Sharifani[a,d]

[a] *School of Mathematics, Institute for Research in Fundamental Sciences* (IPM), *P.O. Box: 19395-5746, Tehran, Iran, E-mails: etesami@gmail.com, ghareghani@ut.ac.ir*

[b] *School of Engineering Science, College of Engineering, University of Tehran, P.O. Box: 11165-4563, Tehran, Iran*

[c] *CNRS -IRIF UMR 8243, Université Paris Diderot - Paris 7 (France). E-mails: habib@irif.fr, reza@irif.fr*

[d] *Department of Computer Science, Yazd University, P.O. Box: 89195-741, Yazd, Iran, E-mails: hooshmandasl@yazd.ac.ir, pouyeh.sharifani@gmail.com*

Abstract

A dominating set is a set S of vertices in a graph such that every vertex not in S is adjacent to a vertex in S. In this paper, we consider the set of all optimal (i.e. smallest) dominating sets S, and ask of the existence of at least one such set S with given constraints. The constraints say that the number of neighbours in S of a vertex inside S must be in a given set ρ, and the number of neighbours of a vertex outside S must be in a given set σ. For example, if ρ is $[1, k]$, and σ is the nonnegative integers, this corresponds to "$[1, k]$-domination."

First, we consider the complexity of recognizing whether an optimal dominating set with given constraints exists or not. We show via two different reductions that this problem is NP-hard for certain given constraints. This, in particular, answers a question of [M. Chellali et al., $[1, 2]$-dominating sets in graphs, Discrete Applied Mathematics 161 (2013) 2885-2893] regarding the constraint that the number of neighbours in the set be upper-bounded by 2. We also consider the corresponding question regarding "total" dominating sets.

Next, we consider some well-structured classes of graphs, including permutation and interval graphs (and their subfamilies), and determine exactly the smallest k such that for all graphs in that family an optimal dominating set exists where every vertex is dominated at most k times. We also consider the problem for trees (with implications for chordal and comparability graphs) and graphs with bounded "asteroidal number".

Keywords:
Dominating set, total dominating set, NP-hardness, permutation graphs, interval graphs.

1. Introduction

Let $G = (V, E)$ be a graph. For $S \subseteq V$ and a vertex v, let $d_S(v)$ denote the number of neighbours of v in S. Let ρ and σ be two sets of numbers such that $0 \notin \rho$. A (ρ, σ)-dominating set is a set $S \subseteq V$ such that for every vertex $v \in V$, we have $d_S(v) \in \rho$ if $v \notin S$, and $d_S(v) \in \sigma$ if $v \in S$. (See [15, 16])

An ordinary dominating set is then nothing other than a $([1, \infty), [0, \infty))$-dominating set, while a total dominating set is a $([1, \infty), [1, \infty))$-dominating set. Finally, a $[1, k]$-dominating set and a total $[1, k]$-dominating set, which are defined in [7], are respectively a $([1, k], [0, \infty))$-dominating set and a $([1, k], [1, k])$-dominating set. For a survey on domination the reader is referred to the books [10] and [11].

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[*]Corresponding author.

We denote the size of the minimum (ρ,σ)-dominating set by $\gamma_{\rho,\sigma}(G)$. By convention if no (ρ,σ)-dominating set exists, $\gamma_{\rho,\sigma}(G)$ is defined to be ∞. In the literature of dominating sets, the following notation is used: $\gamma(G)$ denotes $\gamma_{[1,\infty),[0,\infty)}(G)$, $\gamma_t(G)$ denotes $\gamma_{[1,\infty),[1,\infty)}(G)$, $\gamma_{[1,k]}(G)$ denotes $\gamma_{[1,k],[0,\infty)}(G)$, and finally $\gamma_{t[1,k]}(G)$ denotes $\gamma_{[1,k],[1,k]}(G)$.

In this paper, we consider (ρ,σ)-sets which are (ρ,σ)-dominating sets (i.e. the case when $0\notin\rho$) in a graph G. First, we note that for given sets ρ,ρ',σ and σ' satisfying $\rho'\subseteq\rho$, $0\notin\rho\cup\rho'$ and $\sigma'\subseteq\sigma$, we have $\gamma_{\rho,\sigma}(G)\leq\gamma_{\rho',\sigma'}(G)$, particularly we have $\gamma(G)\leq\gamma_{(\rho,\sigma)}(G)$. So, it is natural to ask when the equality $\gamma_{(\rho,\sigma)}(G)=\gamma(G)$ holds. Here, for certain (ρ,σ)'s, we consider the problem of recognizing when a graph G has an optimal dominating set which is also a (ρ,σ)-dominating set. The computational complexity of this question (i.e. whether $\gamma_{\rho,\sigma}(G)=\gamma(G)$) lies in the second level of the polynomial hierarchy, more specifically in $\Sigma_2\cap\Pi_2$. A special case of this question was posed by [7], Question 8, where the authors ask to characterize graphs for which $\gamma_{[1,2]}(G)=\gamma(G)$. We show that for certain classes of (ρ,σ), in particular for (ρ,σ) corresponding to $[1,k]$-dominating sets for any fixed $k\geq 1$, it is NP-hard to check whether $\gamma_{\rho,\sigma}=\gamma$. Hence, it is impossible to efficiently characterize graphs where $\gamma_{[1,k]}(G)=\gamma(G)$ unless $P=NP$. (For precise statement of our results, see Theorems 2.9 and 2.17.)

In Question 14 of [7], the authors ask the corresponding question about total $[1,2]$-dominating sets. In this paper, we consider the more general question of finding when $\gamma_{t[1,k]}(G)=\gamma_t(G)$ for any fixed integer $k\geq 1$, showing that checking whether a graph has this property is also NP-hard.

We note that our results regarding the complexity of $\gamma_{\rho,\sigma}(G)=\gamma(G)$ are a first step, because the complexity for many (ρ,σ) are not addressed in this paper. For example, $\rho=[2,k]$ and $\sigma=[0,\infty)$ is an interesting open case.

Question 1.1. *Classify the complexity of checking $\gamma_{\rho,\sigma}(G)=\gamma(G)$ given a graph G as an input for every possible (ρ,σ) pair. In particular, what is the complexity of checking $\gamma_{\rho,\sigma}(G)=\gamma$ in cases where $1\notin\rho$?*

In Section 3 we consider the question of whether $\gamma(G)=\gamma_{[1,k]}(G)$ for special classes of graphs, including permutation graphs (defined in Subsection 3.1) and interval graphs. For these two families of graphs we show that $\gamma(G)=\gamma_{[1,3]}(G)$ and give examples of permutation graphs and intervals graphs for which $\gamma_{[1,2]}(G)>\gamma(G)$. However we show that for the subclasses of unit interval graphs and bipartite permutation graphs $\gamma_{[1,2]}(G)=\gamma(G)$ also holds. We also show that for arbitrary large k, there exist trees such that $\gamma_{[1,k]}(G)>\gamma(G)$. Finally, we consider the relationship between the asteroidal number of the graph (defined in Subsection 3.4) and $\gamma(G)=\gamma_{[1,k]}(G)$.

