Circular coloring of signed bipartite planar graphs

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Abstract. In this work, we study the notion of circular coloring of signed graphs which is a refinement of 0-free $2k$-coloring of signed graphs. The main question is that given a positive integer ℓ, what is the smallest even value $f(\ell)$ such that for every signed bipartite (simple) planar graph (G, σ) of negative-girth at least $f(\ell)$, we have $\chi_c(G, \sigma) \leq \frac{2\ell}{\ell-1}$. We answer this question when ℓ is small: $f(2) = 4, f(3) = 6$ and $f(4) = 8$. The results fit into the framework of the bipartite analogue of the Jaeger-Zhang conjecture.

Keywords: circular coloring, homomorphism, signed bipartite graphs

1 Introduction

The theory of graph homomorphism is a natural generalization of the notion of proper coloring of graphs. It's well-known that the C_{2k+1}-coloring problem captures the $(2k+1)$-coloring problem via a basic graph operation: Given a graph G, let G' be the graph obtained from G by subdividing each edge into a path of length $2k-1$. Then G' admits a homomorphism to C_{2k+1} if and only if G is properly $(2k+1)$-colorable (see [6]). Moreover, a graph admits a homomorphism to C_{2k+1} if and only if its circular chromatic number is at most $\frac{2k+1}{k}$.

A famous question relevant to the C_{2k+1}-coloring of planar graphs is the Jaeger-Zhang conjecture (introduced in [17] and studied in [16],[1],[8],[5] among others):

Conjecture 1. Every planar graph G of odd-girth at least $4k+1$ admits a homomorphism to C_{2k+1}, or equivalently, $\chi_c(G) \leq \frac{2k+1}{k}$.

Using the notion of circular coloring of signed graphs, we explore the theory for (negative) even cycles. A *signed graph* (G, σ) is a graph G (allowing loops and multi-edges) together with an assignment $\sigma : E(G) \to \{+, -\}$. The *sign of a closed walk* is the product of signs of all its edges (allowing repetition). Given a signed graph (G, σ) and a vertex v of (G, σ), a *switching at* v is to switch the signs of all the edges incident to v. We say a signed graph (G, σ') is *switching equivalent* to (G, σ) if it is obtained from (G, σ) by a series of switchings at vertices. It's proven in [15] that two signed graphs (G, σ_1) and (G, σ_2) are switching equivalent if and only if they have the same set of negative cycles.

A *(switching) homomorphism* of a signed graph (G,σ) to (H,π) is a mapping of $V(G)$ and $E(G)$ to $V(H)$ and $E(H)$ (respectively) such that the adjacencies, the incidences and the signs of closed walks are preserved. When there exists a homomorphism of (G,σ) to (H,π), we write $(G,\sigma)\to(H,\pi)$. A homomorphism of (G,σ) to (H,π) is said to be *edge-sign preserving* if it, furthermore, preserves the signs of the edges. When there exists an edge-sign preserving homomorphism of (G,σ) to (H,π), we write $(G,\sigma)\xrightarrow{s.p.}(H,\pi)$. The connection between these two kinds of homomorphism is established as follows: Given two signed graphs (G,σ) and (H,π), $(G,\sigma)\to(H,\pi)$ if and only if there exists an equivalent signature σ' of σ such that $(G,\sigma')\xrightarrow{s.p.}(H,\pi)$.

Observe that the parity of the lengths and the signs of closed walks are preserved by a homomorphism. Given a signed graph (G,σ) and an element $ij\in\mathbb{Z}_2^2$, we define $g_{ij}(G,\sigma)$ to be the length of a shortest closed walk whose number of negative edges modulo 2 is i and whose length modulo 2 is j. When there exists no such a closed walk, we say $g_{ij}(G,\sigma)=\infty$. By the definition of homomorphism of signed graphs, we have the following no-homomorphism lemma.

