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A New Approach Towards a Conjecture on Intersecting Three Longest Paths

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In 1966, T. Gallai asked whether every connected graph has a vertex that appears in all longest paths. Since then this question has attracted much attention and many work has been done in this topic. One important open question in this area is to ask whether any three longest paths contain a common vertex in a connected graph. It was conjectured that the answer to this question is positive. In this paper, we propose a new approach in view of distances among longest paths in a connected graph, and give a substantial progress towards the conjecture along the idea.

1. Introduction

In [5] Gallai asked whether every connected graph has a vertex that appears in all longest paths. This question has attracted much attention and many work has been done around this area of study. The answer to this question is false as stated; actually several counterexamples were given in [11, 12, 13]. A graph G is *hypotraceable* if G has no Hamiltonian path but every vertex-deleted subgraph $G - v$ has. Note that hypotraceable graphs constitute a large class of counterexamples. Thomassen [10] showed that there exist infinitely many planar hypotraceable graphs, meaning that there exist infinitely many counterexamples towards the question.

Yet there are classes of graphs for which the answer to Gallai's question is positive. To see this, note that, in a tree, all longest paths must contain its center(s). Klavžar and Petkovšek [8] showed that the answer is also positive

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for split graphs, cacti, and some other classes of graphs. Balister et al. [2], Joos [7] and Chen et al. [3] obtained similar results for the class of interval graphs, circular arc graphs, series-parallel graphs, respectively. Some other related results were obtained by Rautenbach and Sereni [9].

Regarding Gallai's question, what happens if we consider the intersection of a smaller number of longest paths? While we can easily check that every two longest paths share a vertex, it is not known whether every three longest paths also share a vertex. In [6] it appears as a conjecture, which has originally been asked by Zamfirescu since the 1980s (see [14]).

Conjecture 1.1. *For every connected graph, any three of its longest paths have a common vertex.*

So far, very little progress has been made on this conjecture. Axenovich [1] proved that Conjecture 1.1 is true for connected outerplanar graphs, and de Rezende et al. [4] proved that Conjecture 1.1 is true for connected graphs in which all nontrivial blocks are Hamiltonian.

In this paper, we introduce a new graph parameter in view of distances among longest paths in a connected graph. To state this, we give some basic definitions. For a graph G, let P be a path in G, and let x and y be the end-vertices of P. Note that $|V(P)| = 1$ if and only if $x = y$. For $X, Y \subseteq V(G)$, P is called an *X-Y path* if $V(P) \cap X = \{x\}$ and $V(P) \cap Y = \{y\}$. Let $u, v \in V(P)$. We let uPv denote the $\{u\}$-$\{v\}$ path on P. Furthermore, we let $\check{u}Pv = uPv - u$, $uP\check{v} = uPv - v$ and $\check{u}P\check{v} = uPv - \{u, v\}$.

Let G be a connected graph. Let $l(G)$ be the length of any longest path in G, and let $\mathcal{L}(G)$ be the set of longest paths of G; thus $\mathcal{L}(G) = \{P \mid P$ is a path in G with $|V(P)| = l(G) + 1\}$. For $x, y \in V(G)$ let $d_G(x, y)$ be the distance between x and y in G (i.e., the length of a shortest path joining x and y in G). Also for a vertex $x \in V(G)$ and a subset $U \subseteq V(G)$, let $d_G(x, U) = \min\{d_G(x, y) \mid y \in U\}$. For $\mathcal{P} \subseteq \mathcal{L}(G)$, the *distance sum* of a vertex v from $\mathcal{P}$ is defined as $\sum_{P \in \mathcal{P}} d_G(v, V(P))$, and let $f(G, \mathcal{P})$ be the minimum of the distance sum from $\mathcal{P}$ over all vertices in G.

Using this graph parameter, we can formulate Conjecture 1.1 as follows.

Conjecture 1.2. *Let G be a connected graph, and let $\mathcal{P}$ be a subset of $\mathcal{L}(G)$ with $|\mathcal{P}| = 3$. Then $f(G, \mathcal{P}) = 0$.*

As mentioned before, it is easy to check that any two longest paths of a connected graph have a common vertex. We now give the proof in this context.

