Mapping planar graphs into Coxeter graph

Ararat Harutyunyan[a], Reza Naserasr[a], Mirko Petruševski[b], Riste Škrekovski[c], Qiang Sun[a]

[a] *LRI, CNRS and Université Paris Sud, F-91405 Orsay Cedex, France*
[b] *Department of Mathematics and Informatics, Faculty of Mechanical Engineering, Skopje, Macedonia*
[c] *Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia & Faculty of Information Studies, 8000 Novo mesto, Slovenia & University of Primorska, FAMNIT, 6000 Koper, Slovenia.*

Abstract

We conjecture that every planar graph of odd-girth at least 11 admits a homomorphism to the Coxeter graph. Supporting this conjecture, we prove that every planar graph of odd-girth at least 17 admits a homomorphism to the Coxeter graph.

Keywords: homomorphisms, planar graphs, projective cubes, Coxeter graph.

1. Introduction

In this paper, considering a question of the second author [Nas13], we conjecture that:

Conjecture 1. *Every planar graph of odd-girth 11 admits a homomorphism to the Coxeter graph.*

Supporting this conjecture we then prove that:

Theorem 2. *Every planar graph of odd-girth 17 admits a homomorphism to the Coxeter graph.*

We start with notation and motivation. For standard terminology of graph theory we simply refer to [GR01]. The length of a shortest odd-cycle of a non-bipartite graph is called *odd-girth*. The subclass of planar graphs of odd-girth at least $2k+1$ will be denoted by $\mathcal{P}_{2k+1}$. A *homomorphism* of a graph G to another graph H is a mapping $\varphi : V(G) \to V(H)$ which preserves adjacency. If there exists a homomorphism of G to H, we write $G \to H$. Given a class $\mathcal{C}$ of graphs and a graph H, if every graph in $\mathcal{C}$ admits a homomorphism to H we write $\mathcal{C} \prec H$ and we say H *bounds* $\mathcal{C}$.

The *projective cube* of dimension d, denoted $\mathrm{PC}(d)$, is the Cayley graph $(\mathbb{Z}_2^d, \{e_1, e_2, \ldots, e_d, J\})$ where e_i's are standard basis and J is the all 1-vector. It is called projective cube because it is isomorphic to the graph obtained from the hypercube of dimension $d+1$ by identifying antipodal vertices. It is easy to verify that $\mathrm{PC}(2d+1)$ is bipartite. In contrast, $\mathrm{PC}(2d)$ is of chromatic number 4, while its shortest odd-cycle is of length $2d+1$, see [Nas07] for a proof. $\mathrm{PC}(2)$ is isomorphic to K_4, $\mathrm{PC}(3)$ is $K_{4,4}$ and $\mathrm{PC}(4)$ is the well-known Clebsch graph.

Given n and k such that $n \geq 2k$, the Kneser graph $K(n,k)$ is defined to be a graph whose vertices are k-subsets of an n-set where two such vertices are adjacent if they have no intersection. The Kneser graph $K(2d+1, d)$ is an induced subgraph of $\mathrm{PC}(2d)$ (see [Nas13]).

The existence of a homomorphism from a class of graphs to a projective cube is of special importance. Generally, it captures a certain packing problem (see [NRS13]). In particular, we have the following conjecture in extension of the four-color theorem:

Conjecture 3. *[Nas07] The class $\mathcal{P}_{2k+1}$ is bounded by* $\mathrm{PC}(2k)$.

The conjecture can be seen as an optimization for the following result of J. Nešetřil and P. Ossona De Mendez:

Theorem 4 ([NOdM12])**.** *Let $\mathcal{F} = \{F_1, F_2, \dots, F_r\}$ be a finite set of connected graphs. Let $\mathcal{M}$ be a minor-closed family of graphs and $\mathcal{M}_{\mathcal{F}}$ be the subclass consisting of those members of $\mathcal{M}$ which admit no homomorphism from a member of $\mathcal{F}$. Then there is a graph $H_{\mathcal{F}}$ which admits homomorphism from no member of $\mathcal{F}$, but bounds $\mathcal{M}_{\mathcal{F}}$.*

The bounds $H_{\mathcal{F}}$ built in known proofs of this theorem in most cases are from being optimal (in number of elements of graph). The question of finding an optimal $H_{\mathcal{F}}$ captures some of the most well-known problems in graph theory. For instance, the simplest case of $\mathcal{C} = \{K_n\}$ with $\mathcal{M}$ being the class of K_n-minor free graphs captures the Hadwiger conjecture. For $\mathcal{F} = \{C_{2k-1}\}$ and $\mathcal{M}$ being the class of planar graphs, we have $\mathcal{M}_{\mathcal{C}} = \mathcal{P}_{2k+1}$. In this case, it is recently shown in [NSS] that $H_{\mathcal{F}}$ must have at least 2^{2k} vertices. Thus, if Conjecture 3 holds, then $\mathrm{PC}(2k)$ is an optimal answer. This conjecture generalizes the four-color theorem as the case $k = 1$ is the four-color theorem. This conjecture is shown, in [Nas07], to be equivalent to a special case of a conjecture of P. Seymour on determining the edge-chromatic number of planar multi-graphs.

For $r > k$, since $\mathcal{P}_{2r+1}$ is included in $\mathcal{P}_{2k+1}$, if $\mathrm{PC}(2k)$ bounds $\mathcal{P}_{2k+1}$, then it also bounds $\mathcal{P}_{2r+1}$. However, in this case we believe that a proper subgraph of $\mathrm{PC}(2k)$ would suffice to bound $\mathcal{P}_{2r+1}$. Therefore it is asked:

Problem 5. *[Nas13] Given $r > k$, what are the optimal subgraphs of* $\mathrm{PC}(2k)$ *which bound $\mathcal{P}_{2r+1}$?*

It is shown in [Nas13] that this problem captures several interesting theories. In particular, if $K(2k+1, k)$ is an answer for $r = k+1$, it would determine the fractional chromatic number of $\mathcal{P}_{2r+1}$. In this regard, while the case $r = 2$ and $k = 1$ is implied by Grötzsch's theorem, the best result for $r = 3$ and $k = 2$ is that of [DŠV08] where it is proved that $\mathcal{P}_9$ is bounded by $K(5, 2)$. Note that $K(5, 2)$ is the well-known Petersen graph. For $r \geq 2k$ it is claimed by Jeager-Zhang conjecture that C_{2k+1} is the optimal answer for Problem 5. This case would determine the circular chromatic number of $\mathcal{P}_{4k+1}$. While Grötzsch's theorem is a special case here, the best result for general r is that of X. Zhu [Zhu01] who proved that $\mathcal{P}_{8k-3}$ is bounded by C_{2k+1}.

