Extremal graphs for the identifying code problem☆

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Abstract

An identifying code of a graph G is a dominating set C such that every vertex x of G is distinguished from other vertices by the set of vertices in C that are at distance at most 1 from x. The problem of finding an identifying code of minimum possible size turned out to be a challenging problem. It was proved by N. Bertrand that if a graph on n vertices with at least one edge admits an identifying code, then a minimal identifying code has size at most $n-1$. They introduced classes of graphs whose smallest identifying code is of size $n-1$. Few conjectures were formulated to classify the class of all graphs whose minimum identifying code is of size $n-1$.

In this paper, disproving these conjectures, we classify all finite graphs for which all but one of the vertices are needed to form an identifying code. We classify all infinite graphs needing the whole set of vertices in any identifying code. New upper bounds in terms of the number of vertices and the maximum degree of a graph are also provided.

Key words: Identifying codes, Dominating sets, Infinite graphs.

1. Introduction

Given a graph G, an identifying code of G is a subset C of vertices of G such that the subset of C at distance at most 1 from a given vertex x is nonempty and uniquely determines x. Identifying codes have been widely studied since the introduction of the concept in [14], and have been applied to problems such as fault-diagnosis in multiprocessor systems [14], compact routing in networks [15], emergency sensor networks in facilities [17] or the analysis of secondary RNA structures [13].

The concept of identifying codes of graphs is related to several other concepts, such as locating-dominating sets [19, 20] for graphs and the well celebrated theorem of Bondy [1] on set systems.

The purpose of this paper is to classify extremal cases in some previously known upper bounds for the minimum size of identifying codes and thus also improving those upper bounds. We begin by introducing our terminology.

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Unless specifically mentioned $G = (V, E)$ will be a finite simple graph with $n = |V|$ being the number of vertices. The degree of a vertex x is denoted $deg(x)$. By $\Delta(G)$ we denote the maximum degree of G.

For two vertices x and y of G, we denote by $d_G(x, y)$ (or $d(x, y)$ if there is no ambiguity) the distance between x and y in G. The *ball* of radius r centered at x, denoted $B_r(x)$, is the set vertices at distance at most r of x. We note that x belongs to $B_r(x)$ for every r. A vertex x of G is *universal* if $B_1(x) = V(G)$. Given a subset S of $V(G)$, we say that a vertex x is S-universal if $S \subseteq B_1(x)$. The symmetric difference of two sets A and B is denoted by $A \ominus B$. Given a pair of vertices of a graph G, we write $\ominus_r(x, y) = B_r(x) \ominus B_r(y)$. Two vertices x and y are called *twins* in G if $B_1(x) = B_1(y)$. A graph is called *twin-free* if it has no pair of twin vertices. The complement of a graph G is denoted by $\overline{G}$. For $r \geq 2$, the r^{th}*-power* of G, is the graph $G^r = (V, E')$ with $E' = \{xy \mid x, y \in V, d_G(x, y) \leq r\}$. Conversely if $H^r \cong G$, then we say H is an r*-root* of G. We denote by $G - x$ the graph obtained from G by removing x from $V(G)$ and all edges containing x from $E(G)$. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, $G_1 \bowtie G_2$ is the *join graph* of G_1 and G_2. Its vertex set is $V_1 \cup V_2$ and its edge set is $E_1 \cup E_2 \cup \{x_1x_2 \mid x_1 \in V_1, x_2 \in V_2\}$. We denote by K_n, the complete graph on n vertices, by P_n, the path on n vertices, and by $K_{a,b}$, the complete bipartite graph with bipartitions of sizes a and b.

Given a graph G and an integer $k \geq 2$, a subset I of vertices of G is called a k*-independent set* if for all distinct vertices x, y of I, $d_G(x, y) \geq k$. A 2-independent set is simply an *independent set*. Given an integer $r \geq 1$, a subset S of vertices of G is called an r*-dominating set* if for every vertex x of G, $B_r(x) \cap S \neq \emptyset$. We say that S r*-separates* two vertices x and y, if $B_r(x) \cap S \neq B_r(y) \cap S$. A subset S of vertices is an r*-separating set* if it r-separates all distinct vertices x, y of G. If S is both r-dominating and r-separating, S is an r*-identifying code* [14]. If S is r-dominating and r-separates vertices of $V(G) \setminus S$, it is called an r*-locating-dominating set* [20]. Given a bipartite graph G with a partition $V = I \cup A$, a subset S of A is said to be an r*-discriminating code* [5] if S r-separates all pairs of distinct vertices of I.

In each of the previous concepts when $r = 1$, we simply use the name of the concept without specifying the value of r.

Note that a set C is an r-separating set of G (resp. r-identifying code) if and only if it is a separating set (resp. identifying code) of G^r. A graph G admits a separating set (resp. identifying code) if and only if it is twin-free, as a consequence it admits an r-separating set (resp. r-identifying code) if and only if G^r is twin-free [6].

For a graph G, the minimum cardinalities of an r-dominating set and of an r-locating-dominating set are commonly denoted by $\gamma_r(G)$ and $\gamma_r^{\text{LD}}(G)$. If G^r is twin-free, we denote by $\gamma_r^{\text{ID}}(G)$ (respectively $\gamma_r^{\text{S}}(G)$) the minimum cardinality of an r-identifying code (r-separating set) of G. It is clear from the definition that $\gamma_r^{\text{S}}(G) \leq \gamma_r^{\text{ID}}(G) \leq \gamma_r^{\text{S}}(G) + 1$.

While the exact value of γ^{ID} for some classes of graphs has been determined [3, 4], finding the value of $\gamma_r^{\text{ID}}(G)$ for a general graph G is known to be NP-hard for any $r \geq 1$ [7, 9].