Proposition 1.3. *Let G be a connected graph, and let $\mathcal{P}$ be a subset of $\mathcal{L}(G)$ with $|\mathcal{P}| = 2$. Then $f(G, \mathcal{P}) = 0$.*

Proof. Write $\mathcal{P} = \{P_1, P_2\}$, and for each $i \in \{1, 2\}$, let u_i and v_i be the end-vertices of P_i. Since G is connected, G has a $V(P_1)$-$V(P_2)$ path Q. Note that $V(P_1) \cap V(P_2) \neq \emptyset$ if and only if $|V(Q)| = 1$. For each $i \in \{1, 2\}$, write $V(P_i) \cap V(Q) = \{w_i\}$. We may assume that $|V(u_iP_iw_i)| \geq |V(v_iP_iw_i)|$ for each $i \in \{1, 2\}$. Then the length of the path $u_1P_1w_1Qw_2P_2u_2$ in G is $(|V(u_1P_1w_1)|-1)+(|V(Q)|-1)+(|V(u_2P_2w_2)|-1)$. On the other hand, for each $i \in \{1, 2\}$, $|V(u_iP_iw_i)|-1 \geq ((|V(u_iP_iw_i)|-1)+(|V(v_iP_iw_i)|-1))/2 = (|V(P_i)|-1)/2 = l(G)/2$. Consequently,

$$\begin{aligned}
(|V(u_1P_1w_1)| - 1) &+ (|V(u_2P_2w_2)| - 1) \\
&\geq \frac{l(G)}{2} + \frac{l(G)}{2} \\
&= l(G) \\
&\geq (|V(u_1P_1w_1)| - 1) + (|V(Q)| - 1) + (|V(u_2P_2w_2)| - 1).
\end{aligned}$$

This leads to $|V(Q)| = 1$, and hence $V(P_1) \cap V(P_2) \neq \emptyset$. □

In this paper, we give an upper bound of $f(G, \mathcal{P})$ with $|\mathcal{P}| = 3$, which is linear in terms of $|V(G)|$.

Theorem 1.4. *Let G be a connected graph of order n, and let $\mathcal{P}$ be a subset of $\mathcal{L}(G)$ with $|\mathcal{P}| = 3$. Then $f(G, \mathcal{P}) \leq (n + 6)/13$.*

After proving this bound in Section 2, in the follow-up section we show that to prove the conjecture it would be enough to improve our linear bound to any nondecreasing sublinear bound. Namely, we propose an equivalent conjecture towards Conjecture 1.1 in terms of the function $f(G, \mathcal{P})$.

2. Proof of Theorem 1.4

We start with some lemmas.

For a set $\mathcal{P}$ of graphs and $P \in \mathcal{P}$, set $X_{\mathcal{P}}(P) = V(P) - (\bigcup_{P' \in \mathcal{P} - \{P\}} V(P'))$.

Lemma 2.1. *Let G be a connected graph of order n, and let $\mathcal{P} \subseteq \mathcal{L}(G)$ with $|\mathcal{P}| = 3$. If $f(G, \mathcal{P}) > 0$, then $n \geq (3l(G) + \sum_{P \in \mathcal{P}} |X_{\mathcal{P}}(P)| + 3)/2$.*

Proof. Write $\mathcal{P} = \{P_1, P_2, P_3\}$. Since $\bigcap_{1 \leq i \leq 3} V(P_i) = \emptyset$,

$$n \geq |\bigcup_{1 \leq i \leq 3} V(P)| = \sum_{1 \leq i \leq 3} |X_{\mathcal{P}}(P_i)| + \sum_{1 \leq i < j \leq 3} |V(P_i) \cap V(P_j)|. \tag{2.1}$$

Since $l(G)+1 = |V(P_i)| = |X_{\mathcal{P}}(P_i)| + \sum_{j\neq i} |V(P_i)\cap V(P_j)|$ for each $1 \le i \le 3$,

$$\begin{aligned} 3l(G)+3 &= \sum_{1\le i\le 3} |X_{\mathcal{P}}(P_i)| + \sum_{1\le i\le 3} (\sum_{j\neq i} |V(P_i)\cap V(P_j)|) \\ &= \sum_{1\le i\le 3} |X_{\mathcal{P}}(P_i)| + 2\sum_{1\le i<j\le 3} |V(P_i)\cap V(P_j)|. \end{aligned} \tag{2.2}$$

By (2.1) and (2.2),

$$\begin{aligned} n &\ge \sum_{1\le i\le 3} |X_{\mathcal{P}}(P_i)| + \sum_{1\le i<j\le 3} |V(P_i)\cap V(P_j)| \\ &= \sum_{1\le i\le 3} |X_{\mathcal{P}}(P_i)| + (3l(G)+3 - \sum_{1\le i\le 3} |X_{\mathcal{P}}(P_i)|)/2 \\ &= (3l(G)+3+\sum_{1\le i\le 3} |X_{\mathcal{P}}(P_i)|)/2. \end{aligned}$$

Thus we get the desired conclusion. □

For a set $\mathcal{P}$ of three paths and $P \in \mathcal{P}$, let $t_{\mathcal{P}}(P)$ be the number of $V(P_1)$-$V(P_2)$ paths on P, where $\mathcal{P} - \{P\} = \{P_1, P_2\}$. If $\mathcal{P}$ consists of three longest paths of a connected graph, then $t_{\mathcal{P}}(P) \ge 1$ for every $P \in \mathcal{P}$ by Proposition 1.3.