Lemma 1. [13] *If $(G,\sigma)\to(H,\pi)$, then $g_{ij}(G,\sigma)\geq g_{ij}(H,\pi)$ for each $ij\in\mathbb{Z}_2^2$.*

1.1 Circular coloring of signed graphs

The notion of circular coloring of signed graphs defined in [14] is a common extension of circular coloring of graphs and $2k$-coloring of signed graphs.

Given a real number r, a *circular r-coloring* of a signed graph (G,σ) is a mapping $\varphi: V(G)\to C^r$ such that for each positive edge uv of (G,σ), $\varphi(u)$ and $\varphi(v)$ are at distance at least 1, and for each negative edge uv of (G,σ), $\varphi(u)$ and the antipodal of $\varphi(v)$ are at distance at least 1. The *circular chromatic number* of a signed graph (G,σ) is defined as

$$\chi_c(G,\sigma)=\inf\{r\geq 1:(G,\sigma)\text{ admits a circular }r\text{-coloring}\}.$$

For integers $p\geq 2q>0$ such that p is even, the *signed circular clique* $K^s_{p;q}$ has the vertex set $[p]=\{0,1,\dots,p-1\}$, in which ij is a positive edge if and only if $q\leq|i-j|\leq p-q$ and ij is a negative edge if and only if either $|i-j|\leq\frac{p}{2}-q$ or $|i-j|\geq\frac{p}{2}+q$. Moreover, let $\hat{K}^s_{p;q}$ be the signed subgraph of $K^s_{p;q}$ induced by vertices $\{0,1,\dots,\frac{p}{2}-1\}$. The following statements are equivalent:

1. (G,σ) has a circular $\frac{p}{q}$-coloring;
2. (G,σ) admits an edge-sign preserving homomorphism to $K^s_{p;q}$;
3. (G,σ) admits a homomorphism to $\hat{K}^s_{p;q}$.

The next lemma is a straightforward consequence of the transitivity of the homomorphism relation.

Lemma 2. *If $(G,\sigma)\to(H,\pi)$, then $\chi_c(G,\sigma)\leq\chi_c(H,\pi)$.*

1.2 Homomorphism of signed bipartite graphs

Given a signed graph (G, σ), we define $T_l(G, \sigma)$ to be the signed graph obtained from (G, σ) by replacing each edge with a path of length l with the sign $-\sigma(uv)$. For a non-zero integer ℓ, we denote by C_ℓ the cycle of length $|\ell|$ whose sign agrees with the sign of ℓ. The *negative-girth* of a signed graph is defined to be the shortest length of a negative closed walk of it. In the following lemma, the k-coloring problem of graphs is captured by C_{-k}-coloring problem of signed graphs.

Lemma 3. [10] *A graph G is k-colorable if and only if $T_{k-2}(G, +)$ is C_{-k}-colorable.*

In particular, the $2k$-coloring problem of graphs is captured by the C_{-2k}-coloring problem of signed bipartite graphs. Thus we could restate the Four-Color Theorem as follows:

Theorem 1. *For any planar graph G, the signed bipartite planar graph $T_2(G, +)$ admits a homomorphism to C_{-4}.*

Moreover, the problem of mapping signed bipartite graphs to negative even cycles is equivalent to the question of bounding the circular chromatic number of signed bipartite graphs.

Proposition 1. *A signed bipartite graph (G, σ) admits a homomorphism to C_{-2k} if and only if $\chi_c(G, \sigma) \leq \frac{4k}{2k-1}$.*

Proof. Observe that the signed graph $\hat{K}^s_{4k;2k-1}$ is obtained from C_{-2k} by adding a negative loop at each vertex and $\chi_c(C_{-2k}) = \frac{4k}{2k-1}$. It suffices to prove that if $\chi_c(G, \sigma) \leq \frac{4k}{2k-1}$, then $(G, \sigma) \to C_{-2k}$. Let $\{x_1, \ldots, x_{4k}\}$ be the vertex set of $K^s_{4k;2k-1}$ and let φ be an edge-sign preserving homomorphism of (G, σ) to $K^s_{4k;2k-1}$. For the rest of the proof, addition in indices of vertices are considered mod $4k$.

