A note on Δ-critical graphs

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A *k-critical* graph is a k-chromatic graph whose proper subgraphs are all $(k-1)$-colourable. An old open problem due to Borodin and Kostochka asserts that for $k \geq 9$, no k-critical graph G with $k = \Delta(G)$ exists, where $\Delta(G)$ denotes the maximum degree of G. We show that if a certain special list-colouring property holds for every 8-critical graph with $\Delta = 8$ (which is true for the apparently only known example), then the Borodin-Kostochka Conjecture holds. We also briefly survey constructions of Δ-critical graphs with $\Delta \leq 8$, highlighting the apparent scarcity of such graphs once Δ exceeds 6.

1 Introduction

Our main motivation in this note is a famous open problem about extending Brooks' classical theorem [3], which was posed by Borodin and Kostochka [2] in 1977. They conjectured that for every $k \geq 9$, if a graph G with $\Delta(G) = k$ satisfies $\chi(G) \geq \Delta(G)$, then G contains a complete subgraph K_k with k vertices. Here $\chi(G)$ denotes the chromatic number of G, and $\Delta(G)$ denotes its maximum degree.

We begin our discussion here by reformulating the Borodin-Kostochka Conjecture (BKC) in terms of critical graphs. A *k-critical* graph is a graph G with the property that $\chi(G) = k$ but $\chi(G') \leq k-1$ for every proper subgraph G' of G.

Conjecture 1. *For each $k \geq 9$, the only k-critical graph G with $\Delta(G) \leq k$ is K_k.*

Since every graph G with $\chi(G) \geq k$ contains a k-critical subgraph, Conjecture 1 clearly implies the BKC. Conversely, since K_k is k-chromatic, the BKC implies that any k-critical graph G with $\Delta(G) \leq k$ contains K_k, and hence is K_k. Thus the BKC implies Conjecture 1.

Since $\Delta(K_k) = k-1$, Conjecture 1 is equivalent to asserting the non-existence of Δ-critical graphs once $\Delta \geq 9$. Phrased in similar language, Brooks' Theorem implies that for each $k \geq 4$, the only k-critical graph with $\Delta + 1 \leq k$ is K_k. The condition $k \geq 9$ in Conjecture 1 is necessary, due in particular to the graph $C_5[K_3]$ shown in Figure 1.

Much progress has been made towards a solution to the BKC, for example [2, 5–8, 12, 14, 16] (see e.g. [6] for a detailed description of previous work). Despite these developments, the conjecture remains open in general. Here we mention just two of the strongest results to date, namely, that the conjecture holds provided Δ is a sufficiently large (unspecified) constant [16], and that any Δ-chromatic graph with $\Delta \geq 13$ must contain $K_{\Delta-3}$ [6]. In [7] it was shown that the conjecture is equivalent to the seemingly much weaker assertion that every Δ-chromatic graph with $\Delta \geq 9$ contains the graph obtained from K_Δ by removing the edges of a $K_{\Delta-3}$.

One simple but important step in addressing the BKC, shown in [4, 12], has been to reduce the general problem to graphs with $\Delta = 9$. (Similar arguments were used in e.g. [6, 7, 14] as well.) In this note we continue the same theme, linking the general conjecture to the behaviour of graphs with $\Delta \leq 8$.

We will refer to a particular restricted notion of list colouring, called *non-identical* list colouring (abbreviated NIL), defined as follows. For a set S, an integer k, and a graph G, an (S,k)-NIL *assignment* of G is an assignment $\mathcal{L}$ of lists $L(v) \subseteq S$ to each vertex $v \in V(G)$ such that each $|L(v)| = k$, and $L(v) \neq L(w)$ for some pair $v, w \in V(G)$. We say that G is (S,k)-*NIL colourable* if every (S,k)-NIL assignment $\mathcal{L}$ of G admits an $\mathcal{L}$-list colouring of G. For example, it is easy to verify (see Section 2) that the complete graph K_n is $(S, n-1)$-NIL colourable for every S.

In this note we prove the following theorem, where $[n]$ denotes $\{1,\dots,n\}$. The proof appears in Section 3.