Related works: Our work fits into the general framework of (ρ,σ)-sets of graphs, which was first introduced by Telle in [15]. (Our study restricts to the case where $0\notin\rho$ so that the set is dominating.) Concerning (ρ,σ)-dominating sets, in Telle's thesis [17], it is conjectured that whenever $0\notin\rho$, the problem of computing minimum (ρ,σ)-dominating set for an arbitrary given graph is NP-complete. He can prove this result for certain (ρ,σ), most significantly whenever ρ and σ are both finite, and also for dominating induced matching ($\rho=[1,\infty),\sigma=\{1\}$), and for independent n-dominating set ($\rho=[n,\infty),\sigma=\{0\}$). Previous to his work, NP-completeness was proved for n-dominating set, with $n\geq 2$ ($\rho=[n,\infty),\sigma=[0,\infty)$), independent dominating set ($\rho=[1,\infty),\sigma=\{0\}$), perfect dominating set ($\rho=\{1\},\sigma=[0,\infty)$, and total dominating set ($\rho=[1,\infty),\sigma=[1,\infty)$]). (See the references in [17]). Later [7] also showed that for $[1,2]$-dominating set (i.e. $\rho=[1,2],\sigma=[0,\infty)$), the problem of minimum (ρ,σ)-dominating set is NP-complete.

We note that the problem of existence of a (ρ,σ)-dominating set is easier than the corresponding optimization problem, and for some of the above (ρ,σ) it can be solved in polynomial time. But for example, for independent [1,1]-dominating sets also known as "perfect codes", i.e. for $\rho = \{1\}, \sigma = \{0\}$, the existence problem is also NP-complete (and even on the subclass of cubic planar graphs [12]) because all perfect codes of a graph are of the same size.

On the other hand it is proved in [6, 13, 18], among other results, that if each of the sets ρ and σ is either finite or is a complement of a finite set, and if G is a graph of bounded MIM-width[2], cliquewidth, or treewidth, then the smallest (ρ, σ)-set can be calculated in polynomial time (with ∞ as an accepted answer when there is no such a set). For the definition of these classes of graphs, we refer to the three papers just mentioned. The class of graphs of bounded MIM-width in particular contains interval graphs and permutation graphs.

We note that special cases of (ρ, σ)-dominating set has also been considered separately. The concept of the $[j, k]$-domination number of a graph (which is $\gamma_{[j,k],[0,\infty)}$ using our notation) has been introduced by Chellali et al. in [7]. In [8] the author introduced the quasi-perfect domination number of a graph which is the same as the $[1, 2]$-domination number of a graph. In [7] the authors proved $\gamma_{[1,2]}(G) = \gamma(G)$ when G belongs to the family of claw-free graphs, P_4-free graphs or caterpillar graphs. They also proved $\gamma_{[1,2]}(G) = \gamma(G)$ for graphs with $\Delta(G) \geq n - 3$, where n is the number of vertices of G and $\Delta(G)$ is the maximum degree of G. Finally, the authors of [7] posed some open questions at the end of their paper. Some of these questions have been answered and investigated in [3, 9, 20]. More precisely, in [20], the authors considered the problem of finding graphs for which $\gamma_{[1,2]}(G) = n$ and they showed that there are planar graphs and bipartite graphs which satisfy this condition. They also proved that $\gamma_{[1,2]}(T) = n - k$, when T is a tree with k leaves and $d_T(v) \geq 4$ for every non-leaf vertex v. In [9], the authors gave a linear time algorithm to compute $\gamma_{[1,2]}(T)$ as well as $\gamma_{t[j,k]}(T)$, for every tree T. Then they recursively generated all trees that do not have total $[1, 2]$-set. A linear time algorithm for computing $\gamma_{[1,j]}(T)$ for a tree T and a polynomial time algorithm for computing $\gamma_{[1,j]}(G)$ for a fixed j in a split graph G is given in [3].

In [1, 2], the authors defined the set-restricted domination number of a graph, which is the generalization of the $[j, k]$-domination number of a graph. The difference between that work and our work is that different vertices have individual, probably different, constraints on the number of their neighbours in the dominating set.

2. Complexity of checking whether $\gamma_{\rho,\sigma} = \gamma$ or whether $\gamma_{t[1,k]} = \gamma_t$

In this section, we provide two reductions from the 3-SAT problem to the existence of optimal dominating sets with certain given (ρ, σ) constraints.

Definition 2.1 (3-SAT problem)**.** *In the 3-SAT problem, the input is a 3-CNF formula over the variables $x_1, \ldots, x_n$, i.e., a conjunction (AND) of clauses, where each clause is a disjunction (OR) of three literals, where a literal is a variable x_i or the negation of the variable. The output is YES/NO according to whether there exists a satisfying truth assignment, i.e., an assignment of boolean values to the variables such that the formula becomes true.*

Here are some properties that separate the two reductions:

- The corresponding (ρ, σ)'s for which the two reductions work are different.

[2] Maximum Induced Matching width, introduced in[19], is a measure of how well it is possible to decompose a graph along vertex cuts with bound on the size of maximum induced matching on the bipartite graph of edges crossing the cut. It is a stronger notion than cliquewidth and treewidth, but many problems are tractable on graphs of bounded MIM-width.

- Curiously, in the first reduction, we have $\gamma_{\rho,\sigma} = \gamma$ exactly when the formula is satisfiable, whereas in the second reduction, we have $\gamma_{\rho,\sigma} = \gamma$ exactly when the formula is not satisfiable. So in complexity theory terms, the first reduction is a Karp reduction to $\gamma_{\rho,\sigma} = \gamma$, whereas the second reduction is a Karp reduction to $\gamma_{\rho,\sigma} \neq \gamma$ (and a Cook reduction to $\gamma_{\rho,\sigma} = \gamma$).
- The first reduction proves hardness also for the subclass of bipartite graphs.
- The first reduction also gives corresponding results concerning optimal total dominating sets, not only optimal dominating sets.
- The first reduction is via the 1-in-3-SAT problem defined below.

2.1. First reduction

Definition 2.2 (1-in-3-SAT problem)**.** *In the 1-in-3-SAT problem, the input is a 3-CNF formula, and the YES/NO output is given according to whether there exists an assignment such that for every clause exactly one literal is true.*

Schaefer [14] proved that 1-in-3-SAT is NP-complete:

Definition 2.3. *Define the function f mapping 3-CNF formulas to 3-CNF formulas according to the following rule. It maps $C_1 \wedge \cdots \wedge C_m$ over the variables $x_1, \ldots, x_n$ to*

$$C_{1,1} \wedge C_{1,2} \wedge C_{1,3} \wedge \cdots \wedge C_{m,1} \wedge C_{m,2} \wedge C_{m,3}$$

over the set of variables

$$\{x_1, \cdots, x_n\} \cup \{t_1, \cdots, t_m\} \cup \{u_1, \cdots, u_m\} \cup \{v_1, \cdots, v_m\} \cup \{w_1, \cdots, w_m\},$$

where if $C_i = l_1^i \vee l_2^i \vee l_3^i$ we have:

$$\begin{aligned} C_{i1} &= \neg l_1^i \vee t_i \vee u_i, \\ C_{i2} &= l_2^i \vee u_i \vee v_i, \\ C_{i3} &= \neg l_3^i \vee v_i \vee w_i. \end{aligned}$$

The mapping f defined above is a valid reduction in the following sense:

Proposition 2.4. *[14] For every 3-CNF φ, $f(\varphi)$ is a YES instance of 1-in-3-SAT if and only if φ is a YES instance of 3-SAT.*

For our purposes, the following observation will be very useful.