Lemma 2.2. *Let G be a connected graph, and let $\mathcal{P} \subseteq \mathcal{L}(G)$ with $|\mathcal{P}| = 3$. Then $|X_{\mathcal{P}}(P)| \ge t_{\mathcal{P}}(P)(f(G,\mathcal{P})-1)$ for each $P \in \mathcal{P}$.*

Proof. We may assume that $f(G,\mathcal{P}) \ge 1$. Write $\mathcal{P} - \{P\} = \{P_1, P_2\}$, and let $\mathcal{Q}$ be the set of $V(P_1)$-$V(P_2)$ paths on P. Note that every path in $\mathcal{Q}$ has order at least two and $|\mathcal{Q}| = t_{\mathcal{P}}(P)$. Let $Q \in \mathcal{Q}$, and let u and v be the end-vertices of Q with $u \in V(P_1)$ and $v \in V(P_2)$. Then $V(Q) \cap X_{\mathcal{P}}(P) = V(Q) - \{u, v\}$. Since $u \in V(P) \cap V(P_1)$, $f(G,\mathcal{P}) \le \sum_{P'\in\mathcal{P}} d_G(u, V(P')) = d_G(u, V(P_2)) \le d_G(u, v) \le |V(Q)| - 1$. Hence $|V(Q) \cap X_{\mathcal{P}}(P)| = |V(Q)| - 2 \ge f(G,\mathcal{P}) - 1$. Since Q is arbitrary,

$$\sum_{Q\in\mathcal{Q}} |V(Q)\cap X_{\mathcal{P}}(P)| \ge t_{\mathcal{P}}(P)(f(G,\mathcal{P})-1). \tag{2.3}$$

Clearly, each vertex in $X_{\mathcal{P}}(P)$ belongs to at most one path in $\mathcal{Q}$. This together with (2.3) implies that $|X_{\mathcal{P}}(P)| \ge |\bigcup_{Q\in\mathcal{Q}} (V(Q) \cap X_{\mathcal{P}}(P))| = \sum_{Q\in\mathcal{Q}} |V(Q) \cap X_{\mathcal{P}}(P)| \ge t_{\mathcal{P}}(P)(f(G,\mathcal{P})-1)$. □

We also need the following lemma, which was originally proven in [1]. We give the proof of this lemma for the convenience of readers.

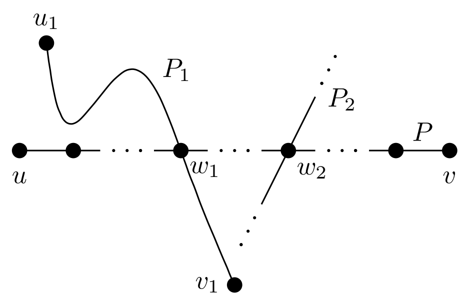

Figure 1: paths in $\mathcal{P}$

Lemma 2.3. *Let G be a connected graph, and let $\mathcal{P} \subseteq \mathcal{L}(G)$ with $|\mathcal{P}| = 3$. If there exists a path $P \in \mathcal{P}$ with $t_{\mathcal{P}}(P) = 1$, then $f(G, \mathcal{P}) = 0$.*

Proof. Suppose that $f(G, \mathcal{P}) > 0$. Let u and v be the end-vertices of P. Write $\mathcal{P} - \{P\} = \{P_1, P_2\}$, and for each $i \in \{1, 2\}$, let w_i be the vertex which is contained in P_i and the unique $V(P_1)$-$V(P_2)$ path on P (see Figure 1). We may assume that $|V(uPw_1)| \leq |V(uPw_2)|$. Since $f(G, \mathcal{P}) > 0$, $w_1 \neq w_2$, and hence $|V(w_1Pv)| > |V(w_2Pv)|$. Furthermore, we may assume that $|V(uPw_1)| \leq |V(vPw_2)|$. Since $l(G) = |V(uPw_1)| + |V(w_1Pv)| - 2$,

$$\begin{aligned} |V(w_1Pv)| &> \frac{|V(w_1Pv)|}{2} + \frac{|V(w_2Pv)|}{2} \\ &= \frac{l(G) - |V(uPw_1)| + 2}{2} + \frac{|V(w_2Pv)|}{2} \\ &\geq \frac{l(G) - |V(vPw_2)| + 2}{2} + \frac{|V(w_2Pv)|}{2} \\ &= \frac{l(G) + 2}{2}. \end{aligned} \tag{2.4}$$