Theorem 2. *Suppose that every 8-critical graph with $\Delta \leq 8$ is regular and $([8],7)$-NIL colourable. Then the only 9-critical graph with $\Delta \leq 9$ is K_9 (i.e. the BKC holds).*

Recall that the lexicographic product of graphs G and H is the graph $G[H]$ with vertex set $V(G) \times V(H)$, in which (x,u) is adjacent to (y,v) whenever $xy \in E(G)$, or $x = y$ and $uv \in E(H)$. The (regular) 8-critical graph $C_5[K_3]$ (see Figure 1) is shown in Section 2 to be $([8],7)$-NIL colourable. Since we are unaware of any 8-critical graphs with $\Delta \leq 8$ besides $C_5[K_3]$ and K_8, the following corollary tempts us to conjecture that there are no others, thus possibly strengthening Conjecture 1.

Corollary 3. *Suppose that the only 8-critical graphs with $\Delta \leq 8$ are $C_5[K_3]$ and K_8. Then the BKC holds.*

Indeed, the assumptions of Theorem 2 precisely identify the properties of $C_5[K_3]$ and K_8 that make a simple proof of Theorem 2 possible.

We end this brief note with a discussion on constructing Δ-critical graphs for small Δ in Section 4. In particular, we point out there that while infinite families of such graphs exist when $\Delta \leq 6$, it seems more difficult to construct examples for $\Delta = 7$, and (as mentioned) even more so for $\Delta = 8$.

2 Non-identical list colouring

In this section we verify that the graph $C_5[K_3]$ shown in Figure 1 is $([8],7)$-NIL colourable. Observe that $C_5[K_3]$ has 15 vertices and independence number 2, so it is not 7-colourable. An 8-colouring can be easily found ($\chi(C_5[K_3]) \le 8$ also follows from Brooks' theorem). To see that it is critical, observe that the graph is vertex transitive and there are only two types of edges (those in triangles $x_iy_iz_i$ and those connecting them). Given an [8]-colouring ϕ, let us assume that 8 is the unique colour that appears exactly once, and that $\phi(x_1) = 8$. By symmetry, we may assume that $\phi(z_2) = \phi(z_5) = 7$, and that colors $1, 2, \ldots, 6$ each appear on neighbours of x_1 exactly once. Then in a subgraph missing x_1y_1 we may recolour x_1 to $\phi(y_1)$, and in a subgraph missing x_1x_2 we may recolour x_1 to $\phi(x_2)$. This shows that $C_5[K_3]$ is also 8-critical.