The first case not covered by any of these theorems and conjectures is $k = 3$ and $r = 5$. For this case we introduce Conjecture 1. Following [GR01], we will use a definition of the Coxeter graph based on Fano plane.

Given a set U of size 7, a Fano plane is a set of seven 3-subsets of U such that each pair of elements from U appears exactly in one 3-subsets. It can be checked that there is a unique such collection up to isomorphism. This collection then satisfies the axioms of finite geometry and triples would be called lines. Throughout this paper we will use the labeling of Figure **??** to denote the Fano plane.

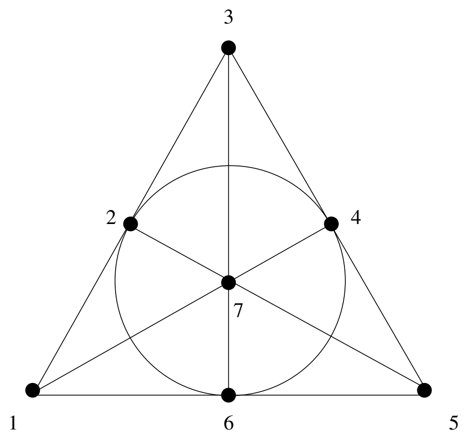

Figure 1: Fano plane

The Coxeter graph, denoted by Cox, is a subgraph of $K(7,3)$ obtained by deleting the vertices corresponding to the lines of the Fano plane. Therefore, Cox is an induced subgraph of PC(6). Hence, we propose that Cox is an answer for the case $k = 3$ and $r = 5$ of Problem 5:

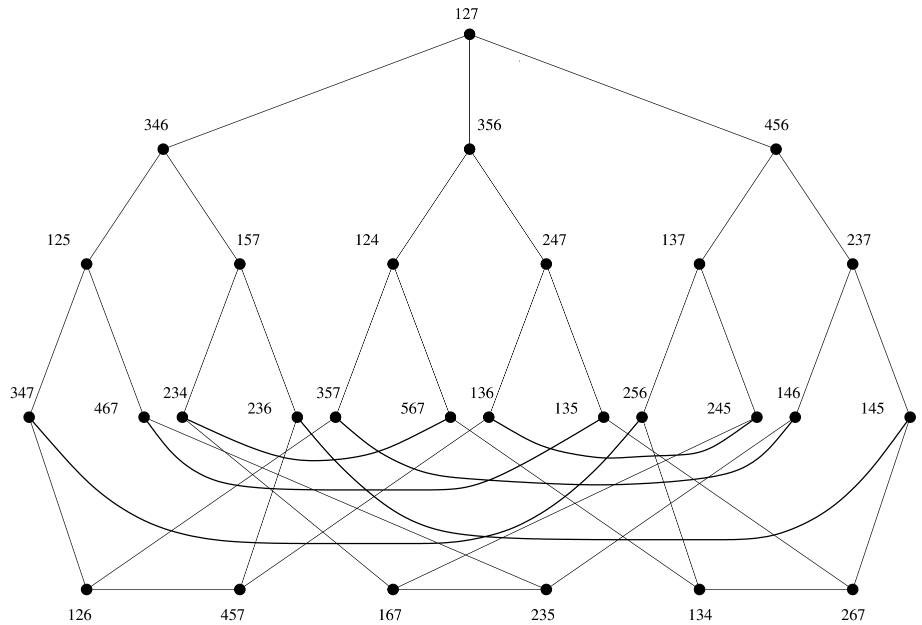

Figure 2: A presentation of the Coxeter graph

Since C_7 is a subgraph of Cox, the result of Zhu [Zhu01] implies that Cox also bounds $\mathcal{P}_{21}$. The aim of this work is to provide an improvement in this direction by proving Theorem 2.

Before proceeding further, we would like to mention the following interesting interpretation of a homomorphism to Cox. Given a graph G, a *Fano-coloring* of G is to associate to each vertex of G a triplet of elements of Fano satisfying the following two conditions:

(i) each triplet is in general position, i.e., it is not a line in Fano;

(ii) adjacent vertices have no element in common.

It simply follows from the definition that Fano-coloring is equivalent to homomorphism to Cox.

Our proof of Theorem 2 is based on discharging technique. Assuming that there is an element of $\mathcal{P}_{17}$ not mapping to Cox we choose X to be such an element with the smallest value of $|V(X)| + |E(X)|$. Hence, no proper homomorphic image of X is in $\mathcal{P}_{17}$. Since Cox is a vertex-transitive graph, X is 2-connected. Furthermore, repeated application of the folding Lemma of [KZ00], implies that each face of X is a 17-cycle.

The paper is organized as follows: in the next section we collect a list of properties of the Coxeter graph and further notation. In the following section we provide a list of small trees none of which can be an induced subgraph of X. Finally, in the last section we use discharging technique to obtain a contradiction. Some larger configurations that show up during our discharging process are shown to be reducible in this section.

Some more notation we will use are as follows. Given a graph G, a vertex of degree d is called a d-vertex. Analogously, a d^+-vertex is a vertex whose degree is d or more. Vertices x and y are said to be *weakly adjacent* if there exists an x-y path all of whose internal vertices are of degree 2 in G. Given a 3^+-vertex x, the number of 2-vertices weakly adjacent to x is denoted by $d_{weak}(x)$. Given a positive integer i and a vertex x of a graph G, $N_i(x)$ denotes the set of vertices at distance exactly i from x. When $i = 1$, we simply write $N(x)$. For a subset, U, of vertices of G we write $N_i(U) = \bigcup_{x\in U} N(x)$.

2. Coxeter graph and Cox-coloring

The Coxeter graph was discovered and is well-known for its highly symmetric structure. There are many symmetric presentations of it, but we will use the presentation of Figure 2 which is labeled by the definition we will use based on the labeling of the Fano plane given in Figure 1. The main properties of this graph we will need are collected in the following lemma.