Upper bounds, in terms of basic graph parameters, have been given for the minimum sizes of the corresponding sets for most of the previously defined concepts. In particular it has been shown that $\gamma_r^{\text{LD}}(G) \le |V(G)| - 1$ and $\gamma_r^{\text{ID}}(G) \le |V(G)| - 1$ unless $G \cong \overline{K_n}$ (see [8, 12, 19]).

For the case of locating-dominating sets, it was proved in [19] that for a connected graph G we have $\gamma^{\text{LD}}(G) = |V(G)| - 1$ if and only if G is either a star or a complete graph.

In this paper, we do the analogous classification for identifying codes. In the case of identifying codes, the class of graphs reaching this bound is a much richer family. Thus we answer, in negative, the two attempted conjectures for such classification [5, 18]. This gives a partial answer to a question posed in [5]. This is done in Section 3.

Examples of nontrivial infinite graphs for which the whole vertex set is needed to form an identifying code are given in [8]. We classify all such infinite graphs in Section 4. In Section 5 we introduce new upper bounds for γ^{ID} in terms of n and Δ. In all these sections we address the problem of identifying codes only for $r = 1$. In Section 6 we consider general r-identifying codes.

The next section provides a set of preliminary results.

2. Preliminary results

In this section we have put together some basic results necessary for our main work. These results could be useful in the study of identifying codes in general. We start by recalling the following theorem.

Theorem 1 ([2, 12]). *Let G be a twin-free graph on n vertices having at least one edge. Then $\gamma^{ID}(G) \le n-1$.*

It is shown in [8] that this bound is tight. In particular it is shown that for any $t \ge 2$, $\gamma^{\text{ID}}(K_{1,t}) = t$. A stronger result is proved in Section 5 (see Lemma 19).

The next lemma is an obvious but a crucial one.

Lemma 2. *Let G be a twin-free graph and let C be an identifying code of G. Then, any set $C' \subseteq V(G)$ such that $C \subseteq C'$ is an identifying code of G.*

The next proposition is useful in proving upper bounds on minimum identifying codes by induction.

Proposition 3. *Let G be a twin-free graph and $S \subseteq V(G)$ such that $G - S$ is twin-free. Then $\gamma^{ID}(G) \le \gamma^{ID}(G - S) + |S|$.*

Proof. Take a minimum code C_0 of $G - S$. Consider the vertices of S in an arbitrary order $(x_1, \ldots, x_{|S|})$. Using induction we extend C_0 to a subset C_i of G which identifies the vertices in $V_i = V(G) \setminus \{x_{i+1}, \ldots, x_{|S|}\}$. To do this, if C_{i-1} identifies all the vertices of V_i, we are done. Otherwise, since all the vertices in V_{i-1} are identified, either $B_1(x_i) \cap C_{i-1} = B_1(y) \cap C_{i-1}$ for exactly one vertex y in V_{i-1}, or x_i is not dominated by C_{i-1}. In the first case x_i and y are separated in G by some vertex, say u, so let $C_i = C_{i-1} \cup \{u\}$. In the second case, let $C_i = C_{i-1} \cup \{x_i\}$. Now, in both cases, C_i identifies all the vertices of V_i. At step $|S|$, $C_{|S|}$ is an identifying code of G of size at most $|C_0| + |S| \leq \gamma^{\text{ID}}(G - S) + |S|$. □

We will need the following special case of the previous proposition.

Corollary 4. *Let G be a graph with $\gamma^{ID}(G) = |V(G)| - 1$, $G \ncong K_{1,2}$, then there is a vertex x of G such that $\gamma^{ID}(G - x) = |V(G - x)| - 1$.*

Proof. If $G \cong K_{1,t}$, $t \neq 2$, then any leaf vertex works. Thus, we may suppose $G \ncong K_{1,t}$. Then by Theorem 1, there is a vertex x of G such that $V(G - x)$ is an identifying code of G and thus $G - x$ is twin-free and $G - x \ncong \overline{K_n}$. By Proposition 3, we have $\gamma^{\text{ID}}(G - x) \geq \gamma^{\text{ID}}(G) - 1 = |V(G - x)| - 1$. Equality holds since otherwise $\gamma^{\text{ID}}(G) = |V(G)|$. □

Lemma 5. *Let G be a twin-free graph and let $v \in V(G)$. Let x, y be a pair of twins in $G - v$. If $G - x$ or $G - y$ has a pair of twins, then v must be one of the vertices of the pair.*

Proof. Since v separates x and y, it is adjacent to one of them (say x) and not to the other. Suppose z, t are twins in $G - x$. Suppose z is adjacent to x and t is not. If $z \neq v$ then y is also adjacent to z and, therefore, t is also adjacent to y which implies x being adjacent to t. This contradicts the fact that x separates z and t. The other case is proved similarly. □

Proposition 6. *Let G_1 and G_2 be twin-free graphs such that for every minimum separating set S there is an S-universal vertex. If $G_1 \bowtie G_2$ is twin-free, then we have $\gamma^S(G_1 \bowtie G_2) = \gamma^S(G_1) + \gamma^S(G_2) + 1$. Furthermore, if S is a separating set of size $\gamma^S(G_1) + \gamma^S(G_2) + 1$ of $G_1 \bowtie G_2$, then there is an S-universal vertex.*

Proof. Let S be a minimum separating set of $G_1 \bowtie G_2$. Since vertices of G_2 do not separate any pair of vertices in G_1 then $S \cap V(G_1)$ is a separating set of G_1. By the same argument $S \cap V(G_2)$ is a separating set of G_2. Therefore, $|S| \geq \gamma^{\text{S}}(G_1) + \gamma^{\text{S}}(G_2)$. But if $|S| = \gamma^{\text{S}}(G_1) + \gamma^{\text{S}}(G_2)$, then there is a $[S \cap V(G_1)]$-universal vertex x in G_1 and a $[S \cap V(G_2)]$-universal vertex y in G_2. But then, x and y are not separated by S.