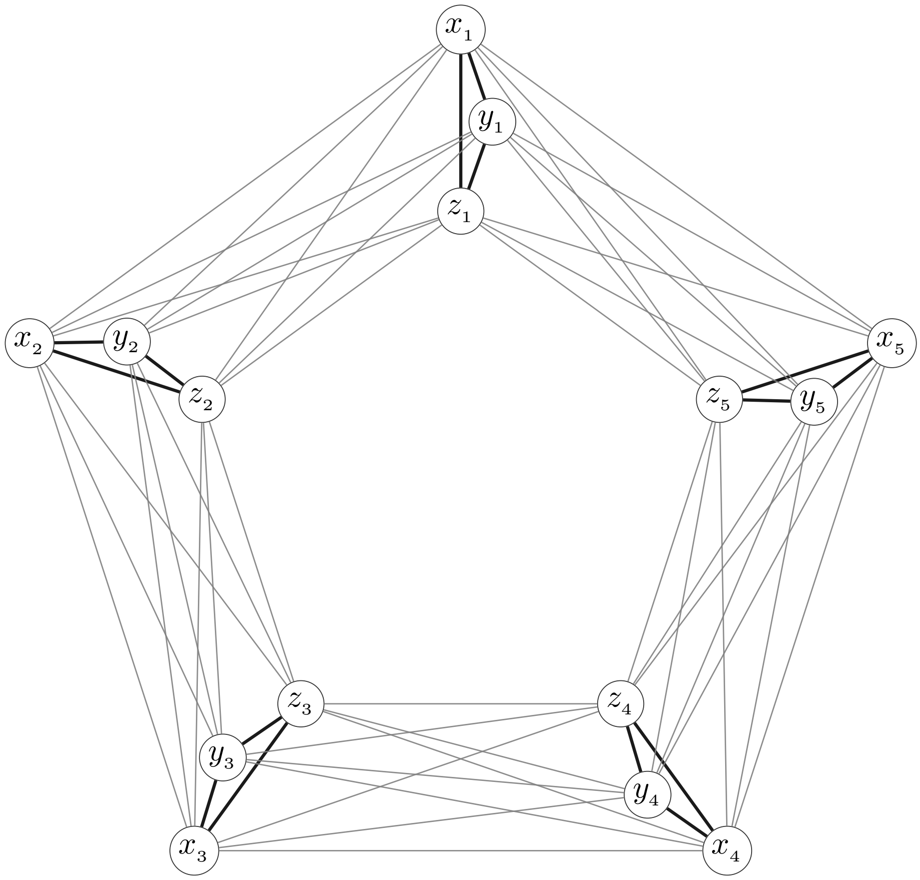

Figure 1: $C_5[K_3]$

First we note the following simple consequence of the well-known fact (equivalent to Hall's Theorem) that a collection of sets $\{L(v) : v \in V\}$ has a system of distinct representatives $\{s_v \in L(v) : v \in V\}$ if and only if $|\bigcup_{v \in T} L(v)| \ge |T|$ for each $T \subseteq V$.

Lemma 4. *For every $n \ge 2$ and every set S, the complete graph K_n is $(S, n-1)$-NIL colourable.*

Proof. Let $\mathcal{L}$ be an $(S, n-1)$-NIL assignment of G. Each nonempty $T \subseteq V(G)$ clearly satisfies $|\bigcup_{v \in T} L(v)| \geq n-1$, so the above condition fails only if $|\bigcup_{v \in V(G)} L(v)| = n-1$, i.e. only if all lists are identical. □

Lemma 5. *The graph $C_5[K_3]$ is $([8], 7)$-NIL colourable.*

Proof. Fix an $([8], 7)$-NIL assignment $\mathcal{L}$ of $G = C_5[K_3]$. By symmetry, we may assume that two (adjacent) vertices with different lists are among x_1, x_2, y_1, y_2. Let K denote the K_4 induced by these four vertices. Since every pair of vertices of G have at least 6 common colours in their lists, by renaming colours if necessary we may choose the following partial colouring φ:

- $\varphi(z_1) = \varphi(z_3) = 7$.
- $\varphi(z_2) = \varphi(z_5) = 8$.
- $\varphi(x_3) = \varphi(x_5) = 6$ and $\varphi(y_3) = \varphi(y_5) = 5$.
- For each $v \in K \cup \{x_4, y_4, z_4\}$ set $L'(v) = L(v) \setminus \{5, 6, 7, 8\}$. Then we may easily colour each vertex $v \in \{x_4, y_4, z_4\}$ from $L'(v)$ since $|L'(v)| \geq 3$ for each.

It remains to colour the vertices v of K from their current lists $L'(v)$. If $|L'(v)| = 3$ for each $v \in K$, then each $L(v)$ contains all of $\{5, 6, 7, 8\}$ and hence, by our choice of K, we find that $\{L'(v) : v \in K\}$ is a $([4], 3)$-NIL assignment of K_4. If $|L'(x)| \geq 4$ for some $x \in K$, then we may assign to each v in K a 3-subset of $L'(v)$ such that not all are identical. Hence, in either case, by Lemma 4, we may complete the colouring of K and hence of G. □

3 Proof of Theorem 2

As mentioned in the Introduction, and shown in [4, 12] (see also [6, 7, 14]), the BKC can be reduced to the specific case of $k = 9$. A useful tool for such purposes is the following result of King [11] (which is based on [9]). Here $\omega(G)$ denotes the maximum size of a clique in G.

Theorem 6. *If a graph G satisfies $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$, then G contains an independent set I such that $\omega(G - I) = \omega(G) - 1$.*

Our proof of Theorem 2 begins with an application of Theorem 6, which in this context is quite standard. For example it essentially repeats the proof of Lemma 1.8 in [7], but we include this argument in order to establish how the 8-critical graph H sits in G. This is needed for the rest of the proof.

Proof of Theorem 2. Suppose that there exist 9-critical graphs G with $\Delta(G) \leq 9$ that are distinct from K_9, and choose one such G with the smallest number of vertices. Then by Brooks' Theorem we know $\Delta(G) = 9$, since otherwise G would be 8-colourable. By Theorem 6 we may choose a maximal independent set I in G that intersects every 8-clique. (Note that if $\omega(G) \leq 7$ then any maximal independent set will do.)

By maximality of I we know $\Delta(G-I) \leq 8$, by 9-criticality of G we know $\chi(G-I) \leq 8$ (and hence $\chi(G-I) = 8$), and by choice of I we have $\omega(G-I) \leq 7$. Let H be an 8-critical subgraph of $G - I$. Then clearly we have $\Delta(H) \leq 8$ and $\omega(H) \leq 7$ as well. Hence in particular H is not K_8, so again by Brooks' Theorem $\Delta(H) = 8$. Thus, by the assumption of the theorem, H is 8-regular and $([8], 7)$-NIL colourable.