Lemma 6. *The Coxeter graph satisfies the following:*

- *(i) It is distance-transitive.*
- *(ii) It is of diameter four.*
- *(iii) Its girth is seven.*
- *(iv) Given a vertex A, we have $|N(A)| = 3$, $|N_2(A)| = 6$, $|N_3(A)| = 12$ and $|N_4(A)| = 6$.*
- *(v) The independence number of* Cox *is* 12.
- *(vi) No proper homomorphic image of* Cox *is a subgraph of* Cox.
- *(vii) Given an edge A_1A_2, there exist exactly two vertices B_1 and B_2 such that $d(A_i, B_j) = 4$ for $i, j \in \{1, 2\}$. Furthermore, B_1 and B_2 are adjacent vertices of* Cox.
- *(viii) Let $\mathcal{A}$ and $\mathcal{B}$ be two (not necessarily distinct) subsets of $V(\text{Cox})$ each of size* 14*. Then there are $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $AB \in E(\text{Cox})$.*
- *(ix) For any two distinct vertices A and B of* Cox *we have $|N(A) \cap N(B)| \leq 1$.*
- *(x) For any pair A and C of vertices of* Cox *we have that $|N_2(A) \cap N(C)| \leq 2$ with equality only when $A \sim C$.*
- *(xi) For any pair A and C of vertices of* Cox *we have $|N_3(A)\cap N_3(C)| \geq 4$. Furthermore, when equality holds, there does not exist a second-neighbor B of A and a second-neighbor D of C such that $N_3(A) \cap N_3(C) \subseteq N(B) \cup N(D)$.*

Proof. The properties (i) through (v) are well known. We comment on the remaining six.

(vi) By contradiction, let ϕ be a homomorphism of Cox to a proper subgraph of itself. Then ϕ must identify at least two vertices, say A_1 and A_2. Using the distance-transitivity of the Coxeter graph, it is readily checked that there exists an A_1-A_2 path P with odd length less than 7. Hence, the image of P under ϕ contains a closed odd walk of length less then 7, contradicting that odd-girth of Cox equals 7.

(vii) Since Cox is edge-transitive, without loss of generality, we may assume $A_1 = 127$ and $A_2 = 346$. It is then implied that $\{B_1, B_2\} = \{134, 267\}$.

$(viii)$ Suppose the subsets $\mathcal{A}$ and $\mathcal{B}$ provide a counter-example, and let $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$. Note that, by connectivity of Cox, $\mathcal{C}$ is not empty. Let $\mathcal{C}' = (\mathcal{A} \cup \mathcal{B})^c$. By our assumption, $\mathcal{C}$ is an independent set of Cox, thus $|\mathcal{C}| \leq 12$. Furthermore, for each vertex C in $\mathcal{C}$ all three neighbors of C are in $\mathcal{C}'$. Since Cox is 3-regular and $|\mathcal{C}'| = |\mathcal{C}|$, $\mathcal{C} \cup \mathcal{C}'$ induces a proper 3-regular subgraph of Cox, contradicting the connectivity of Cox.

(ix) For otherwise, a 4-cycle would appear in Cox.

(x) If A is not adjacent to C, then existence of two elements in $N_2(A) \cap N(C)$ would result in a cycle of length at most 6 which is a contradiction. If A is adjacent to V, then $N_2(A) \cap N(C) = N(C) - A$.

(xi) Using the distance-transitivity of Cox, this is proved by considering the five possibilities for $d(A, C)$. If $d(A, C) = 0$ then $|N_3(A) \cap N_3(C)| = |N_3(A)| = 12$. If $d(A, C) = 1$, we may assume $A = 127$ and $C = 346$. Then $N_3(A) \cap N_3(C) = \{567, 135, 256, 145\}$ and it is readily checked that each of the second-neighbors of A has at most one neighbor among these four vertices, implying the last part of the statement. If $d(A, C) = 2$, we may assume $A = 127$ and $C = 125$. Then $N_3(A) \cap N_3(C) = \{234, 236, 357, 146\}$ and it is readily checked that the vertex 146 is respectively at distances $1, 3, 3$ from the vertices $357, 236, 234$. This clearly implies the last part of the statement. If $d(A, C) = 3$, we may assume $A = 127$ and $C = 347$. Then $N_3(A) \cap N_3(C) = \{236, 567, 136, 135, 245, 146\}$. Finally, if $d(A, C) = 4$, we may assume $A = 127$ and $C = 126$. Then $N_3(A) \cap N_3(C) = \{467, 567, 245, 145\}$ and each of the second-neighbors of A has at most one neighbor among these four vertices, implying the last part of the statement.

□

We may refer to a mapping of a graph H to the Coxeter graph as a Cox-coloring of H. A *partial* Cox-*coloring* of H is a mapping from a subset of vertices of H to vertices of the Coxeter graph which preserves adjacency among the colored vertices. Let H be a graph, ϕ be a partial Cox-coloring of H and u be a vertex of H not colored yet. We define $ad_{H,\phi}(u)$ to be the number of distinct choices $A \in V(\text{Cox})$ such that the assignment $\phi(u) = A$ is extendable to a Cox-coloring of H. When H and ϕ are clear from the context, we will simply write $ad(u)$.

3. Reducible configurations

Given an induced subgraph T of X, let *boundary* of T, denoted $\mathrm{Bdr}(T)$, be the set of vertices in T each with at least one neighbor from $X - T$. Let *interior* of T be $\mathrm{Int}(T) = T - X$

Let $X_T = X - Int(T)$ be a subgraph of X induced by vertices not in $\mathrm{Int}(T)$. If at least one Cox-coloring of X_T can be extended to a Cox-coloring of X, then $(T, \mathrm{Bdr}(T))$ is a called a *reducible* configuration. Each reducible configurations we will consider in this work is a tree having all its leaf vertices as its boundary, thus we will simply use T to denoted $(T, \mathrm{Bdr}(T))$.

We note that, by the minimality, X cannot contain any reducible configuration.

In this section, we provide a list of ten reducible configurations, all of which are trees of small order. Sometimes to prove that a configuration is reducible, we will consider smaller configurations and prove that most of the local Cox-colorings on the boundary are extendable.

Our first lemma is about paths. Given a u-v path P of length at most five we characterize all possible partial Cox-colorings of $\{u, v\}$ which are extendable to P.

Lemma 7. *Let P_l be a path of length $l \leq 5$ with u and v as its end-vertices. Let $\phi(u) = A$ and $\phi(v) = B$ where $\{A, B\} \subset V(Cox)$. Then, the partial* Cox*-coloring ϕ is extendable to a mapping of P_l to* Cox *if and only if:*

(i) $l = 2$ *and* $d(A, B) \in \{0, 2\}$, *or*

(ii) $l = 3$ *and* $d(A, B) \in \{1, 3\}$, *or*

(iii) $l = 4$ *and* $d(A, B) \neq 1$, *or*

(iv) $l = 5$ *and* $A \neq B$.