Recall that in $K^s_{4k;2k-1}$, each x_i is adjacent with positive edges to three vertices furthest from it, namely x_{i+2k-1}, x_{i+2k} and x_{i+2k+1}, and it is adjacent with negative edges to three vertices closest to it, namely x_{i-1}, x_i and x_{i+1}. Let $K'^s_{4k;2k-1}$ be the signed graph obtained from $K^s_{4k;2k-1}$ by removing negative loops and positive edges $x_i x_{i+2k}$ for each i. We claim that $(G, \sigma) \to K'^s_{4k;2k-1}$. Let (A, B) be a bipartition of vertices of (G, σ) and let (X, Y) be the bipartition of $K'^s_{4k;2k-1}$ where $X = \{x_1, x_3, \ldots, x_{4k-1}\}$ and $Y = \{x_2, x_4, \ldots, x_{4k}\}$. For any $u \in V(G)$ with $\varphi(u) = x_i$, we define $\phi(u)$ as follows: If either $u \in A$ and i is even or $u \in B$ and i is odd, then $\phi(u) = x_{i+1}$; Otherwise, $\phi(u) = x_i$. It's easy to verify that ϕ is an edge-sign preserving homomorphism of (G, σ) to $K'^s_{4k;2k-1}$. Since $K'^s_{4k;2k-1} \to C_{-2k}$, it completes the proof.

Restricted to signed planar graphs, there is a bipartite analogue question of Jaeger-Zhang conjecture proposed in [12]: Given an integer k, what is the smallest value $f(k)$ such that every signed bipartite planar graph of negative-girth at least $f(k)$ admits a homomorphism to C_{-2k}?

2 Circular coloring of signed bipartite planar graphs

For a class $\mathcal{C}$ of signed graphs, we define $\chi_c(\mathcal{C}) = sup\{\chi_c(G, \sigma) : (G, \sigma) \in \mathcal{C}\}$. In the sequel, we denote the class of signed bipartite planar graph of negative-girth at least $2k$ by $\mathcal{SBP}_{2k}$. For this special class of signed graphs, some bounds have been already studied. It has been proved in [14] that $\chi_c(\mathcal{SBP}_4) = 4$.

2.1 Signed bipartite planar graphs of negative-girth at least 6

In this section, we will show that every signed bipartite planar graph of negative-girth at least 6 admits a homomorphism to $(K_{3,3}, M)$ in which the negative edges form a matching. Its connection with circular coloring of signed bipartite planar graphs is presented in the following lemma.

Lemma 4. *A signed bipartite graph (G, σ) admits a homomorphism to $(K_{3,3}, M)$ if and only if $\chi_c(G, \sigma) \leq 3$.*

Proof. As $\chi_c(K_{3,3}, M) = 3$, it remains to show that if $(G, \sigma) \xrightarrow{s.p.} K^s_{6;2}$, then $(G, \sigma) \to (K_{3,3}, M)$. Let φ be an edge-sign preserving homomorphism of (G, σ) to $K^s_{6;2}$ and let (A, B) be the bipartition of (G, σ). Let $\{x_1, x_2, \ldots, x_6\}$ be the vertex set of $K^s_{6;2}$. Let $K'^s_{6;2}$ be the signed graph obtained from $K^s_{6;2}$ by deleting all the negative loops and positive edges $x_i x_{i+2}$ for all the i (the index addition is taken (mod 6)). First of all, $K'^s_{6;2}$ is bipartite and let (X, Y) be its bipartition where $X = \{x_1, x_3, x_5\}$ and $Y = \{x_2, x_4, x_6\}$. Secondly, $K'^s_{6;2}$ is switching equivalent to $(K_{3,3}, M)$.

For any vertex u with $\varphi(u) = x_i$, we define ϕ to be as follows: if $u \in A$ and i is even or $u \in B$ and i is odd, then $\phi(u) = x_i$; otherwise, we switch at u and $\phi(u) = x_{i+3}$. It's easy to verify that ϕ is a homomorphism of $(G, \sigma) \to K'^s_{6;2}$. Hence $(G, \sigma) \to (K_{3,3}, M)$.