Let u_1 and v_1 be the end-vertices of P_1. We may assume that $|V(u_1P_1w_1)| \geq |V(w_1P_1v_1)|$. Since $l(G) = |V(u_1P_1w_1)| + |V(w_1P_1v_1)| - 2$,

$$|V(u_1P_1w_1)| \geq \frac{|V(u_1P_1w_1)| + |V(w_1P_1v_1)|}{2} = \frac{l(G) + 2}{2}. \tag{2.5}$$

By (2.4) and (2.5), $|V(u_1P_1w_1)| + |V(w_1Pv)| - 2 > (l(G) + 2)/2 + (l(G) + 2)/2 - 2 = l(G)$. By the assumption that $t_{\mathcal{P}}(P) = 1$, the path $\breve{w}_1Pv$ contains no vertex in $V(P_1)$. Hence $P_1^{(1)} = u_1P_1w_1Pv$ is a path in G with length $|V(u_1P_1w_1)| + |V(w_1Pv)| - 2 > l(G)$, which is a contradiction. $\square$

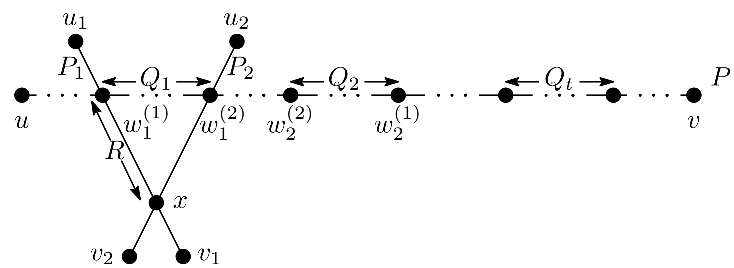

Figure 2: paths in $\mathcal{P}$

Proof of Theorem 1.4. We may assume that $f(G,\mathcal{P}) \geq 1$. Choose $P \in \mathcal{P}$ so that $t = t_{\mathcal{P}}(P)$ is as small as possible. Then $t_{\mathcal{P}}(P) \geq 2$ by Lemma 2.3. Let u and v be the end-vertices of P. Write $\mathcal{P} - \{P\} = \{P_1, P_2\}$, and let u_i and v_i be the end-vertices of P_i for each $i \in \{1,2\}$. Let $Q_1, Q_2, \cdots, Q_t$ be the $V(P_1)$-$V(P_2)$ paths on P which are aligned on P in order of indices with initial point u (i.e. for each $2 \leq i \leq t$, the unique $\{u\}$-$V(Q_i)$ path on P contains $\bigcup_{1\leq j \leq i-1} V(Q_j)$). We may assume that the length of the unique $\{u\}$-$V(Q_1)$ path on P is at least that of the unique $\{v\}$-$V(Q_t)$ path on P. For each $1 \leq i \leq t$ and each $j \in \{1,2\}$, write $V(Q_i) \cap V(P_j) = \{w_i^{(j)}\}$. We may assume that $|V(uPw_1^{(1)})| \leq |V(uPw_1^{(2)})|$. Let R be a $\{w_1^{(1)}\}$-$V(P_2)$ path on P_1, and write $V(R) \cap V(P_2) = \{x\}$. For each $i \in \{1,2\}$, we may assume that $|V(u_iP_iw_1^{(i)})| \leq |V(u_iP_ix)|$ (see Figure 2).