Given a separating set S_1 of G_1 and a separating set S_2 of G_2, the set $S_1 \cup S_2$ separates all pairs of vertices except the S_1-universal vertex of G_1 from the S_2-universal vertex of G_2. But since $G_1 \bowtie G_2$ is twin-free, we could add one more vertex to $S_1 \cup S_2$ to obtain a separating set of $G_1 \bowtie G_2$ of size $\gamma^{\text{S}}(G_1) + \gamma^{\text{S}}(G_2) + 1$.

For the second part assume S is a separating set of size $\gamma^{\text{S}}(G_1) + \gamma^{\text{S}}(G_2) + 1$ of $G_1 \bowtie G_2$. Then we have either $|S \cap V(G_1)| = \gamma^{\text{S}}(G_1)$ or $|S \cap V(G_2)| = \gamma^{\text{S}}(G_2)$. Without loss of generality assume the former. Then there is a $[S \cap V(G_1)]$-universal vertex z of G_1. Since z is also adjacent to all the vertices of G_2, it is an S-universal vertex of $G_1 \bowtie G_2$. □

In Proposition 6 if $G_1 \ncong K_1$ and $G_2 \ncong K_1$, then $\gamma^{\text{ID}}(G_1 \bowtie G_2) = \gamma^{\text{S}}(G_1 \bowtie G_2) = \gamma^{\text{S}}(G_1) + \gamma^{\text{S}}(G_2) + 1$.

The following lemma was discovered in a discussion between the first author, R. Klasing and A. Kosowski. We include a proof for the sake of completeness.

Lemma 7 ([11])**.** *Let G be a connected twin-free graph, and I be a 4-independent set such that for every vertex x of I, the set $V(G) \setminus \{x\}$ is an identifying code of G. Then $C = V(G) \setminus I$ is an identifying code of G.*

Proof. Clearly C is a dominating set of G. Let x, y be a pair of vertices of G. If they both belong to I, $C \cap B_1(x) \neq C \cap B_1(y)$ because of the distance between x and y. Otherwise, one of them, say x, is in C. If they are not separated by C, then they must be adjacent. Thus, together they could have only one neighbour in I, call it u. This is a contradiction because $V(G) \setminus \{u\}$ identifies G. □

We note that 4 is the best possible in the previous lemma. For example, let $G = P_4$ and assume x and y are the two ends of G. It is easy to check that $V(G) \setminus \{x\}$ and $V(G) \setminus \{y\}$ are both identifying codes of G but $V(G) \setminus \{x, y\}$ is not.

3. Graphs with $\gamma^{\text{ID}}(G) = |V(G)| - 1$

In this section we classify all graphs G for which $\gamma^{\text{ID}}(G) = |V(G)| - 1$. As already mentioned, stars are examples of such graphs. To classify the rest we show that special powers of paths are the basic examples of such graphs. Then we show that any other example is mainly obtained from the join of some basic elements.

Definition 8. *For an integer $k \geq 1$, let $A_k = (V_k, E_k)$ be the graph with vertex set $V_k = \{x_1, \ldots, x_{2k}\}$ and edge set $E_k = \{x_i x_j \mid |i - j| \leq k - 1\}$.*

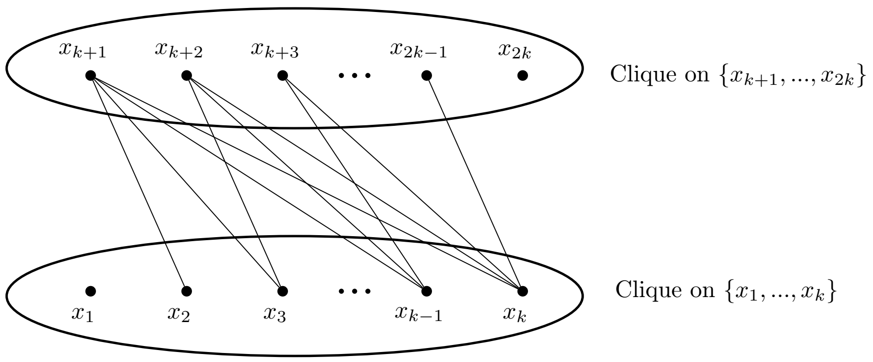

Figure 1: The graph A_k which needs $|V(A_k)| - 1$ vertices for any identifying code

We note that for $k \geq 2$ we have $A_k = P_{2k}^{k-1}$ and $A_1 = \overline{K_2}$. It is also easy to check that the only non trivial automorphism of A_k is the mapping $x_i \to x_{2k+1-i}$. It is not hard to observe that A_k is twin-free, $\Delta(A_k) = 2k - 2$ and that A_k and $\overline{A_k}$ are connected if $k \geq 2$.