We denote by I_H the subset $I \cap N_G(H)$. Since $\Delta(G) = 9$ and H is 8-regular, by maximality of I we know $d_{I_H}(v) = 1$ for each vertex v of H, from which it follows that I_H is a vertex cut in G separating H from $G - I - H$. (We remark that this is the only place in our proof that the assumption of regularity of H is essential rather than simply convenient.) Furthermore since H is 8-regular and not K_9 it has more than 9 vertices, so $|I_H| \geq 2$.

By 9-criticality of G we know that $G-H$ is 8-colourable. First suppose that $G-H$ has an 8-colouring ϕ in which two vertices of I_H receive different colours. Since each vertex v of H has exactly one neighbour v_I in I_H, the list assignment $\mathcal{L}$ given by $L(v) = [8] \setminus \{\phi(v_I)\}$ is an $([8], 7)$-NIL assignment for H. Hence, by the assumption, H has an $\mathcal{L}$-colouring, which together with ϕ shows that G is 8-colourable, giving a contradiction. Thus we may assume that every 8-colouring of $G - H$ must give all vertices of I_H the same colour. In other words, adding any edge xy to $G - H$ where $x, y \in I_H$ results in a 9-chromatic graph $G - H + xy$.

Claim 1. *Every pair of disinct vertices $x, y \in I_H$ lies in a K_9^--subgraph K_{xy} of $G-H$ with $V(K_{xy}) \cap I_H = \{x, y\}$. (Here K_9^- denotes the graph obtained by removing one edge from K_9.)*

To verify the Claim, observe that since $G - H + xy$ is 9-chromatic, it contains a 9-critical subgraph J. Since $|V(J)| \leq |V(G - H)| < |V(G)|$ and $\Delta(J) \leq \Delta(G) = 9$, by minimality of our counterexample G, we conclude that $J = K_9$. Clearly J must contain the edge xy, so $J - xy$ is the claimed K_9^-.

Recalling that $|I_H| \geq 2$ and $\Delta(G) = 9$, Claim 1 immediately implies a contradiction if any $v \in I_H$ satisfies $d_H(v) \geq 3$, since in that case $d_G(v) = d_H(v) + d_{G-H}(v) \geq 3 + (9 - 2) = 10 > \Delta(G)$. Hence we may assume $d_H(v) \leq 2$ for each $v \in I_H$. We know H has more than 9 vertices, so since $d_{I_H}(u) = 1$ for each vertex u of H, we find that $|I_H| \geq \lceil \frac{9}{2} \rceil = 5$. Fix $v \in I_H$.

For four vertices $u_i \in I_H$, $1 \leq i \leq 4$, $u_i \neq v$, by Claim 1, we have copies K_{vu_i} of K_9^- in $G - H$, each of which contains v. Since $d(v) \leq \Delta(G) = 9$, by the pigeonhole principle some neighbour of v lies in $\lceil \frac{4(7)}{9} \rceil = 4$ of these subgraphs. Let w denote such a vertex. Then w is a vertex of $G - I - H$, and w has eight neighbours in K_{vu_1} together with u_2, u_3 and u_4. Thus there are a total of at least 11 neighbours of w in G, contradicting the fact that $\Delta(G) = 9$. This completes the proof of Theorem 2. $\square$

4 Δ-critical graphs for small Δ

The graph $C_5[K_3]$ is one example of a critical graph obtained by blowing up each vertex of an odd cycle into a clique. One can easily construct infinite families of k-critical graphs with $\Delta = k$ for $k \in \{4, 5\}$ by using such blow-ups using clique sizes 1 and 2. For example, for each $t \geq 2$ let U be an independent set of t vertices in C_{2t+1}. Blowing up each vertex of U into a K_2 results in a 4-critical graph with $\Delta = 4$.