Since $w_1^{(1)} \in V(P) \cap V(P_1)$, $f(G,\mathcal{P}) \leq \sum_{P' \in \mathcal{P}} d_G(w_1^{(1)}, V(P')) = d_G(w_1^{(1)}, V(P_2)) \leq \min\{d_G(w_1^{(1)}, w_1^{(2)}), d_G(w_1^{(1)}, x)\} \leq \min\{|V(Q_1)|-1, |V(R)|-1\}$. Hence

$$(2.6) \qquad |V(Q_1)| \geq f(G,\mathcal{P}) + 1 \quad \text{and} \quad |V(R)| \geq f(G,\mathcal{P}) + 1.$$

Since $w_1^{(2)}Q_1\check{w}_1^{(1)}$ contains no vertex in $V(P_1)$, $w_1^{(2)}Q_1w_1^{(1)}Rx$ is a path in G. Furthermore, since $\check{w}_1^{(2)}Q_1w_1^{(1)}P_1\check{x}$ contains no vertex in $V(P_2)$,
(i) $S_1 = v_2P_2w_1^{(2)}Q_1w_1^{(1)}R\check{x}$,
(ii) $S_2 = u_2P_2w_1^{(2)}Q_1w_1^{(1)}RxP_2v_2$ and
(iii) $S_3 = u_2P_2xRw_1^{(1)}Q_1\check{w}_1^{(2)}$.
are paths in G (see Figure 3).

Since the length of S_1 is $(|V(v_2P_2w_1^{(2)})|-1)+(|V(Q_1)|-1)+(|V(w_1^{(1)}R\check{x})|-1)$ and $|V(w_1^{(1)}R\check{x})| = |V(R)|-1$, we have $(|V(v_2P_2w_1^{(2)})|-1)+(|V(w_1^{(2)}P_2u_2)|-1) = |V(P_2)|-1 = l(G) \geq (|V(v_2P_2w_1^{(2)}|-1)+(|V(Q_1)|-1)+(|V(R)|-2)$.

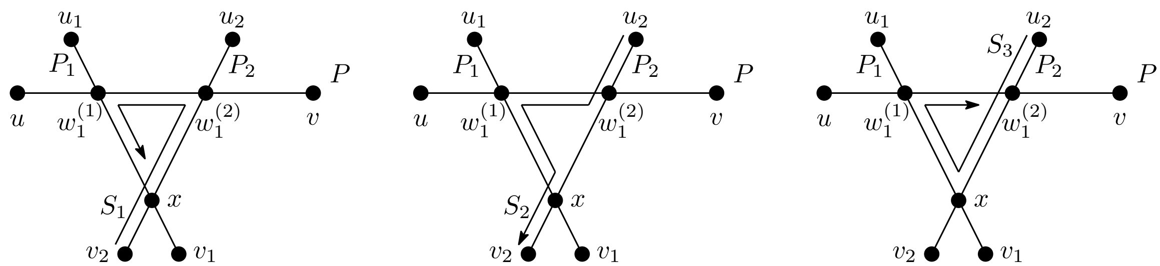

Figure 3: path S_i

This together with (2.6) leads to

$$|V(u_2P_2w_1^{(2)})| \geq |V(Q_1)| + |V(R)| - 2 \geq 2f(G,\mathcal{P}). \tag{2.7}$$

By comparing the length of P_2 and S_2 and (2.6), we have

$$|V(w_1^{(2)}P_2x)| \geq |V(Q_1)| + |V(R)| - 1 \geq 2f(G,\mathcal{P}) + 1. \tag{2.8}$$

By comparing the length of P_2 and S_3 and (2.6), we also have

$$|V(xP_2v_2)| \geq |V(Q_1)| + |V(R)| - 2 \geq 2f(G,\mathcal{P}). \tag{2.9}$$

Therefore

$$\begin{aligned} l(G) &= |V(P_2)| - 1 \\ &= |V(u_2P_2w_1^{(2)})| + |V(w_1^{(2)}P_2x)| + |V(xP_2v_2)| - 3 \\ &\geq 2f(G,\mathcal{P}) + (2f(G,\mathcal{P}) + 1) + 2f(G,\mathcal{P}) - 3 \\ &= 6f(G,\mathcal{P}) - 2. \end{aligned} \tag{2.10}$$

Case 1: $t_{\mathcal{P}}(P) = 2$.

It is easy to check that $|V(vPw_2^{(1)})| \leq |V(vPw_2^{(2)})|$. Since the path $uP\check{w}_1^{(2)}$ contains no vertex in $V(P_2)$, $T = uPw_1^{(2)}P_2v_2$ is a path in G (see Figure 4). Since the length of T is $(|V(uPw_1^{(1)})| - 1) + (|V(Q_1)| - 1) + (|V(w_1^{(2)}P_2v_2)| - 1)$, $(|V(u_2P_2w_1^{(2)})| - 1) + (|V(w_1^{(2)}P_2v_2)| - 1) = |V(P_2)| - 1 = l(G) \geq (|V(uPw_1^{(1)})| - 1) + (|V(Q_1)| - 1) + (|V(w_1^{(2)}P_2v_2)| - 1)$. This together