Proposition 9. *For $k \geq 1$, we have: $\gamma^{S}(A_k) = 2k - 1$ with $B_1(x_k)$ and $B_1(x_{k+1})$ being the only separating sets of size $2k - 1$ of A_k. Furthermore, if $k \geq 2$, $\gamma^{ID}(A_k) = 2k - 1$.*

Proof. Let S be a separating set of A_k. For $i < k$, we have $\ominus(x_i, x_{i+1}) = \{x_{i+k}\}$ and for $k < i \leq 2k - 1$, we have $\ominus(x_i, x_{i+1}) = \{x_{i-k+1}\}$. Thus, $\{x_2, \ldots, x_{2k-1}\} \subset S$. But to separate x_k and x_{k+1}, we must add x_1 or x_{2k}. It is now easy to see that $V_k \setminus \{x_1\} = B_1(x_{k+1})$ and $V_k \setminus \{x_{2k}\} = B_1(x_k)$, each is a separating set of size $2k - 1$. If $k \geq 2$, then they both dominate A_k and therefore are also identifying codes. □

In the previous proof in fact we have also proved that:

Corollary 10. *For $k \geq 1$ every minimum separating set S of A_k has a S-universal vertex.*

Let $\mathcal{A}$ be the closure of $\{A_i \mid i = 1, 2, \ldots\}$ with respect to operation $\bowtie$. It is shown below that elements of $\mathcal{A}$ are also extremal graphs with respect to both separating sets and identifying codes.

Proposition 11. *For every graph $G \in \mathcal{A}$, we have $\gamma^{S}(G) = |V(G)| - 1$. Furthermore, every minimum separating set S of G has an S-universal vertex.*

Proof. The proposition is true for basic elements of $\mathcal{A}$ by Proposition 9 and by Corollary 10. For a general element $G = G_1 \bowtie G_2$ it is true by Proposition 6 and by induction. □

Corollary 12. *If $G \in \mathcal{A}$ and $G \ncong A_1$, then $\gamma^{ID}(G) = |V(G)| - 1$.*

Further examples of graphs extremal with respect to separating sets and identifying codes can be obtained by adding a universal vertex to each of the graphs in $\mathcal{A}$, as we prove below.

Proposition 13. *For every graph G in $\mathcal{A} \bowtie K_1$ we have $\gamma^{ID}(G) = \gamma^{S}(G) = |V(G)| - 1$.*

Proof. Assume $G = G_1 \bowtie K_1$ with $G_1 \in \mathcal{A}$, and assume u is the vertex corresponding to K_1. Suppose S is a minimum separating set of G. We first note that since $S \cap V(G_1)$ is a separating set of G_1, we have $|S \cap V(G_1)| \geq |V(G_1)| - 1$. But if $|S \cap V(G_1)| = |V(G_1)| - 1$, then by Proposition 11, there is a $[S \cap V(G_1)]$-universal vertex y of G_1. Then y is not separated from x. Thus $|S \cap V(G_1)| = |V(G_1)|$ and therefore $S = V(G_1)$. It is easy to check that S is also an identifying code. □

It was proved in [8] that $\gamma^{\text{ID}}(K_n \setminus M) = n - 1$ where $K_n \setminus M$ is the complete graph minus a maximal matching. We note that this graph, for even values of n, is the join of $\frac{n}{2}$ disjoint copies of A_1, thus it belongs to $\mathcal{A}$. For odd values of n, it is built from the previous graph by adding a universal vertex.

So far we have seen that $\gamma^{\text{ID}}(G) = |V(G)| - 1$ for $G \in \{K_{1,t} \mid t \geq 2\} \cup \mathcal{A} \cup (\mathcal{A} \bowtie K_1)$, $G \ncong A_1$. We also know that $\gamma^{\text{ID}}(\overline{K}_n) = n$. More examples of graphs with $\gamma^{\text{ID}}(G) = |V(G)| - 1$ can be obtained by adding isolated vertices. In the next thorem we show that for any other twin-free graph G we have $\gamma^{\text{ID}}(G) \leq |V(G)| - 2$.

Theorem 14. *Given a connected graph G, we have $\gamma^{ID}(G) = |V(G)| - 1$ if and only if $G \in \{K_{1,t} \mid t \geq 2\} \cup \mathcal{A} \cup (\mathcal{A} \bowtie K_1)$ and $G \ncong A_1$.*

Proof. The "if" part of the theorem is already proved. The proof of the "only if" part is based on induction on the number of vertices of G. For graphs on at most 4 vertices this is easy to check. Assume the claim is true for graphs on at most $n-1$ vertices and, by contradiction, let G be a twin-free graph on $n \geq 5$ vertices such that $\gamma^{\text{ID}}(G) = n-1$ and $G \notin \{K_{1,t} \mid t \geq 2\} \cup \mathcal{A} \cup (\mathcal{A} \bowtie K_1)$.

By Corollary 4 there is a vertex $x \in V(G)$ such that $\gamma^{\text{ID}}(G-x) = |V(G-x)| - 1$. By the induction hypothesis we have $G - x \in \{K_{1,t} \mid t \geq 2\} \cup \mathcal{A} \cup (\mathcal{A} \bowtie K_1)$. Depending on which one of these 3 sets $G-x$ belongs to, we will have 3 cases.

Case 1, $G - x \in \{K_{1,t} \mid t \geq 2\}$. In this case we consider a minimum identifying code C of $G-x$. If C does not already identify x then either $deg(x) \leq 3$ or $deg(x) \geq n-2$. We leave to the reader to check that in each of these cases, there is an identifying code of size $n-2$.

Case 2, $G - x \in \mathcal{A}$. We consider two subcases. Either $G - x \cong A_k$ for some k or $G - x = G_1 \bowtie G_2$, with $G_1, G_2 \in \mathcal{A}$.

(1) $G - x \cong A_k$, for some $k \geq 2$. If x is adjacent to all the vertices of $G-x$, then $G \in \mathcal{A} \bowtie K_1$ and we are done. Else there is a pair of consecutive vertices of A_k, say x_i and x_{i+1}, such that one is adjacent to x and the other is not. By the symmetry of A_k we may assume $i \leq k$. We claim that $C = V(G) \setminus \{x_1, x\}$ or $C' = V(G) \setminus \{x_{2k}, x\}$ is an identifying code of G. This would contradict our assumption. We first consider C and note that $C \cap V(A_k)$ is an identifying code of A_k. If x is also separated from all the vertices of $G-x$ then we are done. Else there will be two possibilities.