Remark 2.5. For every 3-CNF φ, $f(\varphi)$ is always a YES instance of 3-SAT, but not necessarily a YES instance of 1-in-3-SAT.

Corollary 2.6. *Problem 1-in-3-SAT is NP-complete even if we are given the promise that the 3-CNF input to the 1-in-3-SAT problem is a YES instance of 3-SAT.*

Next we introduce our main mapping from 3-CNF formulas to graphs.

Definition 2.7. *Given a 3-CNF $\varphi = C_1 \wedge \cdots \wedge C_m$ on variables $x_1, \cdots, x_n$ and a positive integer k, we build the graph $G_{\varphi,k}$ as follow:*

$$V(G_{\varphi,k}) = \bigcup_{i=1}^{n} (X_i \cup \overline{X}_i \cup Y_i \cup Z_i \cup W_i) \cup \{c_1, \cdots, c_m\},$$

where $X_i = \{x_{i_1}, \cdots, x_{i_k}\}$, $\overline{X}_i = \{\overline{x}_{i_1}, \cdots, \overline{x}_{i_k}\}$, $Y_i = \{y_{i_1}, \cdots, y_{i_{2k}}\}$, $Z_i = \{z_i\}$ *and* $W_i = \{w_{i_1}, w_{i_2}\}$. *Also*

$$E(G_{\varphi,k}) = \cup_{i=1}^{n} E_i,$$

where

$$\begin{aligned} E_i = & \bigcup_{j=1}^{k} \{x_{i_j} y_{i_{2j-2}} x_{i_j} y_{i_{2j-1}}, \overline{x}_{i_j} y_{i_{2j-1}}, \overline{x}_{i_j} y_{i_{2j}}\} \\ & \cup \bigcup_{j=1}^{k} \{x_{i_j} z_i, \overline{x}_{i_j} z_i\} \cup \bigcup_{j=1}^{2} \{w_{i_j} z_i\} \\ & \cup \bigcup_{j=1}^{k} \{x_{i_j} c_l : \ x_i \ \textit{in clause} \ C_l\} \\ & \cup \bigcup_{j=1}^{k} \{\overline{x}_{i_j} c_l : \ \overline{x}_i \ \textit{in clause} \ C_l\}. \end{aligned}$$

In the above, when $j = 1$, *by* $y_{i_{2j-2}}$ *we mean* $y_{i_{2k}}$.

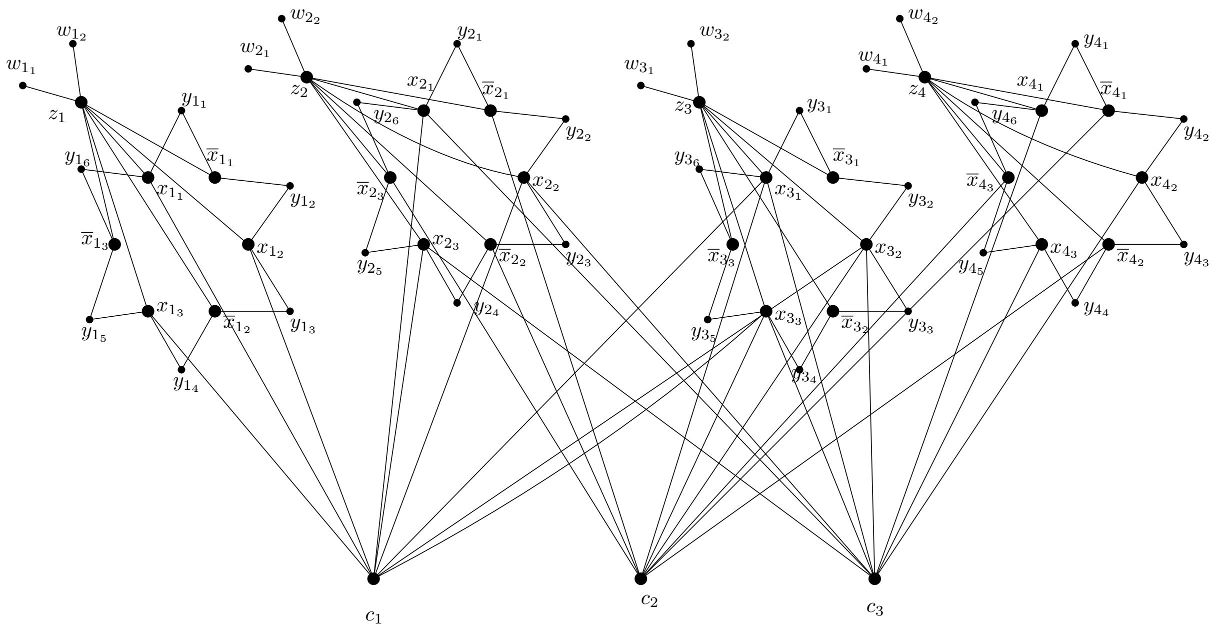


Figure 1: The Construction of $G_{\varphi,3}$.

Example 2.8. Let $\varphi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x}_2 \vee x_3 \vee \overline{x}_4) \wedge (x_2 \vee x_3 \vee x_4)$. The corresponding graph $G_{\varphi,3}$ is shown in Figure 1.

Theorem 2.9. *For every* $k \geq 1$, *and* (ρ, σ) *such that* $\{1, k\} \subseteq \rho \subseteq [1, 2k-1] \cup [2k+1, 3k-1] \cup [3k+1, \infty)$ *and* $\{1, k\} \subseteq \sigma$, *given a bipartite graph* G *it is NP-hard to check whether* $\gamma_{\rho,\sigma}(G) = \gamma(G)$, *and it is NP-hard to check whether* $\gamma_{t[1,k]}(G) = \gamma_t(G)$. *(In particular, checking* $\gamma_{[1,k]}(G) = \gamma(G)$ *is also NP-hard.)*

Proof. It is easy to see that the graph $G = G_{\varphi,k}$ is bipartite: One part consists of X_i's, $\overline{X}_i$'s, and W_i's, and the other part consists of the rest of the vertices.

By the following Lemmas 2.10 and 2.11, $\gamma = \gamma_t = n(k+1)$ under the promise that φ is a YES instance of 3-SAT. By the next Lemmas 2.12 and 2.13, $\gamma_{\rho,\sigma}$ and $\gamma_{t[1,k]}$ are each greater than $n(k+1)$ iff φ is a NO instance of 1-in-3-SAT. Proposition 2.4 and Remark 2.5 complete the proof. $\square$

We note that we do not claim that the decision problem $\gamma_{\rho,\sigma}(G) = \gamma(G)$ is NP-complete. We only claim it is NP-hard. This is because we do not know if the decision problem is in NP: we do not know of an efficiently checkable certificate that $\gamma_{\rho,\sigma}(G) = \gamma(G)$ since we cannot efficiently certify a lower bound on $\gamma(G)$ (though an upper bound on $\gamma(G)$ can be efficiently certified once an dominating set is given). In particular, since the second reduction shows that the decision problem $\gamma_{\rho,\sigma} = \gamma(G)$ is co-NP-hard under Karp reductions for certain (ρ, σ)'s some of which are common with the (ρ, σ)'s of Theorem 2.9, this shows that for these common (ρ, σ)'s the problem is not NP-complete unless NP = co-NP.