A *Signed Projective Cube* of dimension k, denoted $SPC(k)$, is a projective cube $PC(k)$ (the graph with vertex set $\mathbb{Z}_2^k$ where vertices u and v are adjacent if $u - v \in \{e_1, e_2, \ldots, e_d\} \cup \{J\}$) together with an assignment such that all the edges uv satisfying that $u - v = J$ are negative. The following result is implied from an edge-coloring result of [4] and a result of [11]:

Theorem 2. *If (G, σ) is a signed planar graph satisfying that $g_{ij}(G, \sigma) \geq g_{ij}(SPC(5))$, then $(G, \sigma) \to SPC(5)$.*

As $g_{10}(SPC(5)) = 6$ and $g_{01}(SPC(5)) = g_{11}(SPC(5)) = \infty$, it means that every signed bipartite planar graph of negative-girth at least 6 admits a homomorphism to $SPC(5)$.

Theorem 3. [11] *Let (G, σ) be a signed graph. We have that $(G, \sigma) \to SPC(k)$ if and only if there exists a partition of the edges of G, say $E_1, E_2, \ldots, E_{k+1}$, such that for each $i \in \{1, 2, \ldots, k+1\}$, the signature σ_i which assigns $-$ to the edges in E_i is switching equivalent to σ.*

We are now ready to prove our main theorem:

Theorem 4. *Every signed bipartite planar graph of negative-girth at least* 6 *admits a homomorphism to* $(K_{3,3}, M)$. *In other words,* $\chi_c(\mathcal{SBP}_6) \leq 3$.

Proof. Let (G, σ) be a signed bipartite planar graph of negative-girth at least 6 with a bipartition (A, B). By Theorem 2, $(G, \sigma) \rightarrow SPC(5)$. Thus, by Theorem 3, there exists a partition of the edges of G, say $E_1, E_2, \ldots, E_6$, such that for each $i \in [6]$, there is a signature σ_i equivalent to σ satisfying that E_i is the set of all negative edges in (G, σ_i).

We consider the signed graph (G, σ_1) where the set of negative edges is E_1. Contracting all the negative edges in E_1, we obtain a signed graph with only positive edges, denoted by G'. Observe that G' might contain parallel edges. It is also easily observed that a cycle C' of G' is odd if and only if it is obtained from a negative cycle C of (G, σ) by contraction. Since C must contain at least one edge from each of $E_2, E_3, \ldots, E_6$, C' is of length at least 5. In other words, G' is a planar graph with no loops and no triangle.

We conclude that G' is a triangle-free planar graph. Thus, by the Grötzsch theorem, G' admits a 3-coloring, say $\varphi : V(G') \rightarrow \{1, 2, 3\}$. Let (X, Y) be a bipartition of $(K_{3,3}, M)$ where $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ such that $\{x_1y_1, x_2y_2, x_3y_3\}$ is the set of negative edges. We can now define a mapping ψ of (G, σ_1) to $(K_{3,3}, M)$ as follows:

$$\psi(u) = \begin{cases} x_i, & \text{if } u \in A \text{ and } \varphi(u) = i \\ y_i, & \text{if } u \in B \text{ and } \varphi(u) = i. \end{cases}$$

It's easy to verify that ψ is an edge-sign preserving homomorphism of (G, σ_1) to $(K_{3,3}, M)$. It completes the proof.

Examples of signed bipartite graphs of negative-girth 4 which do not map to $(K_{3,3}, M)$ are given in [9]. So the negative-girth 6 is the best possible girth condition for signed bipartite graph (G, σ) to satisfy $\chi_c(G, \sigma) \leq 3$.