    First we consider the possibility: x is not adjacent to x_i and adjacent to x_{i+1}. In this case each vertex x_j, $j > i+k$, is separated from x by x_{i+1} and each vertex x_j, $j < i+k$, is separated from x by x_i. Thus x is not separated from x_{i+k}. In the other possibility, x is adjacent to x_i and not adjacent to x_{i+1}. A similar argument implies that x is separated from every vertex but x_1. In either of these two possibilities, C' would be an identifying code.

(2) $G - x \cong G_1 \bowtie G_2$ with $G_1, G_2 \in \mathcal{A}$. If x is adjacent to all the vertices of $G-x$, then $G \in \mathcal{A} \bowtie K_1$ and we are done. Thus there is a vertex, say y, that is not adjacent to x. Without loss of generality, we can assume $y \in V(G_1)$. Let C_1 be an identifying code of size $\gamma^{\text{ID}}(G_1) = |V(G_1)| - 1$ of G_1 which contains y. The existence of such an identifying code becomes apparent from the proof of Proposition 11. Then $C = C_1 \cup V(G_2)$ is an identifying code of $G_1 \bowtie G_2$ of size $|V(G_1 \bowtie G_2)| - 1 = |V(G)| - 2$. Thus C does not separate a vertex of $G_1 \bowtie G_2$ from x. Call this vertex z. Since $y \in C$, z is not adjacent to y, hence $z \in V(G_1)$. Therefore, z is adjacent to all the vertices of G_2. So x should also be adjacent to all the vertices of G_2. Thus we have $G = (G_1 + x) \bowtie G_2$ and any minimum identifying

code of $G_1 + x$ together with all vertices of G_2 would form an identifying code of G. This proves that $\gamma^{\text{ID}}(G_1 + x) = |V(G_1 + x)| - 1$. Since $G_1 + x$ has less vertices than G, by induction hypothesis, we have $G_1 + x \in \{K_{1,t} \mid t \geq 2\} \cup \mathcal{A} \cup (\mathcal{A} \bowtie K_1)$ and $G \ncong A_1$. Since $G_1 \in \mathcal{A}$, and since x is not adjacent to a vertex of G_1, we should have $G_1 + x \in \mathcal{A}$ but all graphs in $\mathcal{A}$ have an even number of vertices and this is not possible.

Case 3, $G - x \in \mathcal{A} \bowtie K_1$. Suppose $G - x \cong A_{i_1} \bowtie A_{i_2} \bowtie \ldots \bowtie A_{i_j} \bowtie K_1$ and let u be the vertex corresponding to K_1.

If x is also adjacent to u, then u is a universal vertex of G and $G - u$ is also twin-free. In this case we apply the induction on $G - u$: by Proposition 3, $\gamma^{\text{ID}}(G - u) = |V(G - u)| - 1$ and by induction hypothesis $G - u \in \{K_{1,t} \mid t \geq 2\} \cup \mathcal{A} \cup (\mathcal{A} \bowtie K_1)$. But if $G - u \in \{K_{1,t} \mid t \geq 2\} \cup (\mathcal{A} \bowtie K_1)$, there will be two universal vertices, and therefore twins. Thus $G - u \in \mathcal{A}$ and $G \in \mathcal{A} \bowtie K_1$.

We now assume x is not adjacent to u and we repeat the argument with $G - u$ if it is twin-free. In this case if $G - u \in \{K_{1,t} \mid t \geq 2\} \cup \mathcal{A}$, we apply Case 1 or Case 2. If $G - u \in \mathcal{A} \bowtie K_1$ with u' being the vertex of K_1, then u and u' induce an isomorphic copy of A_1 and $G \in \mathcal{A}$.

If $G - u$ is not twin-free then, by Lemma 5, x must be one of the twin vertices. Let x' be its twin and suppose $x' \in V(A_{i_1})$ with $V(A_{i_1}) = \{z_1, z_2, \ldots, z_{2k}\}$. Without loss of generality we may assume $x' = z_l$ with $l \leq k$. If $l \geq 2$, then we claim $C = V(G) \setminus \{z_l, z_{2k}\}$ is an identifying code of G which is a contradiction. To prove our claim notice first that vertices of $A_{i_2} \bowtie \cdots \bowtie A_{i_j}$ are already identified from each other and from the other vertices. Now each pair of vertices of A_{i_1} are separated by a vertex in $V(A_{i_1}) \cap C$ except z_{l+k-1} and z_{l+k} which are separated by x. The vertex x is also separated from all the other vertices by u. It remains to show that u is separated from vertices of A_{i_1}. It is separated from vertices in $\{z_1, \ldots, z_{l+k-1}\}$ by x and from $\{z_{k+1}, \ldots, z_{2k}\}$ by z_1 ($l \geq 2$). Thus $x' = x_1$ and now it is easy to see that the subgraph induced by $V(A_{i_1})$, u and x is isomorphic to A_{i_1+1} and, therefore, $G \cong A_{i_1+1} \bowtie A_{i_2} \bowtie \ldots \bowtie A_{i_j}$. □

Since graphs in $\{K_{1,t} \mid t \geq 2\} \cup (\mathcal{A} \bowtie K_1)$ have maximum degree $n - 1$ and graphs in $\mathcal{A}$ have maximum degree $n - 2$, we have:

Corollary 15. *Let G be a twin-free connected graph on $n \geq 3$ vertices and maximum degree $\Delta \leq n - 3$. Then $\gamma^{ID}(G) \leq n - 2$.*

4. Infinite graphs

It is shown in [8] that Theorem 1 does not have a direct extension to the family of infinite graphs. In other words, there are nontrivial examples of twin-free infinite graphs requiring the whole vertex set for any identifying code. The basic example of such infinite graphs, originally defined in [8], is given below. In this section, we classify all such infinite graphs. This strengthens a theorem of [12], which claims that there are no such infinite graphs in which all vertices have finite degrees.