Lemma 2.10. *If φ is a YES instance of 3-SAT, then $\gamma(G_{\varphi,k}) \le \gamma_t(G_{\varphi,k}) \le n(k+1)$.*

Proof. It is enough to show $\gamma_t(G_{\varphi,k}) \le n(k+1)$. Choose a satisfying assignment for φ. For each i, if x_i is true, include X_i in the total dominating set; otherwise, include $\overline{X}_i$. Finally, add all z_i's to the total dominating set. $c_1, \ldots, c_m$ are dominated since the assignment is satisfying, and all other vertices are clearly dominated. The size of the set is obviously $n(k+1)$. $\square$

Lemma 2.11. $\gamma_t(G_{\varphi,k}) \ge \gamma(G_{\varphi,k}) \ge n(k+1)$.

Proof. It is enough to show $\gamma(G_{\varphi,k}) \ge n(k+1)$. To dominate the w_{i_j}'s for a fixed i, it is optimal to choose z_i. For each fixed i, there are $2k$ vertices y_{i_j}, and each of the x_{i_j}'s dominate only two of those $2k$ vertices. So if we choose l of the x_{i_j}'s and l' of the y_{i_j}'s, we have $l' + 2l \ge 2k$, which implies $l' + l \ge k$. So at least k of the vertices in $Y_i \cup X_i \cup \overline{X}_i$ must be in the dominating set. This implies at least $n(k+1)$ vertices in the whole dominating set. $\square$

Lemma 2.12. *If φ is a Yes instance of 1-in-3-SAT, with (ρ, σ) as in statement of Theorem 2.9, then $\gamma_{\rho,\sigma}(G_{\varphi,k}) \le n(k+1)$. In particular, $\gamma_{t[1,k]}(G_{\varphi,k}) \le n(k+1)$.*

Proof. Consider the total dominating set constructed in the proof of Lemma 2.10. Clearly, it dominates each vertex w_{i_j} only once, each z_i exactly k times, each vertex y_{i_j} exactly once, and each x_{i_j} and $\overline{x}_{i_j}$ exactly once. Furthermore, if each clause is satisfied by exactly one literal, each c_i is dominated by exactly k vertices. Thus, $\gamma_{\rho,\sigma}(G_{\varphi,k}) \le n(k+1)$. In particular we have $\gamma_{t[1,k]}(G_{\varphi,k}) \le n(k+1)$. $\square$

Lemma 2.13. *If φ is a NO instance of 1-in-3-SAT, with (ρ, σ) as in statement of Theorem 2.9, then $\gamma_{\rho,\sigma}(G_{\varphi,k}) > n(k+1)$. In particular, $\gamma_{t[1,k]}(G_{\varphi,k}) > n(k+1)$.*

Proof. Consider a dominating set for $G_{\varphi,k}$. For each i, to dominate W_i, at least one vertex among $W_i \cup Z_i$ should be in the dominating set. For each i, to dominate Y_i, by the same argument as in Lemma 2.11, at least k vertices among $X_i \cup \overline{X}_i \cup Y_i$ should be in the dominating set; and the only case where k vertices is enough when we choose either all of X_i or all of $\overline{X}_i$. So if the dominating set consists of at most $n(k+1)$ vertices, no vertex c_i is chosen, and the dominating set encodes a satisfying truth assignment to the variables $x_1, \ldots, x_n$. Since by assumption, one clause must be satisfied by more than one literal, one of the vertices c_i is dominated either $2k$ or $3k$ times. This proves the claim of the lemma. $\square$

Remark 2.14. We note that whether a formula φ is a YES instance of 2-in-3-SAT is also NP-complete even under the promise that φ is 3-SAT satisfiable. (To see this, negate all the literals in Definition 2.3.) Therefore, by the same reasoning as above, if $1, 2k \in \rho$, $1, k \in \sigma$, and $k, 3k \notin \rho$, then checking $\gamma_{\rho,\sigma} = \gamma$ is NP-hard.

2.2. Second reduction

The following reduction, which is a modification of a known reduction from 3-SAT to domination number (see [10]).

Definition 2.15. *Given a 3-CNF formula φ and $k \geq 2$, let $M = k+1$, and construct the graph $G'_{\varphi,k}$ with vertex set*

$$V = \{w\} \cup \bigcup_{j=1}^{M} \Big(\bigcup_{i=1}^{n} (x_i^j \cup \overline{x}_i^j \cup y_i^j) \cup \{c_1^j, \ldots c_m^j\} \Big),$$

and edge set

$$\begin{aligned} E_i = & \bigcup_{j=1}^{M} \bigcup_{i=1}^{n} \{y_i^j x_i^j, y_i^j \overline{x}_i^j, x_i^j \overline{x}_i^j, w x_i^j, w \overline{x}_i^j\} \\ & \cup \bigcup_{j=1}^{M} \{x_i^j c_l^j : x_i \textit{ in clause } C_l\} \\ & \cup \bigcup_{j=1}^{M} \{\overline{x}_i^j c_l^j : \overline{x}_i \textit{ in clause } C_l\} \\ & \cup \bigcup_{j=1}^{M} \{w c_1^j, \ldots, w c_m^j\}. \end{aligned}$$

Example 2.16. Let $\varphi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_2} \vee x_3 \vee \overline{x_4}) \wedge (x_2 \vee x_3 \vee x_4)$. The corresponding graph $G'_{\varphi,3}$ is shown in Figure 2.

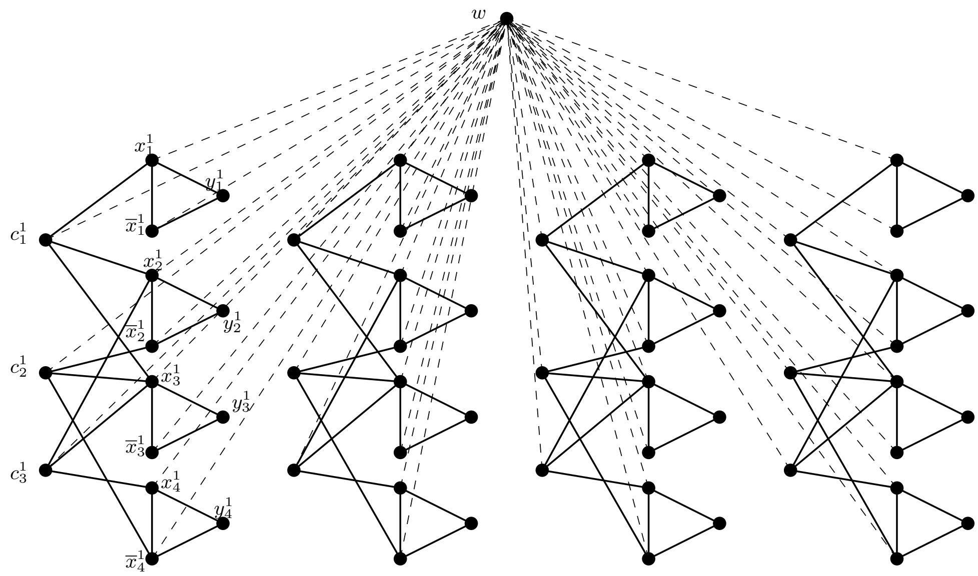


Figure 2: The Construction of $G'_{\varphi,3}$.