2.2 Signed bipartite planar graphs of negative-girth at least 8

In this section, we include the result that every signed bipartite planar graph of negative-girth at least 8 admits a homomorphism to C_{-4}. Generalizing the notion of H-critical graph defined by Catlin [2], we say a signed graph (G, σ) is (H, π)-*critical* if $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ for $ij \in \mathbb{Z}_2^2$, it does not admit a homomorphism to (H, π) but any proper subgraph of it does. The girth condition implies, in particular, that every C_{-4}-critical signed graph is bipartite.

Theorem 5. [10] *If* (G, σ) *is a* C_{-4}*-critical signed graph, then* $|E(G)| \geq \frac{4|V(G)|-1}{3}$.

Corollary 1. [10] *Every signed bipartite planar graph of negative-girth at least* 8 *maps to* C_{-4}.

Here the negative-girth condition 8 is the best possible because there exists a signed bipartite planar graph of negative-girth 6 which does not admit a homomorphism to C_{-4} as shown in [10]. Furthermore, considering the signed bipartite planar graphs Γ_n introduced in [14], we have $\lim_{n\to\infty} T_2(\Gamma_n) = \frac{8}{3}$. Therefore, $\chi_c(\mathcal{SBP}_8) = \frac{8}{3}$.

References

1. Borodin, O. V., Kim, S. J., Kostochka, A. V., & West, D. B. (2004). Homomorphisms from sparse graphs with large girth. Journal of Combinatorial Theory, Series B, 90(1), 147-159.
2. Catlin, P. A. (1988). Graph homomorphisms into the five-cycle. Journal of Combinatorial Theory, Series B, 45(2), 199-211.
3. Charpentier, C., Naserasr, R., & Sopena, É. (2020). Homomorphisms of sparse signed graphs. The Electronic Journal of Combinatorics, P3-6.
4. Dvořák, Z., Kawarabayashi, K. I. & Kral. D. (2016). Packing six T-joins in plane graphs. Journal of Combinatorial Theory, Series B, 116, 287-305.
5. Dvořák, Z., & Postle, L. (2017). Density of 5/2-critical graphs. Combinatorica, 37(5), 863-886.
6. Hell, P., & Nešetřil, J. (1990). On the complexity of H-coloring. Journal of Combinatorial Theory, Series B, 48(1), 92-110.
7. Kardoš, F., & Narboni, J. (2021). On the 4-color theorem for signed graphs. European Journal of Combinatorics, 91, 103215.
8. Lovász, L. M., Thomassen, C., Wu, Y., & Zhang, C. Q. (2013). Nowhere-zero 3-flows and modulo k-orientations. Journal of Combinatorial Theory, Series B, 103(5), 587-598.
9. Naserasr, R., Škrekovski, R., Wang, Z., & Xu, R. (2021). Mapping sparse signed graphs to (K_{2k}, M). arXiv preprint arXiv:2101.08619.
10. Naserasr, R., Pham, L. A., & Wang, Z. (2021). Density of C_{-4}-critical signed graphs. arXiv preprint arXiv:2101.08612.
11. Naserasr, R., Rollová, E., & Sopena, E. (2013). On homomorphisms of planar signed graphs to signed projective cubes. In The Seventh European Conference on Combinatorics, Graph Theory and Applications (pp. 271-276). Edizioni della Normale, Pisa.
12. Naserasr, R., Rollová, E., & Sopena, É. (2015). Homomorphisms of signed graphs. Journal of Graph Theory, 79(3), 178-212.
13. Naserasr, R., Sopena, E., & Zaslavsky, T. (2021). Homomorphisms of signed graphs: An update. European Journal of Combinatorics, 91, 103222.
14. Naserasr, R., Wang, Z., & Zhu, X. (2020). Circular chromatic number of signed graphs. arXiv preprint arXiv:2010.07525.
15. Zaslavsky, T. (1982). Signed graphs. Discrete Applied Mathematics, 4(1), 47-74.
16. Zhu, X. (2001). Circular chromatic number of planar graphs of large odd girth. The electronic journal of combinatorics, R25-R25.
17. Zhang, C. Q. (2002). Circular flows of nearly Eulerian graphs and vertex-splitting. Journal of Graph Theory, 40(3), 147-161.