Definition 16. *Let* $X = \{\ldots, x_{-1}, x_0, x_1, \ldots\}$ *and* $Y = \{\ldots, y_{-1}, y_0, y_1, \ldots\}$. $A_\infty = (X \cup Y, E)$ *is the graph on* $X \cup Y$ *having edge set* $E = \{x_i x_j \mid i \neq j\} \cup \{y_i y_j \mid i \neq j\} \cup \{x_i y_j \mid i < j\}$.

See Figure 2 for an illustration.

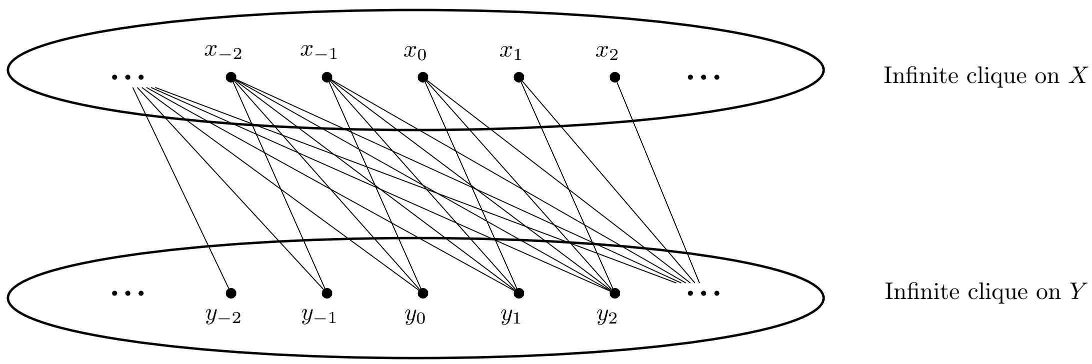

Figure 2: The graph A_∞ which needs all its vertices for any identifying code

It is shown in [8] that the only identifying code of A_∞ is $V(A_\infty)$. One should note that the graph induced by $\{y_1, y_2, \ldots, y_k, x_1, x_2 \ldots, x_k\}$ is isomorphic to the graph A_k.

Before introducing our theorem let us see again why every identifying code of A_∞ needs the whole vertex set: for every i, x_i and x_{i+1} are only separated by y_{i+1}, while y_i and y_{i+1} are separated only by x_i. But this property would still hold if we add a new vertex which is adjacent either to all vertices in X (similarly in Y) or to none. This leads to the following family:

Let H be a finite or infinite simple graph with a perfect matching ρ, that is a mapping $x \to \rho(x)$ of $V(H)$ to itself such that $\rho^2(x) = x$ and $x\rho(x)$ is an edge of H. We define $\Psi(H, \rho)$ to be the graph built as follows: for every vertex x of H we assign $\Phi(x) = \{\ldots x_{-1}, x_0, x_1, \ldots\}$. The vertex set of $\Psi(H, \rho)$ is $\bigcup_{x \in V(H)} \Phi(x)$. For each edge $x\rho(x)$ of H we build a copy of A_∞ on $\Phi(x) \cup \Phi(\rho(x))$ and for every other edge xy of H we join every vertex in $\Phi(x)$ to every vertex in $\Phi(y)$. An example of such construction is illustrated in Figure 3.

It is clear from the previous construction that a vertex not belonging to some copy A of A_∞ corresponding to some $\Phi(x) \cup \Phi(\rho(x))$ is not helpful for separating vertices of A. Thus, we have the following proposition.

Proposition 17. *For every simple, finite or infinite, graph* H *with a perfect matching* ρ*, the graph* $\Psi(H, \rho)$ *can only be identified with* $V(\Psi(H, \rho))$.

In the next theorem we prove that every such extremal connected infinite graph is $\Psi(H, \rho)$ for some connected finite or infinite graph H together with a matching ρ.

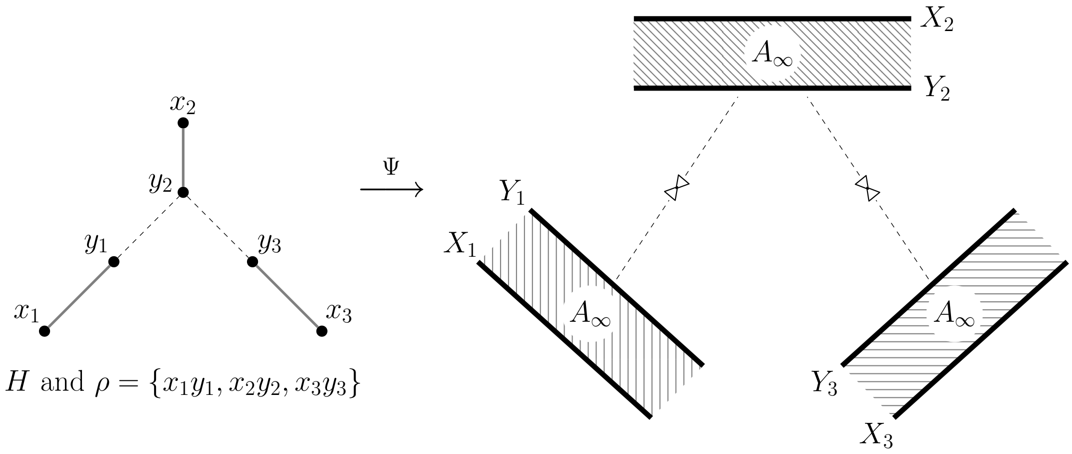

Figure 3: Construction of $\Psi(H, \rho)$ from (H, ρ)