The proof of this lemma follows from the fact that the Coxeter graph is distance-transitive and the following general remark: Let P be a u-v path, and P' be a u'-v' path of lengths l and l', respectively. Then, the mapping $\psi(u) = u'$, $\psi(v) = v'$ is extendable to a mapping of P to P' if and only if $l \equiv l' \pmod 2$ and $l \geq l'$.

Proposition 8. *P_6 is a reducible configuration.*

Proof. Let u and v be the two end-vertices of P_6. Let ϕ be a Cox-coloring of $X - Int(P_6)$ and suppose $\phi(u) = A$ and $\phi(v) = B$. Choose a neighbor C of B distinct from A. Let v' be the neighbor of v in P_6. Extend ϕ to v' by $\phi(v') = C$. Then by Lemma 7 (*iv*) this extends to a Cox-coloring of the rest of P_6. □

It follows immediately that:

Corollary 9. *Given a vertex v of X we have $d_{weak}(v) \leq 4d(v)$.*

For paths of length 5 and 6 we will need to know the number of ways a Cox-coloring of the two end vertices extends to the interior. The next two lemmas are in this regard.

Lemma 10. *Let P be a 5-path on x, v_1, v_2, v_3, v_4, y. Let $\phi(x) = A$ and $\phi(y) = B$, $B \neq A$, be a partial* Cox*-coloring. If $d(A, B) = 2$, then $|ad(v_1)| = |ad(v_2)| = 2$ with the two possible choices for v_2 being at distance three in* Cox*. Otherwise, $|ad(v_1)| = 3$ and $|ad(v_2)| \geq 4$.*

Proof. Since Cox is distance-transitive, the statement can be proven by considering the four possibilities for $d(A, B)$. If $d(A, B) = 1$, we may assume $B = 127$ and $A = 346$. Then $ad(v_1) = N(A)$ and $ad(v_2) = \{346, 356, 456, 347, 467, 234, 236\}$, hence $|ad(v_1)| = 3$ and $|ad(v_2)| = 7$. In case $d(A, B) = 2$, we may assume $B = 127$ and $A = 125$. Then $ad(v_1) = \{347, 467\}$ and $ad(v_2) = \{135, 256\}$, hence $|ad(v_1)| = 2$ and $|ad(v_2)| = 2$, with the two admissible colors for v_2 being at distance three in Cox. If $d(A, B) = 3$, we may assume $B = 127$ and $A = 347$. Then $ad(v_1) = N(A)$ and $ad(v_2) = \{346, 347, 467, 357\}$, hence $|ad(v_1)| = 3$ and $|ad(v_2)| = 4$. Finally, if $d(A, B) = 4$, we may assume $B = 127$ and $A = 126$. Then $ad(v_1) = N(A)$ and $ad(v_2) = \{236, 136, 256, 146\}$, hence $|ad(v_1)| = 3$ and $|ad(v_2)| = 4$. □

Lemma 11. *Let P be a 6-path on $x, v_1, v_2, v_3, v_4, v_5, y$. Let $\phi(x) = A$ and $\phi(y) = B$, be a partial* Cox*-coloring. If $d(A, B) = 1$, then $|ad(v_3)| = 4$, furthermore these four colors constitute the neighbors of an edge of* Cox*. If $d(A, B) \neq 1$, then $|ad(v_3)| \geq 8$.*

Proof. First we consider the case of $d(A, B) = 1$. Since Cox is edge-transitive, we may assume without loss of generality that $A = 127$ and $B = 346$. In this case $ad(v_3) = \{567, 135, 256, 145\}$. Note that these are the neighbors of the edge $A'B'$, where $A' = 134$ and $B' = 267$. We note that each of A' and B' is at distance 4 from both A and B. This property uniquely determines the edge $A'B'$.

If $A = B$, then each vertex in $N(A) \cup N_3(A)$ is an admissible color for v_3, and we have $ad(v_3) = 15$. If $d(A, B) = 2$, since Cox is edge-transitive, we may assume $A = 127$ and $B = 125$. In this case $ad(v_3) = \{346, 356, 456, 347, 467, 234, 236, 357, 146\}$. For the case of $d(A, B) = 3$ we may assume $A = 127$ and $B = 347$, thus $ad(v_3) = \{456, 256, 135, 245, 567, 146, 236, 136\}$. Finally, if $d(A, B) = 4$ we may assume $A = 127$ and $B = 126$. In this case we have $ad(v_3) = \{346, 356, 347, 357, 467, 567, 145, 245\}$. □

We define $T_{k_1 k_2 \dots k_r}$ to be a graph obtained from $K_{1,r}$ by subdividing each edge ut_i, k_i times where t_i's are the leaf vertices and u is the central vertex of $K_{1,r}$. Given an r-vertex u, $r \geq 3$, u together with the set of vertices weakly adjacent to u induces a $T_{k_1 k_2 \dots k_r}$ which is denoted by $T(u)$. The next few lemmas are about the possibilities for $T(u)$ when u is of degree 3 or 4.

Lemma 12. *Let $T = T_{222}$. Then the assignment $\phi(t_i) = A_i$, $i = 1, 2, 3$ is extendable unless $\{A_1, A_2, A_3\}$ induces a P_3.*

Proof. Consider the t_1-t_2 path P in T. Let v be the middle vertex of this path and let $\mathcal{A}$ be the set of colors whose assignment to v is extendable to P. In the proof of Lemma 11, we saw that there are 5 possibilities for $\mathcal{A}$. In three of these possibilities, to be precise, when $d(A_1, A_2) \neq 1, 2$, we have $N(\mathcal{A}) \cup N_3(\mathcal{A}) = V(\text{Cox})$. Thus, in these cases any choice of A_3 is extendable.

If $d(A_1, A_2) = 1$, then $N(\mathcal{A}) \cup N_3(\mathcal{A}) = V(\text{Cox}) \setminus N(\{A_1, A_2\}) \cup \{A_1, A_2\}$. Thus, in this case a choice of A_3 is extendable unless either $A_3 \sim A_1$ and $A_3 \neq A_2$ or $A_3 \sim A_2$ and $A_3 \neq A_1$.

