CIRCULAR $(4-\epsilon)$-COLORING OF SOME CLASSES OF SIGNED GRAPHS*

FRANTIŠEK KARDOŠ†, JONATHAN NARBONI‡, REZA NASERASR§, AND ZHOUNINGXIN WANG¶

Abstract. A circular r-coloring of a signed graph (G,σ) is an assignment ϕ of points of a circle C_r of circumference r to the vertices of (G,σ) such that for each positive edge uv of (G,σ) the distance of $\phi(u)$ from $\phi(v)$ is at least 1 and for each negative edge uv the distance of $\phi(u)$ from the antipode of $\phi(v)$ is at least 1. The circular chromatic number of (G,σ), denoted $\chi_c(G,\sigma)$, is the infimum of r such that (G,σ) admits a circular r-coloring.

This notion is recently defined by Naserasr, Wang, and Zhu who, among other results, proved that for any signed d-degenerate simple graph $\hat{G}$ we have $\chi_c(\hat{G}) \leq 2\lfloor \frac{d}{2} \rfloor + 2$. For $d \geq 3$, examples of signed d-degenerate simple graphs of circular chromatic number $2\lfloor \frac{d}{2} \rfloor + 2$ are provided. But for $d = 2$ only examples of signed 2-degenerate simple graphs of circular chromatic number arbitrarily close to 4 are given, noting that these examples are also signed bipartite planar graphs.

In this work we first observe the following restatement of the 4-color theorem: If (G,σ) is a signed bipartite planar simple graph where vertices of one part are all of degree 2, then $\chi_c(G,\sigma) \leq \frac{16}{5}$. Motivated by this observation, we provide an improved upper bound of $4 - \frac{2}{\lfloor \frac{n+1}{2} \rfloor}$ for the circular chromatic number of a signed 2-degenerate simple graph on n vertices and an improved upper bound of $4 - \frac{4}{\lfloor \frac{n+2}{2} \rfloor}$ for the circular chromatic number of a signed bipartite planar simple graph on n vertices. We then show that each of the bounds is tight for any value of $n \geq 2$.

Key words. Signed graph, circular coloring, planar graph, 4-color theorem

MSC codes. 05C15, 05C22

1. Introduction.

A *signed graph* (G,σ) is a graph G together with a signature σ which assigns to each edge of G one of the two signs, either positive $(+)$ or negative $(-)$. Here $\{+,-\}$ is viewed as a multiplicative group. For simplicity we may use $\hat{G}$ to denote a signed graph based on a graph G. A key notion in the study of signed graphs is the notion of *switching*, which is to multiply the signs of all the edges of an edge-cut by a $-$. Two signatures on a same graph are said to be *equivalent* if one is a switching of the other. The *sign of a closed walk* W of (G,σ) is the product of the signs of all edges of W, counting multiplicity. Note that sign of a closed walk, and in particular a cycle, is invariant under switching. One of the earliest theorems on signed graphs is the following.

THEOREM 1.1. [10] *Two signatures σ_1 and σ_2 on G are equivalent if and only if each cycle C of G has a same sign in (G,σ_1) and (G,σ_2).*

The study of coloring and homomorphisms of signed graphs has gained recent

---

*

Funding: This work is supported by the French ANR project HOSIGRA (ANR-17-CE40-0022) and by the Slovak Research and Development Agency under the contract No. APVV-19-0308. It has also received funding from the European Union's Horizon 2020 research and innovation program under the Marie Sklodowska-Curie grant agreement No 754362.

†Université de Bordeaux, CNRS, LaBRI, F-33400 Talence, France and Comenius University, Bratislava, Slovakia (frantisek.kardos@u-bordeaux.fr).

‡Université de Bordeaux, CNRS, LaBRI, F-33400 Talence, France (jonathan.narboni@u-bordeaux.fr).

§Université Paris Cité, CNRS, IRIF, F-75006, Paris, France, France (reza@irif.fr).