Theorem 18. *Let G be an infinite connected graph. Then a proper subset C of $V(G)$ identifies all pairs of vertices of G unless $G = \Psi(H, \rho)$ for some finite or infinite graph H together with a perfect matching ρ.*

Proof. We already have seen that if $G \cong \Psi(H, \rho)$, then the only identifying code of G is $V(G)$. To prove the converse suppose $G - v$ has a pair of twin vertices for every vertex v of G. It is enough to show that every vertex v of G belongs to a unique induced subgraph A_v of G isomorphic to A_∞ and that if a vertex not in A_v is adjacent to a vertex in the X (respectively, Y) part of A_v then it is adjacent to all the vertices of the X (respectively, Y).

Let x_1 be a vertex of G. The subgraph $G - x_1$ has a pair of twins, let y_1 and y_2 be one such pair. Assume, without loss of generality, that x_1 is adjacent to y_2 and not to y_1. By Lemma 5, x_1 must be one of the vertices of a pair of twins in $G - y_1$. Let the other be x_2. Now consider the subgraph $G - y_1$. This subgraph must have a pair of twins and x_1 must be one of them. Let x_0 be the other one.

Continuing this process in both directions (with negative and positive indices) we build our $A_{x_1} \cong A_\infty$ as a subgraph of G. Since each consecutive pair of vertices in $X \subset A_{x_1}$ is separated only by a vertex in $Y \subset A_{x_1}$, every pair of vertices in X are twins in $G - Y$. Thus each vertex not in A_{x_1}, either is adjacent to all the vertices in X or to none of them. Similarly, every vertex in A_{x_1}, either is adjacent to all the vertices in Y or to none. Hence A_{x_1} is unique. This proves the theorem. □

5. Bounding $\gamma^{\text{ID}}(G)$ by n and Δ

In this section, we introduce new upper bounds on parameter γ^{ID} in terms of both the order and the maximum degree of graph, thus extending a result of [12].

We define A^+_∞ to be the subgraph of A_∞ induced by the vertices of positive indices in X and in Y. The following lemma, which is a strengthening of Theorem 1, has been attributed to N. Bertrand [2]. We give an independent proof as [2] is not accessible.

Lemma 19 ([2])**.** *If G is a twin-free graph (infinite or not) not containing A_∞^+ as an induced subgraph, then for every vertex x of G, there is a vertex $y \in B_1(x)$ such that $G - y$ is twin-free.*

Proof. By contradiction, suppose that x_1 is a vertex that fails the statement of the lemma. Then $G - x_1$ has a pair of twin vertices. We name them y_1 and y_2. Without loss of generality we assume that x_1 is adjacent to y_2 but not to y_1. Now, in $G - y_2$ we must have another pair u, u' of twin vertices. By Lemma 5, $x_1 \in \{u, u'\}$, we name the other element x_2 ($x_2 \in B_1(x_1)$). Note that the subgraph induced on x_1, x_2, y_1, y_2 is isomorphic to A_2. We prove by induction that A_∞^+ is an induced subgraph of G, thus obtaining a contradiction.

To this end suppose A_k on $\{y_1, \ldots y_k, x_1, \ldots, x_k\}$ is already built and x_{k-1}, x_k are twins in $G - y_k$. Similarly y_{k-1}, y_k are twins in $G - x_{k-1}$. Then $x_k \in B_1(x_1)$. Consider $G - x_k$. There must be a pair of twins and, by Lemma 5, y_k must be one of them. Let y_{k+1} be the other. Since y_k and y_{k+1} are twins in $G - x_k$, then y_{k+1} is adjacent to $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$, in particular $y_{k+1} \in B_1(x_1)$. Now, in $G - y_{k+1}$ there must be a pair of twins and again by Lemma 5 one of them must be x_k, let the other one be x_{k+1}. Since x_k and x_{k+1} are twins in $G - y_{k+1}$, then x_{k+1} is adjacent to $x_1, \ldots, x_k$ and not adjacent to $y_1, \ldots, y_k$. Thus the graph induced on $\{y_1, \ldots, y_{k+1}, x_1, \ldots, x_{k+1}\}$ is isomorphic to A_{k+1}. Since this process does not end, we find that A_∞^+ is an induced subgraph of G. $\square$

It was conjectured in [10] that:

Conjecture 20 ([10])**.** *For every connected twin-free graph G of maximum degree $\Delta \geq 3$, we have $\gamma^{ID}(G) \leq \left\lceil |V(G)| - \frac{|V(G)|}{\Delta(G)} \right\rceil$.*

In support of this conjecture, we prove the following weaker upper bound on the size of a minimum identifying code of a twin-free graph. We note that a similar bound is proved in [10].