Finally if $d(A_1, A_2) = 2$, then $N(\mathcal{A}) \cup N_3(\mathcal{A}) = V(\text{Cox}) \setminus \{B\}$, where B is the common neighbor of A_1 and A_2.

□

Proposition 13. *The configurations T_{123} and T_{034} are reducible.*

Proof. We give a proof by contradiction for T_{123}, and the proof for T_{034} is similar. Let X' be the subgraph of X induced by deleting the u-t_3 path of T_{123}. By the minimality of X, there is a Cox-coloring of X'. Thus, by Lemma 7, we may consider a coloring ϕ of $X - Int(T_{123})$ for which $\phi(t_1) \neq \phi(t_2)$. Now, by Lemma 10, there is an extension of ϕ to the t_1-t_2 path of T_{123} such that $\phi(u) \nsim \phi(t_3)$. By Lemma 7, this extends to the rest of T_{123}. □

Corollary 14. *If v is a 3-vertex in X, then $d_{weak}(v) \leq 6$. Furthermore, if $d_{weak}(v) = 6$, then $T(v)$ is one of the following trees: T_{024}, T_{033}, T_{114}, T_{222}.*

Proof. It is easily observed that the only maximal subtrees $T_{k_1k_2k_3}$ with central vertex v, containing neither of the reducible configurations P_6, T_{123}, T_{034}, are T_{024}, T_{033}, T_{114}, T_{222}. □

Proposition 15. *The configurations T_{1334}, T_{2234}, T_{2333} are reducible.*

Proof. For T being one of the three configurations, let ϕ be a Cox-coloring of $X - Int(T)$ and suppose $\phi(t_1) = A_1, \phi(t_2) = A_2, \phi(t_3) = A_3, \phi(t_4) = A_4$.

First we consider T_{1334}. Using Lemma 7, all we need is to find a choice of a color which is at distance 0 or 2 from A_1 (7 choices), adjacent to neither A_2 nor A_3, and distinct from A_4. Thus, by Lemma 6 (x), at least one member of $N_2(A_1)$ satisfies all four conditions.

For the case of T_{2234}, using Lemma 11, if $d(A_1, A_2) \neq 1$, then we have at least eight choices for u each of which is extendable on the u-t_1 and the u-t_2 paths. By Lemma 7, at least four of these choices can be extended also to the u-t_3 and the u-t_4 paths. If $d(A_1, A_2) = 1$, then by Lemma 11 there are exactly four choices for u each of which is extendable on the u-t_1 and the u-t_2 paths, furthermore at most two of these four colors are in $N(A_3)$. Of the remaining two we have a choice distinct from A_4.

For the last case, i.e., T being T_{2333}, by Lemma 7, the number of choices for $\phi(u)$ which are extendable on the u-t_i paths, $i = 1, 2, 3, 4$ are 15, 25, 25 and 25, respectively. Since there are only 28 vertices in the Coxeter graph, there are at least 6 choices for $\phi(u)$ each of which extends on all the four paths. □

Corollary 16. *If v is a 4-vertex in X, then $d_{weak}(v) \leq 12$. Furthermore, if $d_{weak}(v) = 12$, then $T(v)$ is T_{0444}. Otherwise, $d_{weak}(v) \leq 10$.*

The simple proof of this corollary is analogous to the one of the previous corollary, hence we leave it to the reader.

We now would like to investigate weak neighbourhoods of two 3-vertices, say u and v, close to each other.

Proposition 17. *The three trees of Figure 3 are reducible.*

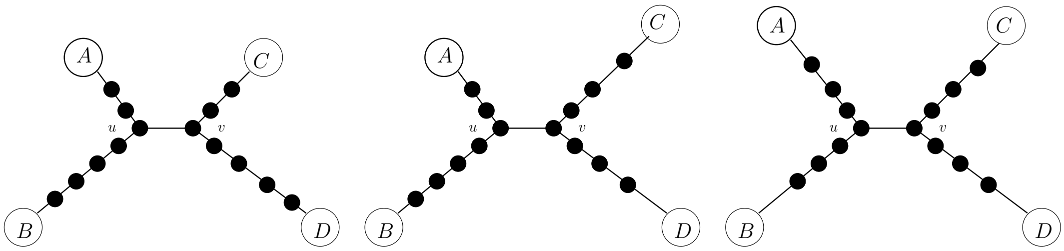

Figure 3: Reducible configurations of adjacent 3-vertices with a partial Cox-coloring of the boundary.

Proof. Consider one of the configurations and let A, B, C and D be colors of the vertices on the boundary as shown in the Figure. By Lemma 7, to color u so that it is extendable to the A-u and B-u paths of the given configuration there are at least 14 choices. Let $\mathcal{A}$ be set of those colors for u. Similarly, we have a set $\mathcal{B}$ of size at least 14 for v with respect to C and D. By Lemma 6 $(viii)$, there are adjacent elements $S \in \mathcal{A}$ and $S' \in \mathcal{B}$. Assignment of S to u and S' to v is now extendable. □

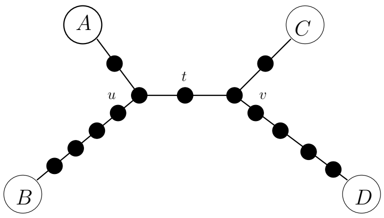

Figure 4: Reducible configuration of close 3-vertices with a partial Cox-coloring of the boundary.

Proposition 18. *The tree of Figure 4 is reducible.*

Proof. Consider a partial Cox-coloring of the leaf vertices, as depicted in Figure 4, and use this colors to denote the leaf vertices. By Lemma 7, to color u so that it is extendable to the A-u path one can choose any vertex in $N_2(A)$. Any such choice, except possibly B, would be extendable on the B-u path and any of their neighbors would be admissible for t. Thus, at least for ten elements $A' \in N_3(A)$ the assignment of A' to t is extendable on the left. Furthermore, two third-neighbors of A are not admissible colors for t if and only if B is a second-neighbor of A, and then these particular two non-admissible colors belong to $N(B)$. Similarly, for at least ten choices of $C' \in N_3(C)$ the assignment of C' to t is extendable on the right with 10 being the exact number of choices if $D \in N_2(C)$. We now apply Lemma 6 (xi) to find a color for t admissible from both sides. □

The next two configurations we consider are not reducible. But we show that, up to isomorphism, there is a unique coloring of the boundary which is not extendable to the interior. This implies, in particular, that if there were a second choice for a color of one of the vertices on the boundary, there the coloring is extendable.