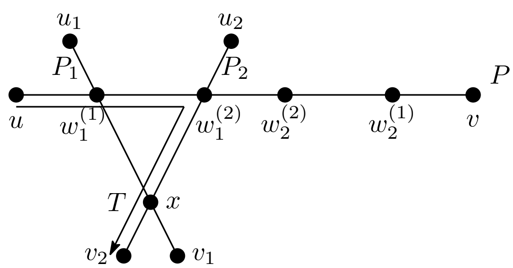

Figure 4: path T

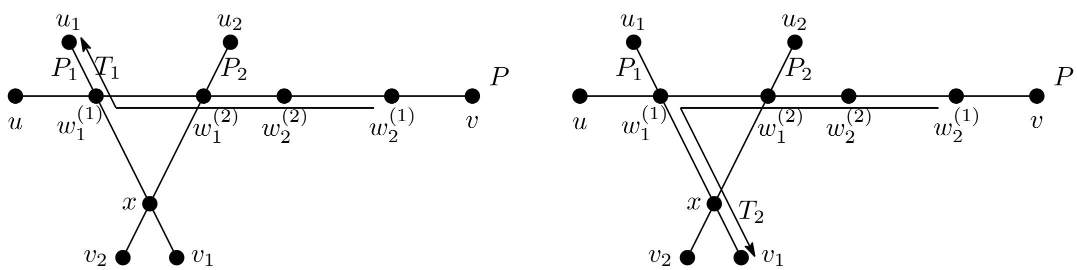

Figure 5: path T_i

with (2.6) leads to

(2.11)
$$|V(u_2P_2w_1^{(2)})| \geq |V(uPw_1^{(1)})| + |V(Q_1)| - 1 \geq |V(uPw_1^{(1)})| + f(G, \mathcal{P}).$$

Since the path $\check{w}_2^{(1)}P\check{w}_1^{(1)}$ contains no vertex in $V(P_1)$, both $T_1 = \check{w}_2^{(1)}Pw_1^{(1)}P_1u_1$ and $T_2 = \check{w}_2^{(1)}Pw_1^{(1)}P_1v_1$ are paths in G (see Figure 5). Since the length of T_1 is $(|V(\check{w}_2^{(1)}Pw_1^{(1)})| - 1) + (|V(w_1^{(1)}P_1u_1)| - 1)$, we have $(|V(vPw_2^{(1)})| - 1) + (|V(w_2^{(1)}Pw_1^{(1)})| - 1) + (|V(w_1^{(1)}Pu)| - 1) = |V(P)| - 1 = l(G) \geq (|V(\check{w}_2^{(1)}Pw_1^{(1)})| - 1) + (|V(w_1^{(1)}P_1u_1)| - 1)$. Consequently, we have $|V(vPw_2^{(1)})| + |V(w_1^{(1)}Pu)| \geq |V(w_1^{(1)}P_1u_1)|$. By comparing the length of P and T_2, we also have $|V(vPw_2^{(1)})| + |V(w_1^{(1)}Pu)| \geq |V(w_1^{(1)}P_1v_1)|$. Hence

$$\begin{aligned} l(G) &= |V(P_1)| - 1 \\ &= |V(u_1P_1w_1^{(1)})| + |V(w_1^{(1)}P_1v_1)| - 2 \\ &\leq 2(|V(vPw_2^{(1)})| + |V(w_1^{(1)}Pu)|) - 2. \end{aligned} \tag{2.12}$$

Recall that the length of the unique $\{u\}$-$V(Q_1)$ path on P (i.e. $uPw_1^{(1)}$) is at least that of the unique $\{v\}$-$V(Q_2)$ path on P (i.e. $vPw_2^{(1)}$). Hence $|V(uPw_1^{(1)})| \geq |V(vPw_2^{(1)})|$. By (2.12), $l(G) \leq 2(|V(vPw_2^{(1)})|+|V(w_1^{(1)}Pu)|)-2 \leq 4|V(uPw_1^{(1)})|-2$, and so $|V(uPw_1^{(1)})| \geq (l(G)+2)/4$. This together with (2.11) implies that

$$|V(u_2P_2w_1^{(2)})| \geq \frac{l(G)+2}{4} + f(G,\mathcal{P}). \tag{2.13}$$

By (2.8), (2.9) and (2.13),

$$\begin{aligned} l(G) &= |V(P_2)|-1 \\ &= |V(u_2P_2w_1^{(2)})| + |V(w_1^{(2)}P_2x)| + |V(xP_2v_2)| - 3 \\ &\geq (\frac{l(G)+2}{4} + f(G,\mathcal{P})) + (2f(G,\mathcal{P})+1) + 2f(G,\mathcal{P}) - 3 \\ &= \frac{l(G)-6}{4} + 5f(G,\mathcal{P}), \end{aligned}$$