Theorem 21. *Let G be a connected, twin-free graph on n vertices and of maximum degree Δ. Then $\gamma^{ID}(G) \leq n(1 - \frac{\Delta-2}{\Delta(\Delta-1)^5-2}) = n - \frac{n}{\Theta(\Delta^5)}$.*

Proof. First, we note that if I is a maximal 6-independent set, then $|I| \geq \frac{n(\Delta-2)}{\Delta(\Delta-1)^5-2}$. This is true because $|B_5(x)| \leq \frac{\Delta(\Delta-1)^5-2}{\Delta-2}$ for every vertex x. Now, let I be a 6-independent set. For each vertex $x \in I$ let $f(x)$ be the vertex found using Lemma 19 and $f(I) = \{f(x) \mid x \in I\}$. Since I is a 6-independent set, $f(I)$ is a 4-independent set of G and $|f(I)| = |I|$. Now, by Lemma 7, we know that $C = V(G) \setminus f(I)$ is an identifying code of G. The bound is now obtained by taking any maximal 6-independent set I. $\square$

It is proved in [12] that in any nontrivial infinite twin-free graph G whose vertices are all of finite degree, there exists a vertex x such that $V(G) \setminus \{x\}$ is an identifying code of G. Using Lemma 19 and similar to the proof of Theorem 21, we can strengthen their result as follows:

Theorem 22. *Let G be a connected infinite twin-free graph whose vertices all have finite degree. Then there exists an infinite set of vertices $I \subseteq V(G)$, such that $V(G) \setminus I$ is an identifying code of G.*

6. General r-identifying codes

To identify the class of graphs with $\gamma_r^{\text{ID}}(G) = n - 1$ one needs to find the r-roots of the graphs in $\{K_{1,t} \mid t \geq 2\}\} \cup \mathcal{A} \cup (\mathcal{A} \bowtie K_1)$. The general problem of finding the r-root of a graph H is an NP-hard problem [16] and it does not seem to be an easy task in this particular case either.

If s divides $k-1$ and $r = \frac{k-1}{s}$, then the graph $G = P_{2k}^s$ is one of the r-roots of A_k. It is easy to see that, in most cases, one can remove many edges of G and still have $G^r \cong A_k$. The difficulty of the problem is that an r-root of A_k is not necessarily a subgraph of P_{2k}^s. An example of such a 2-root of A_5 is given in Figure 4.

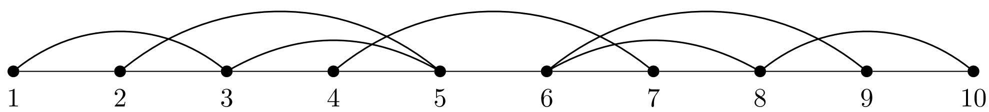

Figure 4: A 2-root of A_5 which is not a subgraph of P_{10}^2

For the case of infinite graphs, we note that there exists a 2-root of A_∞. This graph is defined as follows: it has the same vertex set $X \cup Y$ as A_∞ and the same edges between X and Y, but no edges within neither X nor Y. However, we do not know whether there exist other roots of graphs described in Theorem 18.

We should also note that a $(3r+1)$-independent set in G^r is a 4-independent set in G. Thus we have the following general form of Lemma 7, Theorem 21 and Theorem 22:

Lemma 23. *Let G be a connected graph on n vertices such that G^r is twin-free. Let I be a $(3r+1)$-independent set of G such that for every vertex v of I the set $V(G) \setminus \{v\}$ is an r-identifying code of G. Then $C = V(G) \setminus I$ is an r-identifying code of G.*

Theorem 24. *Let G be a connected graph on n vertices and of maximum degree Δ such that G^r is twin-free. Then $\gamma_r^{ID}(G) \leq n(1 - \frac{\Delta-2}{\Delta(\Delta-1)^{5r}-2}) = n - \frac{n}{\Theta(\Delta^{5r})}$.*

Theorem 25. *Let G be a connected infinite graph whose vertices are of finite degree such that G^r is twin-free. Then there exists an infinite set of vertices $I \subseteq V(G)$, such that $V(G) \setminus I$ is an r-identifying code of G.*

7. Remarks

We conclude our paper by some remarks on related works.

Remark 1 The following two questions were posed in [18]:

1. Does there exist k-regular graphs G of order n with $\gamma^{\text{ID}}(G) = n - 1$ for $k < n - 2$?
2. Does there exist graphs G of odd order n and maximum degree $\Delta < n - 1$ with $\gamma^{\text{ID}}(G) = n - 1$?

As a corollary of Theorem 14, we can now answer these questions in negative. Indeed, for the first question, if G is a k-regular ($k \leq 2$) graph of order n with $\gamma^{\text{ID}}(G) = n - 1$ then G is the join of k disjoint copies of A_1. For the second question, noting that each graph in $\mathcal{A}$ has an even order, we conclude that if a graph G on an odd number, n, of vertices have $\gamma^{\text{ID}}(G) = n - 1$, then $G \in \{K_{1,t} \mid t \geq 2\} \cup (\mathcal{A} \bowtie K_1)$ and, therefore $\Delta(G) = n - 1$.

Remark 2 Given a graph $G = (V, E)$ the *1-ball membership graph* of G is defined to be the bipartite graph $G^* = (I \cup A, E^*)$ where $I = V(G), A = \{B_1(x) \mid x \in V(G)\}$ and $E^* = \{\{u, B_1(v)\} \mid u \in B_1(v), u, v \in V(G)\}$. It is not hard to see that the problem of finding identifying codes in G is equivalent to the one of finding discriminating codes in G^*. But since not every bipartite graph is a 1-ball membership graph, the latter contains the former properly. It is a rephrasing of Bondy's theorem [1], that every bipartite graph $(I \cup A, E)$ has a discriminating code of size at most $|I|$. The class of bipartite graphs $(I \cup A, E)$ in which any discriminating code has size at least $|I|$ are classified in [5]. They further asked for the classification of bipartite graphs in which every discriminating code needs at least $|I| - 1$ vertices of A. In Theorem 14 we answered this question for those bipartite graphs that are isomorphic to a 1-ball membership of a graph.

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