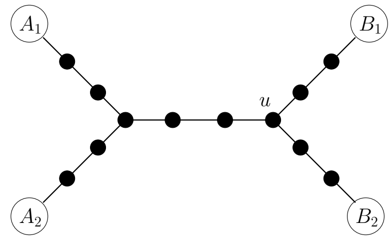

Figure 5: Configuration F_1
with a partial Cox-coloring of the leaf vertices.

Proposition 19. *The partial* Cox*-coloring of configuration F_1 given in Figure 5 is extendable to the whole configuration unless $d(A_1, A_2) = d(B_1, B_2) = 1$ and $d(A_i, B_j) = 4$ for $i, j \in \{1, 2\}$.*

Proof. Consider the T_{222} configuration whose boundary consists of the vertex u and the two vertices colored with A_1 and A_2, respectively. If $d(A_1, A_2) \neq 1$, then by Lemma 12, there are at least 27 choices of a color for u which is extendable to the interior of this T_{222}. On the other hand, by Lemma 11, there are at least 4 choices of a color for u that is extendable to a Cox-coloring of the partially colored 5-path connecting B_1 and B_2. Thus, there are at least three common choices of a color for u which is extendable on the whole configuration.

If $d(A_1, A_2) = 1$, then, again by Lemma 12, there are exactly 4 non-extendable choices of a color for u, these particular four choices are the neighbors of A_1 and A_2 distinct from A_1 and A_2. If any of the other 24 choices is extendable on the B_1-B_2 path, then the coloring is extendable. Otherwise by Lemma 11, or rather the proof of it, we have $d(B_1, B_2) = 1$ and, furthermore, $d(A_i, B_j) = 4$ for $i, j \in \{1, 2\}$. □

Corollary 20. *For the configuration F_1 of Figure 5, if the given partial* Cox*-colouring is not extendable to the whole configuration, then A_1 is uniquely determined by A_2, B_1 and B_2.*

Proof. Note that given an edge A_1A_2, the property $d(A_i, B_j) = 4$ for $i, j \in \{1, 2\}$ determines a unique edge in Cox, as shown in Lemma 6 (vi). □

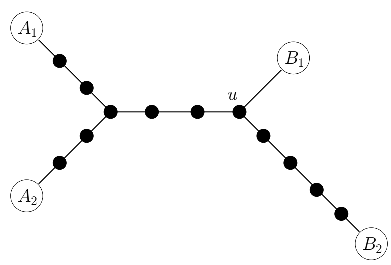

Figure 6: Configuration F_2

Proposition 21. *The partial* Cox*-coloring of configuration F_2 given in Figure 6 is extendable to the whole configuration unless $d(A_1, A_2) = 1$ and $\{B_1, B_2\} = \{A_1, A_2\}$.*

Proof. Our proof is similar to that of the previous proposition. Again we consider the possibilities on u. If $d(A_1, A_2) \neq 1$, then there are at least 27 choices of color for u extendable on the part connecting to A_1 and A_2. Of these 27 vertices in Cox, at least 2 are neighbors of B_1 and of these two, one is distinct from B_2. This is an extendable choice now.

If $d(A_1, A_2) = 1$, then the four neighbors of A_1 and A_2, distinct from A_1 and A_2, are the only choices for u non-extendable on the left side. If $B_1 \notin \{A_1, A_2\}$, then there are at least two neighbors of B_1 whose assignments to u are extendable on the left of u, and at least one of these two is different from B_2. Thus we may assume, without loss of generality, that $B_1 = A_1$. Then A_2 is an extendable choice for u unless $B_2 = A_2$. □

Corollary 22. *If the partial* Cox*-coloring of configuration F_2 given in Figure 6 is not extendable to the whole configuration, then A_1 is uniquely determined by A_2, B_1 and B_2.*

4. Discharging and further reducible configurations

Recall that X, our minimal counterexample, is a 2-connected plane graph in which each face is a 17-cycle.

Being a plane graph, for any choice of real numbers α and β satisfying the condition $\alpha + \beta = 3$, the Euler formula gives the following identity for the vertices and faces of X:

$$\sum_v (\alpha \cdot d(v) - 6) + \sum_f (\beta \cdot l(f) - 6) = -12, \tag{1}$$

where the second sum is taken over all faces of X with $l(f)$ denoting the length of the face f (which in our case is always 17). We remark that the identity (1) is just the well-known Euler formula $V - E + F = 2$ in disguise because of the identity $\sum_v d(v) = 2E = \sum_f l(f)$.

Thus, by setting $\beta = \frac{6}{17}$ and $\alpha = \frac{45}{17}$, the identity (1) reduces to

$$\sum_v (45 \cdot d(v) - 102) + \sum_f 0 = -204. \tag{2}$$

This would lead to the following initial charge on each vertex v of X:

$$w_0(v) = 45 \cdot d(v) - 102.$$

Note that $\sum_{v \in V(X)} w_0(v) = -204 < 0$, and that each 2-vertex has initial charge -12, each 3-vertex has initial charge 33, each 4-vertex has initial charge 78, etc. Our aim is to rearrange charges on vertices so that the at the final step, the charge on each vertex is non-negative, this would contradict the existence of X. We will accomplish this through two phases of discharging. In the first phase, we will take care of vertices of degree 2. Then, in the second phase, we design a discharging rule that would take care of all negatively charged vertices after the first phase. We will then show that each configuration that may lead to a vertex of negative charge is reducible. This would complete our proof.

4.1. First phase of discharging

Here we use the following discharging rule:

(R1) For each pair x, y of weakly adjacent vertices in X, if $d(x) = 2$ and $d(y) \geq 3$, then we give a charge of $+6$ from y to x.

Let $w_1(v)$ denote the new charge at each vertex v. Since X is 2-connected, each 2-vertex receives a total charge of $+12$ ($+6$ from each side), thus $d(v) = 2$ implies $w_1(v) = 0$. If $d(v) = 3$, then by Corollary 14, we have $w_1(v) \geq -3$. Furthermore, if $d_{weak}(v) \neq 6$, then $w_1(v) \geq 3$. For $d(v) \geq 4$ we have $w_1(v) \geq 6$.

A vertex v of X is called *poor* if $w_1(v) < 0$. As a consequence of Corollary 14, we have the following characterization of poor vertices.