Another simple construction of critical graphs from [4] (generalizing examples from [1]) is as follows. Fix $k \geq 4$, let G be a graph with $\Delta(G) \leq k$, and suppose G has a vertex x of degree $k-1$. Form a new graph G_x^k by "evenly splitting" x into 3 vertices (i.e. remove x and add an independent set I_x of three vertices whose degrees differ by at most one and whose neighbourhoods partition $N(x)$), and joining each vertex of I_x to a new clique K_x of order $k-2$. Note that $|V(G_x^k)| = |V(G)|-1+3+(k-2) = |V(G)|+k$. An example of this construction for $k = 7$ with $G = K_7$ is depicted in Figure 2. Here G_x^7 is a 7-critical graph on 14 vertices, with 8 vertices of degree seven and 6 vertices of degree six.

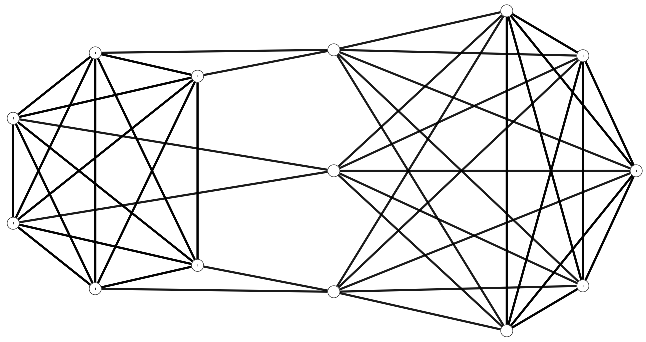

Figure 2: G_x^7 where $G = K_7$

It is straightforward to verify that G_x^k is k-critical if and only if G is k-critical. Since each $v \in V(G) \cap V(G_x^k)$ has degree $d_G(v)$ in G_x^k, each $v \in K_x$ has degree $(k-3)+3 = k$, and each $v \in I_x$ has degree $k-2+\lceil\frac{k-1}{3}\rceil$ or $k-2+\lfloor\frac{k-1}{3}\rfloor$, we see that if $k-2+\lceil\frac{k-1}{3}\rceil \leq k$ then $\Delta(G_x^k)) = k$. This holds for $k \leq 7$. Moreover if $k-2+\lfloor\frac{k-1}{3}\rfloor \leq k-1$, then G_x^k has a vertex of degree less than k, which is true for $k \leq 6$. Hence for $k \in \{4,5,6\}$, we may start for example with the k-critical graph K_k and repeat this construction an arbitrary number of times, thus giving an infinite family of k-critical graphs with $\Delta = k$.

In contrast, for $k = 7$, since no new vertex of degree $k-1$ is introduced by the construction, if we start with K_7 the process will produce 7-critical graphs with $\Delta = 7$ on $7i$ vertices for $i \in [8]$, thus terminating with graphs on 56 vertices. This operation produces various nonisomorphic 7-critical graphs on $7i$ vertices for $i \in \{3,\ldots,8\}$. For example, consider such a graph on 56 vertices. Letting W denote the vertex set of the initial K_7, we note that the subgraph J induced on the set $\bigcup_{w\in W} I_w$ of 21 vertices is a 2-regular graph in which there is (exactly) one edge e_{vw} connecting each pair $\{I_v, I_w\}$ with $v, w \in W$. Thus the edges of J are in one-to-one correspondence with the edges of K_7, and hence the components of J correspond to a decomposition of $E(K_7)$ into connected even-degree subgraphs of K_7. Conversely, any such decomposition $F_1,\ldots,F_s$ of $E(K_7)$ can be realized as such a subgraph J, whose components are cycles of lengths $|E(F_1)|,\ldots,|E(F_s)|$, by following an Euler tour in each of the F_i. There are many such partitions, for example a partition of $E(K_7)$ into three 7-cycles, or a partition into seven 3-cycles (the lines of the Fano plane), or the whole set $E(K_7)$ as a single partition class (for which J is a 21-cycle). The example based on the Fano plane is shown in Figure 3.

Since the graph Q obtained from $C_5[K_3]$ by removing two nonadjacent vertices is also 7-critical, another family of 7-critical graphs with $\Delta = 7$ can be built from Q