Theorem 2.17. *For every $k \geq 2$, and (ρ, σ) such that $\{1,2\} \subseteq \rho \subseteq [1,k]$ and $\{0\} \subseteq \sigma$, given a graph G, it is NP-hard to check whether $\gamma(G) = \gamma_{\rho,\sigma}(G)$.*

Proof. Given a 3-SAT formula φ, w.l.o.g. we assume φ is not empty. We construct $G = G'_{\varphi,k}$. Then by Lemma 2.18, we have $\gamma_{\rho,\sigma}(G) = nM + 1$. On the other hand, by Lemma 2.19, $\gamma(G) = nM$ iff φ is satisfiable. Therefore, $\gamma_{\rho,\sigma}(G'_{\varphi,k}) = \gamma(G'_{\varphi,k})$, Iff φ is not satisfiable. This proves the theorem. $\square$

Lemma 2.18. *For every nonempty formula φ, $k \geq 2$, and (ρ, σ) as in Theorem 2.17, we have $\gamma_{\rho,\sigma}(G'_{\varphi,k}) = nM + 1$.*

Proof. If we choose w and all y_i^j's as a set of dominating vertices of $G'_{\varphi,k}$, clearly the size of the set is $nM + 1$. Furthermore, every vertex outside the set is dominated by only 1 or 2 vertices, and every vertex inside the set is not neighbour to any other vertex inside the set. Thus for every (ρ, σ) satisfying the condition of Theorem 2.17, we have $\gamma_{\rho,\sigma}(G'_{\varphi,k}) \leq nM + 1$.

On the other hand, consider a general dominating set for $G'_{\varphi,k}$. Of each triangle $\{y_i^j, x_i^j, \overline{x}_i^j\}$, at least one vertex must be chosen to dominate y_i^j. If the size of dominating set is at most nM, this is all the vertices we can include in the dominating set without exceeding the size limit. But since φ contains at least one clause, for each j, at least one x_i^j or $\overline{x}_i^j$ must be chosen. Therefore at least M, $M > k$, neighbours of w are chosen, so there is no (ρ, σ)-dominating set of size at most nM. $\square$

Lemma 2.19. $\gamma(G'_{\varphi,k}) = nM$ *iff φ is a YES instance of 3-SAT.*

Proof. By the argument of the previous lemma, any dominating set has size at least nM. And if the size of the dominating set is nM, it includes exactly one vertex from each triangle $\{y_i^j, x_i^j, \overline{x}_i^j\}$. Choosing x_i^j or $\overline{x}_i^j$ instead of y_i^j has the advantage of possibly dominating other vertices, so we assume that for each j and i exactly one of x_i^j or $\overline{x}_i^j$ has been chosen. Now w is covered, and the c_l^j's are covered iff for every j, the chosen vertices correspond to a satisfying assignment to 3-SAT. Q.E.D. $\square$

3. Graphs for which $\gamma_{[1,k]} = \gamma$

Definition 3.1. *For a vertex v in a graph G, we define the open neighbourhood $N_G(v)$ to be the set of neighbours v, and the closed neighbourhood $N_G[v]$ to be $N_G(v) \cup \{v\}$.*

Definition 3.2. *A vertex v belonging to a set S is said to have a* private neighbour *if there exists a vertex $u \in N_G(v) \setminus S$ such that u is not adjacent to any other member of S.*

In some proofs we will use the following lemma regarding the existence of a private neighbour for each element of a minimum dominating set.

Lemma 3.3. *[4] Let G be a graph with no isolated vertices. Then G contains an optimal dominating set S such that each vertex in S has a private neighbour.*

3.1. Permutation graphs

As defined in [5]:

Definition 3.4. *A* permutation graph *is built as follows: let L_t and L_b be two parallel lines on the plane (we imagine L_t on the top and L_b on the bottom). Let $x_1, x_2, \cdots, x_n$ be n distinct points on L_t and $y_1, y_2, \cdots, y_n$ respectively be n distinct points on L_b, noting that these points do not necessarily appear on the two lines from left to right in this order. A permutation graph then has the n line segments $[x_i, y_i]$ of the plane as vertices, and two vertices (i.e. segments) are adjacent if the corresponding segments intersect. In other words, given a permutation σ on n elements, if we consider $x_1, \ldots x_n = [1, n]$ and $y_1, \ldots, y_n = \sigma(1), \ldots \sigma(n)$. A permutation* *graph is just the intersection graph of the lines* $[i, \sigma(i)]$. *A* bipartite permutation graph *is a permutation graph which is also bipartite. Given a vertex* $v = [x_i, y_i]$ *we may write* $t(v) = x_i$ *and* $b(v) = y_i$.

In this section, we prove that $\gamma_{[1,3]} = \gamma$ for permutation graphs. To prove this, we first prove that in fact any minimum dominating set of a permutation graph is also a $[1, 4]$-dominating set. We then show that at least one such dominating set is also a $[1, 3]$-dominating set.

Lemma 3.5. *Let* G *be a permutation graph and* D *be an optimal dominating set for* G. *For each vertex* v *of* G *let* $S(v) = D \cap N_G(v)$. *Then for each* v *we have* $|S(v)| \leq 4$. *Furthermore, if* $|S(v)| = 4$ *then the subgraph induced by* $S(v)$ *is a* 4*-cycle.*

Proof. We partition the set of segments in $S(v)$ into two sets S_1 and S_2 according to the position of their top points on L_t, those on the left of v are in S_1 and those on the right are in S_2. Observe that since all vertices in $S(v)$ are adjacent to v, the partition with respect to L_b (insted of L_t) will be the same after exchanging the role of right and left. To prove the lemma we prove that S_1 and S_2 are both of size at most two and that if any of them is of size 2, then its elements are not adjacent. The proof is based on the fact that D is a dominating set of minimum size. Let u be the vertex in S_1 whose top point is furthest away from that of v and let u' be the vertex in S_1 whose bottom point is furthest away from that of v. Observe that u and u' are not necessarily distinct, and in fact if S_1 is a set of two adjacent vertices, then indeed u and u' are the same vertices. Now consider the set D' which is obtained from D by removing elements of $S_1 - \{u, u'\}$ and adding v. It is easily verified that D' is also a dominating set of G. As D is a dominating set of minimum size, we have $|S_1| \leq 3$ and that if $u = u'$, then $|S_1| \leq 2$. Now suppose contrary to our claim that $|S_1| = 3$ or that $u = u'$ and $|S_1| = 2$. Observe that since $|S| \geq 4$ we have $S_2 \neq \emptyset$. Let z be a vertex in S_2. Observe that z intersects all elements of S_1. Let t be the vertex of S_1 which is distinct from u and u'. Then t is dominated by z and any neighbour of it is dominated by one of u, u' or z, meaning $D - t$ is a dominating set, contradicting the fact that D is a dominating set of a minimum size. $\square$

Corollary 3.6. *Given a permutation graph* G, *any dominating set of minimum size is also a* $[1, 4]$*-dominating set.*

Lemma 3.7. *Let* D *be an optimal dominating set for a permutation graph* G. *Then each vertex of* D *lies in at most one induced* 4*-cycle. In other words every pair of induced* 4*-cycles in* D *are vertex disjoint.*

Proof. Suppose that $u \in D$ lies in a induced 4-cycle C. For the contrary suppose that there is another induced 4-cycle C' which contains u. Let $V(C) = \{s_1, s_2, s_3, s_4\}$ and $t(s_1) < t(s_2) < t(s_3) < t(s_4)$. Since the induced graph on C, is a 4-cycle we have $b(s_3) < b(s_4) < b(s_1) < b(s_2)$. Without loss of generality suppose that $u = s_1$. Since each vertex of C is adjacent to another vertex of C and they are in an optimal dominating set D, each of them should have at least one private neighbour in D: let s'_i be the private neighbour of s_i, $1 \leq i \leq 4$. It is easy to see that the private neighbours of vertices of C should be as shown in Figure 3.