Proposition 23. *A vertex v of X is poor if and only if $d(v) = 3$ and $d_{weak}(v) = 6$.*

Corollary 14 also implies that for each poor vertex v, $T(v)$ is one of the following trees: T_{024}, T_{033}, T_{114}, T_{222}. Our aim is to seek charge for v from its closest leaf vertices of $T(v)$. Thus, a leaf vertex x of $T(v)$ which is closest to v on a shortest path to v in $T(v)$ will be called a *v-support* if x itself is not a poor vertex. Equivalently, we may say *x supports v*.

4.2. Second phase of discharging

In this phase we try to increase the charge of all poor vertices. The discharging rule is as follows:

(R2) If y supports a poor vertex x, then y gives a charge of $+3$ to x if $d(x, y) = 1$, and charge of $+1.5$ to x if $d(x, y) \neq 1$.

Let $w_2(v)$ be the charge of vertex v after this phase. We will show that $w_2(v) \geq 0$, for every vertex v of X.

Observe that the charge of each 2-vertex v remains the same, i.e. $w_2(v) = 0$. If v is a 5^+-vertex, then by Corollary 9 we have $w_1(v) \geq w_0(v) - 24d(v) = 21d(v) - 102 \geq 3$. Furthermore, in applying $(R2)$ if v is a support for a vertex u, then there number of degree 2 vertices on v-u path is at most two. Thus if v supports r vertices then $w_1(v) \geq 3 + 12r$. This implies that $w_2(v) \geq 3$. Now we consider 4-vertices. Let $d(v) = 4$. By Corollary 16, unless $T(v) = T_{0444}$ we have $w_1(v) \geq 18$ and this clearly gives $w_2(v) \geq 6$. If $T(v) = T_{0444}$, then v supports at most one vertex, and, therefore, we have $w_2(v) \geq 3$.

We are left to consider the case of a 3-vertex v. If v is non-poor, originally we have $w_1(v) \geq 3$, and if, furthermore, $d_{weak}(v) \leq 4$, then $w_1(v) \geq 9$. Therefore, if v supports only one poor vertex, or if $d_{weak}(v) \leq 4$, then $w_2(v) \geq 0$ is assured. Hence we need to take care of two kind of vertices: (i) poor vertices (i.e., $d(v) = 3$ and $d_{weak}(v) = 6$, (ii) a 3-vertex v with $d_{weak}(v) = 5$ which must give a support of 4.5 or 6.

To complete our proof we will show that neither of these two scenarios are possible by proving that:

(i) If v is a poor vertex, then either v has an adjacent supporting vertex or it has at least two supporting vertices.

(ii) If $d(v) = 3$ and $d_{weak}(v) = 5$, when applying (R2) it gives at most a total charge of 3.

To prove (i), first we consider the case when v is adjacent to a 3^+-vertex, say x. We claim that x is the adjacent supporting vertex of v. To see this, suppose by contradiction x is poor. Then, the union of $T(x)$ and $T(v)$ must be one of the configurations of Figure 3. But these are reducible configurations as shown in Proposition 17.

Thus, we may assume that all neighbors of v are 2-vertices. Now consider the case when there is a 3^+-vertex, say x, at distance 2 from v. Thus, $T(v)$ must be T_{114}. Hence there are two 3^+-vertices at distance 2 from v. It remains to prove that neither of these two vertices is poor. By contradiction, suppose x is a poor vertex. Then $T(x)$ itself must be T_{114}. Since each face of X is a 17-cycle, the union of $T(x)$ and $T(v)$ must be the configuration of Figure 4, which is shown to be reducible in Proposition 18.

Hence we may assume that $T(v)$ is T_{222}. We prove that at most one vertex in $N_3(v)$ is poor. If $x \in N_3(v)$ is a poor vertex, then $T(x)$ must be either T_{222} or T_{024}. Then, the union $F = T(v) \cup T(x)$ is, respectively, the configuration of Figure 5 or the configuration of Figure 6. Let y be another vertex in $N_3(v)$. If y is also a poor vertex, then in $X - Int(F)$ the vertex y is an internal vertex of an induced path P of length 5. Thus, by Lemma 10, there are at least two choices for extending a Cox-coloring of $(X - Int(F)) - Int(P)$ to y, one of which is extendable to a Cox-coloring of X by Corollary 20 or Corollary 22.

To prove (ii), begin by observing that $T(v)$ must be one of the configurations: T_{014}, T_{023}, T_{113}, T_{122}.

Assume first that $T(v)$ is T_{014} and v supports two poor vertices. There are only two such possible configurations, shown in Figure 7. In this figure, vertices in square are the poor vertices whose support is v. We claim that each configuration of Figure 7 is reducible. To prove this, consider a partial Cox-coloring of the leaf vertices (as depicted), and look first for the minimum number of possible colors for v if we were to extend the partial Cox-coloring by A and B from left until v. This number, which is shown on the left of v, is derived as follows: for the first configuration, by Lemma 7 we have that $ad(\square) = |(N(A) \cup N_3(A)) \backslash \{B\}|$. Using the vertex-transitivity of Cox, it is readily observed that the considered $ad(v)$ surely includes A, all 6 second-neighbors of A, at least 11 third-neighbors of A, and all 6 fourth-neighbors of A, giving the total of 24 choices. For the second configuration, Lemma 7 gives $ad(\square) = |(N(A) \cup N(B))^c|$. Hence, Lemma 6 (ix) implies that the considered $ad(v) = |\{A, B\}^c|$. Similarly, the minimum number of possible extensions of coloring by C and D to v, from right only, is given on the right of v. This number can be easily deduced from Lemma 7. Since the sum of the two numbers in each of the configurations is greater than the number of vertices of Cox, a good common choice for coloring v from both sides exists in each case.

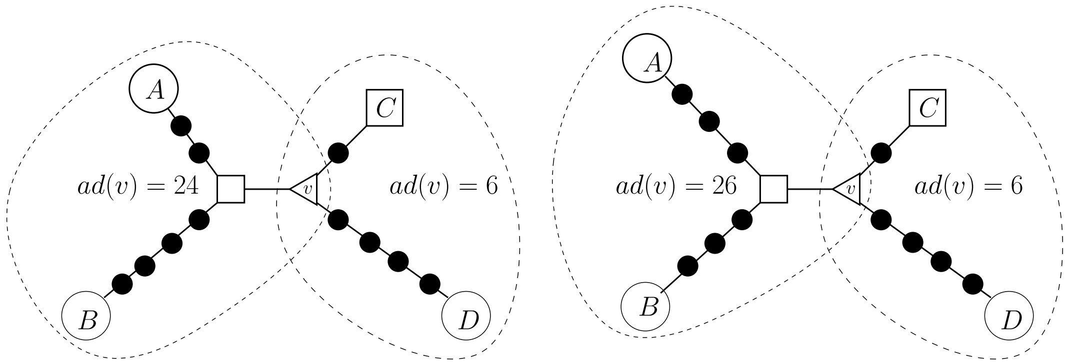

Figure 7: Local configurations of a center of T_{014} supporting two poor vertices.