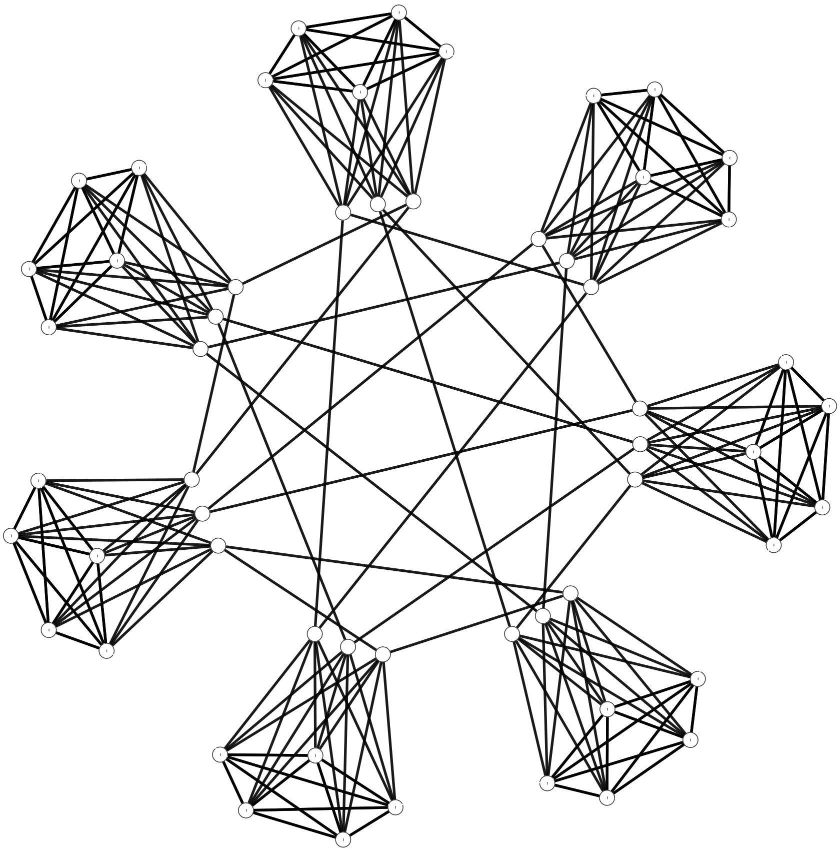

Figure 3: Example of 7-regular 7-critical graph on 56 vertices.

using the splitting operation. This gives further examples, with vertex sets of sizes 20, 27, and 34. To the best of our knowledge this describes all the known 7-critical graphs satisfying $\Delta = 7$. Thus we pose the following natural questions.

Problem 7. *Does there exist an 8-critical graph with $\Delta = 8$ different from $C_5[K_3]$?*

Problem 8. *Does there exist an infinite family of 7-critical graphs with $\Delta = 7$?*

Observe that in all the constructions of 7-critical graphs with $\Delta = 7$ that we have described, the graph Q is the only one that does not contain the graph $K_5 \vee \overline{K_3}$ obtained from K_8 by removing the edges of a triangle. (This is the subgraph induced by $K_x \cup I_x$ after splitting the vertex x.) Thus one potential way of showing that the above list is complete for $\Delta = 7$ might be to show that every 7-critical graph with $\Delta = 7$, aside from Q, contains $K_5 \vee \overline{K_3}$.

We remark that the splitting operation has been generalized and used in various contexts. For example, a generalization to hypergraphs appears in [17], and in [13], a range of more general results on graphs G with chromatic number close to $\Delta(G)$ are obtained using this type of operation.

Recall that the rth power C_ℓ^r of the cycle C_ℓ is the graph obtained from C_ℓ by joining every pair of vertices at each distance $d \in \{2, \ldots, r\}$ in C_t. We conclude by noting that further examples of Δ-critical graphs for $\Delta \in \{4, 6\}$ that are not of either of the types described so far are given by the square C_8^2 of the 8-cycle and the cube C_{11}^3 of the 11-cycle respectively. More generally, the rth power C_{3r+2}^r of the $(3r+2)$-cycle forms the basis for a family of tight examples for a conjecture of Reed [15], that is closely related to Conjecture 1. This conjecture proposes that for every graph G one has $\chi(G) \leq \lceil \frac{\Delta(G)+\omega(G)+1}{2} \rceil$. The graph $C_{3r+2}^r[K_t]$ has $(3r+2)t$ vertices, it is $((2r+1)t-1)$-regular, it has clique number $(r+1)t$, and independence number 2. This implies that its chromatic number is at least

$$\left\lceil \frac{(3r+1)t}{2} \right\rceil = \left\lceil \frac{(2r+1)t-1)+(r+1)t+1}{2} \right\rceil = \left\lceil \frac{\Delta(G)+\omega(G)+1}{2} \right\rceil.$$

It is not difficult to give a colouring using only this many colours. The well-known example $C_5[K_t]$ is the special case $r = 1$.

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