Now, let $u' \in C'$ be a vertex of C' which is not a vertex of C. Since s_i has the private neighbour of s'_i, u' is adjacent to non of the s'_1, s'_2, s'_3 and s'_4. Hence, one of the following three cases can occur for u':
Case 1. $t(u') < t(s'_3)$ and $b(u') < b(s'_1)$;
Case 2. $t(s'_2) < t(u')$ and $b(s'_4) < b(u')$;
Case 3. $t(s_4) < t(u') < t(s'_3)$ and $b(s_3) < b(u') < b(s_2)$.
If Case 1 happens, then since u' and $u = s_1$ are in the 4-cycle C', the other segments of C' would intersect to s'_1 and s'_3, which is a contradiction with the fact that s'_1 and s'_3 are private

neighbours of s_1 and s_3. Hence this case can not be occurred. Case 2 also can not occur: the reason is similar to that of Case 1. If Case 3 occurs, then u' can not have any private neighbour and since u' has a neighbour in D we conclude that $D \setminus \{u'\}$ is also a dominating set which is a contradiction. $\square$

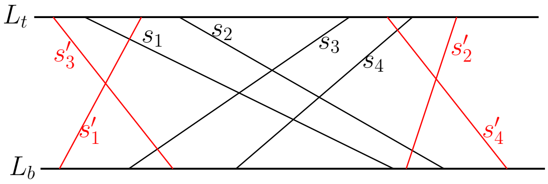


Figure 3: Private neighbours for the vertices of 4-cycle $C = \{s_1, s_2, s_3, s_4\}$

We now prove that at least one minimum dominating sets must be a $[1, 3]$-dominating set.

Theorem 3.8. *Let G be a permutation graph, then $\gamma(G) = \gamma_{[1,3]}(G)$. Moreover, there is a permutation graph H for which $\gamma(H) < \gamma_{[1,2]}(H)$.*

Proof. Given a dominating set D of minimum size, let t_D be the number of induced 4-cycles C of D for which there exists some vertex $v \in V(G) \setminus D$ such that $C = N(v) \cap D$. Let $t = \min\{t_D \mid D \text{ is a} \gamma\text{-set}\}$, and assume for contradiction that $t \geq 1$. Let $\mathcal{F}$ be the set of all minimum dominating sets for which $t_D = t$. Recall that for each vertex v not in D, if $S(v) = N(v) \cap D$ is of size four, then it induces a 4-cycle. Assuming D is in $\mathcal{F}$, there are exactly t such 4-cycles. Let $S(D)$ be vertices of these t 4-cycles, and let $v(D)$ be the vertex in $S(D)$ whose top is most left. Among all members of $\mathcal{F}$ let D' be the one for which $v(D')$ is most left. Let s_1 be the top point of $v(D')$ and let s_2, s_3 and s_4 be the top points of three other vertices of the 4-cycle corresponding to $v(D)$, the points s_i being ordered from left to the right as shown in Figure 4.

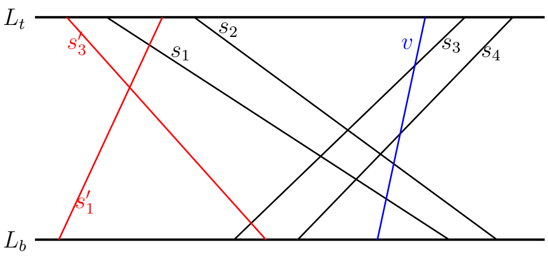


Figure 4: A special 4-cycle appearing in the proof of Theorem 3.8

For simplification, we will use s_1, s_2, s_3, s_4 to denote also the vertices corresponding to these top points. As vertices of a 4-cycle are not isolated in D', and as D' is a minimum dominating set, each of these four vertices must have a private neighbour in $V(G) \setminus D'$. Let s'_3 be the private neighbour of s_3 in $V(G) \setminus D'$. Observe that, based on the position of s_3 in the 4-cycle, and as s'_3 is not adjacent to any of s_1, s_2, s_4, the top point of s'_3 is further left of the top point of s_1. By considering the positions of all five points as shown in Figure 4 we observe that every private neighbour of s_1 in $V(G) \setminus D'$ is also a neighbour of s'_3. Thus D^* obtained from D' by removing s_1 and adding s'_3 is also a dominating set of minimum size. Now, if there exists some induced 4-cycle C in D^*, all whose vertices adjacent to s_1, then C is also contained in d'. By Lemma 3.7, $V(C) \cap \{s_1, s_2, s_3, s_4\} = \emptyset$. Therefore, there are at least six vertices of D^* which are adjacent to $s_1 \in V(G) \setminus D^*$, which is a contradiction with Corollary 3.6. However, since each vertex of D' (and D^*) is at most in one special 4-cycle (Lemma 3.7) we have $t_{D^*} \leq t_{D'}$. But since $t_{D'}$

is minimum it must be the case that s_3' is also in a 4-cycle, but then s_3' is a left top point of a 4-cycle which is to the left of s_1, contradicting the choice of s_1.

To construct a permutation graph H with $\gamma(H) > \gamma_{[1,2]}(H)$, consider the permutation graph with nine vertices shown in Figure 5. Then, it is easy to see that H has three optimal dominating set $D_1 = \{s_5, s_6, s_7\}$, $D_2 = \{s_3, s_5, s_7\}$ and $D_3 = \{s_4, s_5, s_7\}$. None of them is a $[1,2]$-set, since each of the vertex s_3, s_4 have three neighbours in D_1, the vertex s_4 has three neighbours in D_2 and the vertex s_3 has three neighbours in D_3. Therefore $\gamma(H) < \gamma_{[1,2]}(H)$.

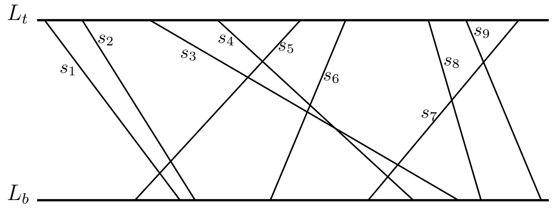


Figure 5: A permutation graph H with $\gamma(H) > \gamma_{[1,2]}(H)$

□

Proposition 3.9. *For every bipartite permutation graph G, $\gamma(G) = \gamma_{[1,2]}(G)$.*

Proof. Let D be an optimal dominating set for which every vertex in it has a private neighbour in $V(G) \setminus D$ (by Lemma 3.3, there exists such an optimal dominating set). We prove that D is a $\gamma_{[1,2]}$-set. Recall that vertices of G are line segments $[x_i, y_i] (1 \leq i \leq |V(G)|)$, where $x_i \in L_t$ and $y_i \in L_b$. Assume by contradiction that D is not a $\gamma_{[1,2]}$-set. This means that there is a vertex $v = [x, y] \in V(G) \setminus D$ and three vertices $v_1 = [x_1, y_1], v_2 = [x_2, y_2], v_3 = [x_3, y_3]$ of D, where $x_1 < x_2 < x_3$, such that $v_i \in N(v), 1 \leq i \leq 3$. Since G is bipartite non of the vertices v_1, v_2, v_3 are adjacent. Therefore, from the inequality $x_1 < x_2 < x_3$ we have $y_1 < y_2 < y_3$. Since v is adjacent to all the vertices v_1, v_2, v_3, without loss of generality, we may assume that $y_3 < y$ and $x < x_1$ (see Figure 6).