and so

$$l(G) \geq \frac{20f(G,\mathcal{P})-6}{3}. \tag{2.14}$$

By the choice of P, $t_{\mathcal{P}}(P') \geq 2$ for every $P' \in \mathcal{P}$. By Lemma 2.2, $\sum_{P'\in\mathcal{P}} |X_{\mathcal{P}}(P')| \geq \sum_{P'\in\mathcal{P}} t_{\mathcal{P}}(P')(f(G,\mathcal{P})-1) \geq 6(f(G,\mathcal{P})-1)$. This together with Lemma 2.1 and (2.14) implies that

$$\begin{aligned} n &\geq \frac{3l(G) + \sum_{P'\in\mathcal{P}} |X_{\mathcal{P}}(P')| + 3}{2} \\ &\geq \frac{3\cdot\frac{20f(G,\mathcal{P})-6}{3} + 6(f(G,\mathcal{P})-1)+3}{2} \\ &= \frac{26f(G,\mathcal{P})-9}{2}, \end{aligned}$$

and hence $f(G,\mathcal{P}) \leq (2n+9)/26$.

Case 2: $t_{\mathcal{P}}(P) \geq 3$.

By the choice of P, $t_{\mathcal{P}}(P') \geq 3$ for every $P' \in \mathcal{P}$. By Lemma 2.2, $\sum_{P'\in\mathcal{P}} |X_{\mathcal{P}}(P')| \geq \sum_{P'\in\mathcal{P}} t_{\mathcal{P}}(P')(f(G,\mathcal{P})-1) \geq 9(f(G,\mathcal{P})-1)$. This together with Lemma 2.1 and (2.10) implies that

$$\begin{aligned} n &\geq \frac{3l(G) + \sum_{P' \in \mathcal{P}} |X_{\mathcal{P}}(P')| + 3}{2} \\ &\geq \frac{3(6f(G,\mathcal{P}) - 2) + 9(f(G,\mathcal{P}) - 1) + 3}{2} \\ &= \frac{27f(G,\mathcal{P}) - 12}{2}, \end{aligned}$$

and hence $f(G,\mathcal{P}) \leq (2n+12)/27$.

This completes the proof of Theorem 1.4. □

To conclude this section, we propose the following conjecture.

Conjecture 2.4. *Let G be a connected graph, and let $\mathcal{P} \subseteq \mathcal{L}(G)$ with $|\mathcal{P}| = 3$. If there exists a path $P \in \mathcal{P}$ with $t_{\mathcal{P}}(P) = 2$, then $f(G,\mathcal{P}) = 0$.*

If Conjecture 2.4 is true, then we can improve the upper bound of $f(G,\mathcal{P})$ in Theorem 1.4 to $(2n+12)/27$ (by the argument in the proof of Theorem 1.4).

3. Bounding the value of $f(G,\mathcal{P})$ by a sublinear function

A function g is *sublinear* if $\lim_{n\to+\infty} \frac{g(n)}{n} = 0$. It follows from the definition that, if g is sublinear, then for any two constants c_0, c_1, we have $g(c_0 t + c_1) < t$ for any large t. Here we pose the following new conjecture, which concerns Conjecture 1.2. Although Conjecture 3.1 is seemingly weaker than Conjecture 1.2, we will show that Conjecture 3.1 is indeed equivalent with Conjecture 1.2.

Conjecture 3.1. *There exists a sublinear non-decreasing function g such that for every connected graph G of order n and every subset $\mathcal{P}$ of $\mathcal{L}(G)$ with $|\mathcal{P}| = 3$, $f(G,\mathcal{P}) \leq g(n)$.*

To prove that this seemingly weaker conjecture is equivalent to Conjecture 1.2, we first show that for a given graph G with a set $\{P_1, P_2, P_3\}$ of three longest paths one can choose a subdivision of G so that subdivisions of P_i's $i = 1, 2, 3$ are the new longest paths and show that the minimum distance from these three subdivided paths in the subdivided graph grows linearly in the order of subdivision. For the exact statement we introduce the following notation.