For the case that $T(v)$ is isomorphic to T_{023}, using the Figure 8, a similar argument is applied.

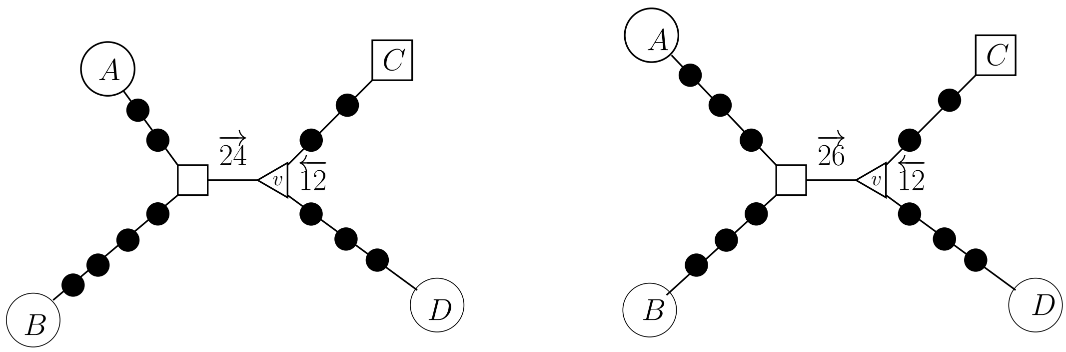

Figure 8: Local configurations of a center of T_{023} supporting two poor vertices.

If $T(v)$ is T_{113}, then v is a support of at most two poor vertices to each of which it may send a charge of $+1.5$. Hence, in the second phase v gives away at most a charge of $+3$.

Finally, assume that $T(v)$ is T_{122}. If $w_2(v) < 0$, then v must have given charges to three weakly adjacent poor vertices. In this case, we have a unique local configuration, given in Figure 9. To prove its reducibility, consider a partial Cox-coloring of its leaf vertices. In this figure the minimum number of possible choices of colors for v extending partial Cox-colorings from the three different directions to v are as given in Figure 9. The first of these three numbers, namely the 23 choices for extending to v the partial coloring by A and B, is derived as follows: If $d(A, B) = 2$, the set of admissible colors for the relevant neighbor of v comprises of the neighbors of A and the 10 third-neighbors of A that are not adjacent to B, hence it is readily checked (using the distance-transitivity of Cox) that the considered $|ad(v)| = 23$. Otherwise $d(A, B) \neq 2$, and each third-neighbor of A is an admissible color for the relevant neighbor of v, which readily gives at least 24 choices for the considered $ad(v)$. The remaining two numbers in the Figure 9 follow from Lemma 7 and Lemma 12, respectively. From these three minimum numbers for possible choices of colors for v, easily follows that there is a common choice.

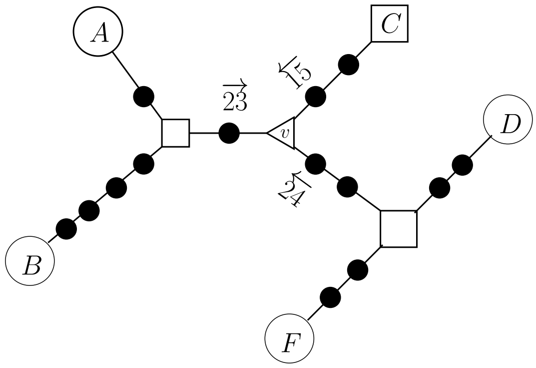

Figure 9: Local configuration of a center of T_{122} supporting three poor vertices.

5. Concluding remarks

1. We have shown in this paper that one may use the existence of a combinatorial design to provide answer for special cases of the Problem 5. Our primary concern in this paper was the case $r = 5$ and $k = 3$ of this question and we proposed an answer using the Fano plane. At a 2011 summer workshop in Prague, Peter Cameron proposed a similar conjecture for the case of $k = 5$ and $r = 7$ based on the existence of a unique Steiner quintuple system of order 11.

2. Using a larger set of trees as reducible configurations our proof can be extended to show that $\mathcal{P}_{15}$ is bounded by Cox.

3. We have considered Problem 5 for PC($2k$). However, using recent developments on signed graphs, a similar question could be asked for PC($2k-1$), see [NRS13]. We will address this question in forth coming works.

References

[DŠV08] Z. Dvořák, R. Škrekovski, and T. Valla, *Planar graphs of odd-girth at least 9 are homomorphic to the Petersen graph*, SIAM J. Discrete Math. **22** (2008), no. 2, 568–591.

[GR01] Chris Godsil and Gordon Royle, *Algebraic graph theory*, Graduate Texts in Mathematics, vol. 207, Springer-Verlag, New York, 2001.

[KZ00] William Klostermeyer and Cun Quan Zhang, $(2+\epsilon)$*-coloring of planar graphs with large odd-girth*, J. Graph Theory **33** (2000), no. 2, 109–119.

[Nas07] Reza Naserasr, *Homomorphisms and edge-colourings of planar graphs*, J. Combin. Theory Ser. B **97** (2007), no. 3, 394–400. MR 2305893 (2007m:05099)

[Nas13] ———, *Mapping planar graphs into projective cubes*, J. of Graph theory **74** (2013), no. 3, 249–259.

[NOdM12] Jaroslav Nešetřil and Patrice Ossona de Mendez, *Sparsity*, Algorithms and Combinatorics, vol. 28, Springer, Heidelberg, 2012, Graphs, structures, and algorithms.

[NRS13] Reza Naserasr, Edita Rollová, and Eric Sopena, *Homomorphisms of planar signed graphs to signed projective cubes*, Discrete Mathematics & Theoretical Computer Science **15** (2013), no. 3, 1–12.

[NSS] Reza Naserasr, Sagnik Sen, and Qiang Sun, *Cliques in walk-powers of planar graphs*, Preprint.

[Zhu01] Xuding Zhu, *Circular chromatic number of planar graphs of large odd girth*, Electron. J. Combin. **8** (2001), no. 1, Research Paper 25, 11.