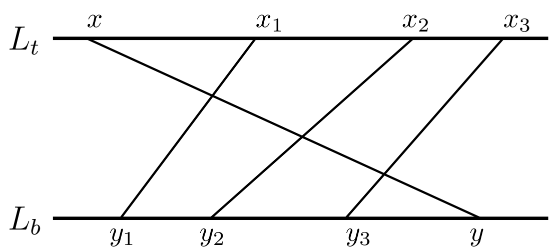


Figure 6: A case in a bipartite permutation graph in which a vertex $[x, y]$ is dominated three times

Now, every private neighbour v_4 of v_2 should be adjacent to v, so v_4, v_2 and v make a triangle, which is a contradiction by the fact that G is bipartite. Hence, v_2 could not have any private neighbour, which is a contradiction. Hence D is a $[1,2]$-set, as desired.

□

3.2. Interval graphs

Definition 3.10. *An* interval graph *is a graph where each vertex is an (open) interval of a given line, and two vertices are adjacent if they intersect. If furthermore all intervals are of unit length then the graph is called a* unit interval graph*.*

In this section we want to prove that for every interval graph G, $\gamma_{[1,3]}(G) = \gamma(G)$. Moreover we give an example of an interval graph G for which the stronger equality $\gamma_{[1,2]}(G) = \gamma(G)$ does not hold. In addition we prove that for every unit interval graph G we have $\gamma_{[1,2]}(G) = \gamma(G)$.

Theorem 3.11. *Let G be an interval graph, then $\gamma(G) = \gamma_{[1,3]}(G)$. Moreover there is an interval graph H for which $\gamma(H) < \gamma_{[1,2]}(H)$.*

Proof. We prove that for every interval graph G, every optimal dominating set is a $\gamma_{[1,3]}$-set. For the contrary suppose that D is an optimal dominating set and there exist a vertex $v \in V(G) \setminus D$ and vertices s_1, s_2, s_3, s_4 in D, which all are adjacent to v, and are ordered increasingly according to their left endpoints. For $1 \le i \le 4$, let a_i (resp. b_i) be the left (resp. right) endpoint of s_i. Let a (resp. b) be the left (resp. right) endpoint of v. Since a_1, a_2, a_3, a_4 are ordered increasingly and s_1, s_2, s_3, s_4 is contained in an optimal dominating set we conclude that b_1, b_2, b_3, b_4 are also ordered increasingly (because otherwise one interval for example $s_i \in D$ will included in another interval for example $s_j \in D$ $(j \neq i)$ which means that $N_G[s_i] \subseteq N_G[s_j]$. Hence, $D \setminus \{s_i\}$ is also a dominating set, which is a contradiction). Since v adjacent to all s_1, s_2, s_3, s_4, we have $a < b_1$ and $a_4 < b$. By these facts, it is easy to see that

$$N_G[\{s_1, s_2, s_3, s_4\}] \subseteq N_G[\{s_1, v, s_4\}].$$

This shows that $(D \cup \{v\}) \setminus \{s_2, s_3\}$ is also a domination set, whose cardinality is less than $|D|$, which is an contradiction. So, for every interval graph G, every optimal dominating set D is also a $\gamma_{[1,3]}$-set.

For the second part of the theorem consider the following example. Let H be an interval graph with vertex set

$$\begin{aligned} V(H) = \{&(1,7),(2,3),(3,4),(4,5),(5,13),(6,14),(8,9),\\ &(9,10),(10,11),(12,19),(15,16),(16,17),(17,18)\} \end{aligned}$$

Interval graph H is shown in Figure 7.

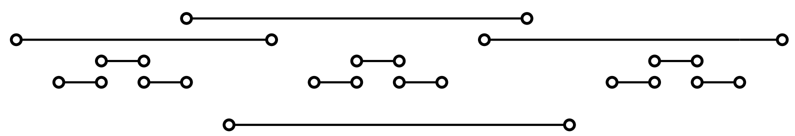


Figure 7: An interval graph H with $\gamma(H) > \gamma_{[1,2]}(H)$

Then it is easy to see that H has two optimal dominating set D_1 and D_2, where $D_1 = \{(1,7),(5,13),(12,19)\}$ and $D_2 = \{(1,7),(6,14),(12,19)\}$. None of them is a $[1,2]$-set, since the vertex $(5,13)$ has three neighbours in D_2 and the vertex $(6,14)$ has three neighbours in D_1. Therefore $\gamma(H) < \gamma_{[1,2]}(H)$. □

Theorem 3.12. *For every unit interval graph G, $\gamma(G) = \gamma_{[1,2]}(G)$.*

Proof. Let D be a optimal dominating set. We prove that D is $\gamma_{[1,2]}$-set. For the contrary suppose that there exist s_1, s_2, s_3 in D and $v \in V(G) \setminus D$ such that v adjacent to all s_1, s_2, s_3. Let $\{a_1, a_2, a_3\}$ $(\{b_1, b_2, b_3\})$ be the set of left endpoints (right endpoints) of s_1, s_2, s_3 and suppose that $\{a_1, a_2, a_3\}$ are ordered increasingly, therefore $\{b_1, b_2, b_3\}$ are also ordered increasingly. If a and b denote the left and right endpoints of v, respectively, then $a < b_1$ and $a_3 < b$. Therefore, $|a_3 - b_1| < b - a = 1$. Since all intervals have unit length, we conclude that $N_G[s_2] \subseteq N_G(s_1) \cup N_G(s_3)$, which is a contradiction with the assumption that D be a optimal dominating set. □

3.3. Trees and superfamilies of permutation graphs and interval graphs

Theorem 3.13. *For every integer $k \geq 1$, there exists a tree T_k such that $\gamma_{[1,k]}(T_k) > \gamma(T_k)$.*

Proof. Let T_k be the tree shown in Figure 8. It is easy to see that the only optimal dominating set consists of the parents of the leave. So, $\gamma(T_k) = k + 1$ and $\gamma_{[1,k]} > k + 1$. □

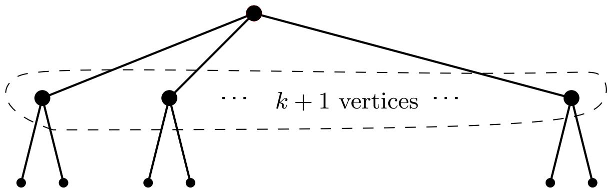


Figure 8: A tree T_k with $\gamma_{[1,k]}(T_k) > \gamma(T_k)$

Definition 3.14. *A comparability graph is a graph with the vertex set equal to the underlying set of a finite partially ordered set, where two vertices are adjacent if they are comparable in the partial order. A chordal graph is a graph where every cycle of length more than* 3 *has a chord, i.e. the graph induced on the cycle has another edge also.*

The above result shows that for other families like comparability graphs or chordal graphs, which are superfamilies of trees, we cannot have $\gamma_{[1,k]} = \gamma$ for fixed k. Note that interval graphs are a subfamily of chordal graphs, and permutation graphs are a subfamily of comparability graphs.