Let G be a connected graph and let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a set of three longest paths. Let G' be obtained by adding a new edge to each end-vertex of P_i's, $i = 1, 2, 3$; thus, minimum of two and maximum of six new vertices and edges are added. Let P_i', $i = 1, 2, 3$ be the path corresponding to P_i with two new edges at the two ends. We define G^t to be the graph obtained from G' by subdividing each edge t times. Let P_i^t, $i = 1, 2, 3$ be the corresponding path of P_i' in G^t. We write $\mathcal{P}^t = \{P_1^t, P_2^t, P_3^t\}$. Also, let $V_{f(G,\mathcal{P})} = \{v \in V(G) \mid \sum_{P \in \mathcal{P}} d_G(v, V(P)) = f(G, \mathcal{P})\}$.

We have the following proposition.

Proposition 3.2. *Given a connected graph G and a set $\mathcal{P} = \{P_1, P_2, P_3\}$ of three longest paths, the set $\mathcal{P}^t = \{P_1^t, P_2^t, P_3^t\}$ is a set of three longest paths of G^t. Furthermore, $f(G^t, \mathcal{P}^t) = (t+1)f(G, \mathcal{P})$.*

Proof. Note that for each $i \in \{1, 2, 3\}$, P_i^t has of length $(t+1)(|V(P_i)|+1)$. If there is a path P^t in G^t that is longer than P_1^t, then by replacing all subdivided paths to the original edges in G, P^t corresponds to a path in G of length at least $|V(P_1)|$. However, it is longer than P_1, a contradiction. Therefore, the first assertion holds.

To prove the second assertion, we show that a vertex of $V_{f(G^t,\mathcal{P}^t)}$ could be chosen as an original vertex of G. The assertion then would follow, as the vertex of G attaining the distance sum $f(G, \mathcal{P})$ from $\mathcal{P}$ has the distance sum $(t+1)f(G, \mathcal{P})$ from $\mathcal{P}^t$ in G^t.

Now let u be a vertex attaining the distance sum $f(G^t, \mathcal{P}^t)$ from $\mathcal{P}^t$. It is easy to check that u is not an end-vertex of P_i^t for any i. If $u \in V(G)$, then we have nothing to prove. Otherwise u is a new vertex subdividing an edge, say xy, of G. We may assume, without loss of generality, that at least two of the shortest paths from u to P_i^t go through x. However, x has a smaller distance sum than $f(G^t, \mathcal{P}^t)$, a contradiction. We note that if u belongs to one or two of these paths then so does x and y, and hence this would not affect the argument. The contradiction proves that u must be a vertex of G and we have $f(G^t, \mathcal{P}^t) = (t+1)f(G, \mathcal{P})$. $\square$

Keeping the above proposition in mind, we can prove the following theorem.

Theorem 3.3. *Conjecture 1.2 is true if and only if Conjecture 3.1 is true.*

Proof. The "only if" part is trivial, and hence we only show the "if" part.

Suppose that G together with $\mathcal{P} = \{P_1, P_2, P_3\}$ is a counterexample for Conjecture 1.2, i.e., $f(G, \mathcal{P}) \geq 1$. The subgraph of G induced by edges and vertices of P_1, P_2, P_3 is also a counterexample (where $\mathcal{P}$ is also a set of non-intersecting three longest paths). Note that, in view of Proposition 1.3, such a subgraph is connected. Thus we may assume from the start that vertices and edges of G are union of vertices and edges of P_1, P_2, P_3.

Let n_0 be the number of vertices of G. Since G is the union of three paths of length at most $n_0 - 1$, we conclude that G has at most $3(n_0 - 1)$ edges. Therefore, G' has at most $n_0 + 6$ vertices and at most $3(n_0 + 1)$ edges. Since G^t is obtained by subdividing edges of G' (each edge exactly t times), we have $|V(G^t)| \leq n_0 + 6 + 3(n_0 + 1)t$. On the other hand, we have $f(G^t, \mathcal{P}^t) = (t+1)f(G, \mathcal{P}) \geq t + 1$. Hence for constants $c_0 = 3(n_0 + 1)$ and $c_1 = n_0 + 6$, it follows from Conjecture 3.1 that

$$g(c_0 t + c_1) \geq g(|V(G^t)|) \geq f(G^t, \mathcal{P}^t) \geq t + 1.$$

(The first inequality follows from the condition that g is non-decreasing). Therefore, the inequality $g(c_0 t + c_1) \geq t+1$ holds for any t, which contradicts the fact that g is a sublinear function. $\square$

In conclusion, Theorem 3.3 tells us that giving a substantial improvement on the magnitude of the upper bound of $f(G, \mathcal{P})$ in Theorem 1.4 settles the longstanding conjecture on intersecting three longest paths in a connected graph.

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