Bipartite graphs are a subclass of comparability graphs, and the fact that a fixed k does not work for comparability graphs is not surprising given our hardness result of Theorem 2.9 for bipartite graphs. Regarding chordal graphs, the following problem looks interesting.

Question 3.15. *What is the complexity of checking $\gamma_{[1,k]}(G) = \gamma(G)$ for a given chordal graph G?*

3.4. Graphs with bounded asteroidal number

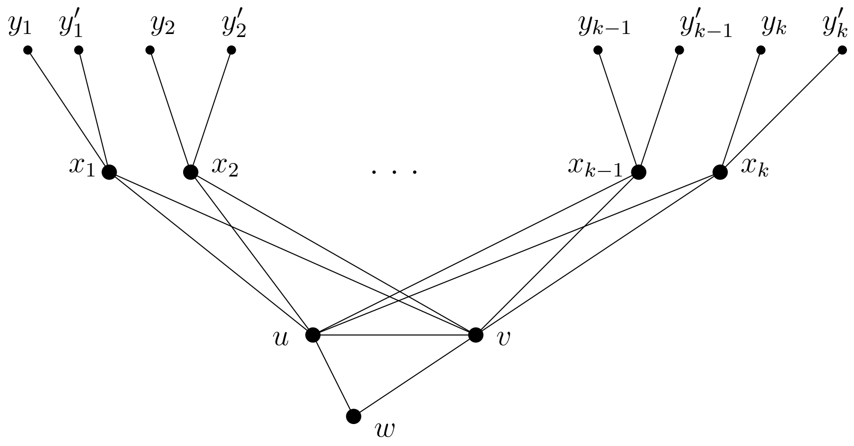
Definition 3.16. *An independent set A of vertices of G is an* asteroidal set *if for each vertex $a \in A$, when we remove a and its neighbours from the graph G, the set $A \setminus \{a\}$ is contained in one connected component of the remaining graph. The* asteroidal number *of G, denoted $AT(G)$ is the maximum cardinality of an asteroidal set of G. Graphs with $AT(G) \leq 2$ are called* asteroidal triple-free *graphs, and include interval graphs, permutation graphs, and complements of comparability graphs known as cocomparability graphs.*

Theorem 3.17. *For each integer $k \geq 2$, there exists a graph G with $AT(G) = k$ and $\gamma_{[1,k]}(G) > \gamma(G)$.*

Proof. Let k be a positive integer and G be the graph shown in the Figure 9. Then the set $Y = \{y_1, y_2, \ldots, y_k\}$ is an asteroidal set of size k. It can be shown, after checking the different cases, that there is no asteroidal set of size $k + 1$. On the other hand it is easily seen that every optimal dominating set of G is of the form $\{x_1, x_2, \ldots, x_k, a\}$, where $a \in \{u, v, w\}$ and so $\gamma(G) = k + 1$. Hence, for each optimal dominating set S, at least one of the vertices u or v does not belong to S. Let $u \notin S$, then u has more than k neighbours in S. Therefore, $\gamma_{[1,k]}(G) > \gamma(G) = k + 1$. Moreover we can show that $\gamma_{[1,k]}(G) = k + 2$. Consider the set $S' = \{x_1, x_2, \ldots, x_{k-1}, y_k, y'_k, w\}$, then S' is a $[1, k]$-dominating set of size $k + 2$. □

Question 3.18. *Is it true that for every graph G with $AT(G) \leq k$, we have $\gamma_{[1,k+1]}(G) = \gamma(G)$?*

We have not yet found a graph G with $AT(G) \leq k$ and $\gamma(G) \neq \gamma_{[1,k+1]}(G)$.



$\gamma(G) = k + 1$ and $\gamma_{[1,k]}(G) = k + 2$.

Figure 9: Graph G with $AT(G) = k$ and $\gamma_{[1,k]}(G) \neq \gamma(G)$

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[1] Amin, A., and Slater, P. J. Neighborhood domination with parity restrictions in graphs. *Congressus Numerantium* (1992), 19–19.

[2] Amin, A., and Slater, P. J. All parity realizable trees. *Journal of Combinatorial Mathematics and Combinatorial Computing 20* (1996), 53–64.

[3] Bishnu, A., Dutta, K., Gosh, A., and Pual, S. $(1, j)$-set problem in graphs. *Discrete Mathematics 339*, 1 (2016), 2215–2525.

[4] Bollobás, B., and Cockayne, E. J. Graph-theoretic parameters concerning domination, independence, and irredundance. *Journal of Graph Theory 3*, 3 (1979), 241–249.

[5] Brandstadt, A., Le, V. B., and Spinrad, J. *Graph classes: a survey*, vol. 3. Siam, 1999.

[6] Bui-Xuan, B.-M., Telle, J. A., and Vatshelle, M. Fast dynamic programming for locally checkable vertex subset and vertex partitioning problems. *Theoretical Computer Science 511* (2013), 66–76.

[7] Chellali, M., Haynes, T. W., Hedetniemi, S. T., and McRae, A. [1, 2]-sets in graphs. *Discrete Applied Mathematics 161*, 18 (2013), 2885–2893.

[8] Dejter, I. J. Quasiperfect domination in triangular lattices. *Discussiones Mathematicae Graph Theory 29*, 1 (2009), 179–198.

[9] Goharshady, A. K., Hooshmandasl, M. R., and Meybodi, M. A. [1, 2]-sets and [1,2]-total sets in trees with algorithms. *Discrete Applied Mathematics 198*, 10 (2016), 136–146.

[10] Haynes, T. W., Hedetniemi, S., and Slater, P. *Fundamentals of domination in graphs.* CRC Press, 1998.

[11] Haynes, T. W., Hedetniemi, S. T., and Slater, P. J. *Domination in graphs: advanced topics.* 1998.

[12] Kratochvíl, J. *Perfect codes in general graphs.* Academia, 1991.

[13] Oum, S., Sæther, S. H., and Vatshelle, M. Faster algorithms for vertex partitioning problems parameterized by clique-width. *Theoretical Computer Science 535* (2014), 16–24.

[14] Schaefer, T. J. The complexity of satisfiability problems. In *Proceedings of the tenth annual ACM symposium on Theory of computing* (1978), ACM, pp. 216–226.

[15] Telle, J. A. Characterization of domination-type parameters in graphs. *Proceedings SCCGC'93 - 24th Southeastern International Conference on Combinatorics, Graph Theory and Computing -Congressus Numerantium 94* (1993), 9–16.

[16] Telle, J. A. Complexity of domination-type problems in graphs. *Nordic Journal of Computing 1*, 1 (1994), 157–171.

[17] Telle, J. A. *Vertex partitioning problems: characterization, complexity and algorithms on partial k-trees.* PhD thesis, University of Oregon, 1994.

[18] Telle, J. A., and Proskurowski, A. Algorithms for vertex partitioning problems on partial k-trees. *SIAM Journal on Discrete Mathematics 10*, 4 (1997), 529–550.

[19] Vatshelle, M. *New width parameters of graphs.* PhD thesis, The University of Bergen, 2012.

[20] Yang, X., and Wu, B. [1, 2]-domination in graphs. *Discrete Applied Mathematics 175* (2014), 79–86.