¶Corresponding author. School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China and Université Paris Cité, CNRS, IRIF, F-75006, Paris, France, France (wangzhou@nankai.edu.cn).

attention for various reasons, in particular because it provides a frame for a better connection between minor theory and graph coloring. A *homomorphism* of a signed graph (G, σ) to a signed graph (H, σ) is a mapping of vertices and edges of G (respectively) to the vertices and edges of H which preserves adjacencies, incidences, and signs of closed walks. As an application of Theorem 1.1, one can show that this definition is equivalent to finding an equivalent signature σ' of σ and a mapping of vertices and edges of G to the vertices and edges of H (respectively) which preserves adjacencies, incidences and signs of edges with respect to σ' and π.

When G and H are simple graphs, then the edge mapping is implied from the vertex mapping. This is mostly the case in this work. The *core* of a signed graph (G, σ) is the smallest subgraph of it to which (G, σ) admits a homomorphism. For a proof that core is well defined and unique up to a switch-isomorphism we refer to [7].

In the theory of homomorphisms of signed graphs two restrictions stand out: I. restriction to signed graphs $(G, -)$ where all edges are negative, II. restriction to signed bipartite graphs. In the former class of signed graphs, the existence of a homomorphism between two members is solely based on a homomorphism of the underlying graphs. Thus the theory of homomorphisms restricted to this subclass is the theory of homomorphisms of graphs. However, we note that as signed graphs, even this restricted case provides a much stronger connection to the minor theory. The restriction to the class of signed bipartite graphs is a focus of this work. It is shown in [6, 7] that the restriction of the theory to this subclass is already at least as rich as the study of graph homomorphisms. The following construction is the key for this observation.

Definition 1.2. *Given a graph G, the signed (bipartite) graph $S(G)$ is the signed graph obtained from G by replacing each edge of G with a pair of paths of length 2 and assigning a signature such that the resulting 4-cycle is negative.*

It is then shown in [6] that:

Theorem 1.3. *Given graphs G and H there exists a homomorphism of G to H if and only if $S(G)$ admits a homomorphism to $S(H)$.*

In this work we consider the recent definition of the circular chromatic number of signed graph as defined in [9], noting that a similar definition was given earlier in [1] but that the two parameters behave differently with respect to fine details. The difference of the two definitions can be reduced down to the difference of the definitions of an antipodal pair. We take the most standard definition where the two ends of a diameter form an antipodal pair.

Definition 1.4. *Given a real number r and signed graph (G, σ), a mapping φ of the vertices of G to a circle of circumference r is said to be a* circular r-coloring *of (G, σ) if for each positive edge uv of (G, σ), the distance between $\varphi(u)$ and $\varphi(v)$ is at least 1 and for each negative edge uv of (G, σ), the distance between $\varphi(u)$ and the antipode of $\varphi(v)$ is at least 1. The* circular chromatic number *of a signed graph (G, σ) is defined as*

$$\chi_c(G, \sigma) = \inf\{r \geq 1 : (G, \sigma) \text{ admits a circular } r\text{-coloring}\}.$$

In practice we view a circle of circumference r to be the set $[0, r)$. This leads to the following equivalent definition of circular r-coloring of a signed graph as follows.

Definition 1.5. *A* circular r-coloring *of a signed graph (G, σ) is a mapping $f : V(G) \to [0, r)$ such that for each positive edge uv, $1 \leq |f(u) - f(v)| \leq r - 1$ and for each negative edge uv, either $|f(u) - f(v)| \leq \frac{r}{2} - 1$ or $|f(u) - f(v)| \geq \frac{r}{2} + 1$.*

One of the first theorems in the study of circular coloring is the notion of *tight cycle* defined as follows. Given a signed graph (G, σ) and a circular r-coloring φ of it, a cycle $C = v_1 v_2 \cdots v_k$ is said to be a *tight cycle* if for each positive edge $v_i v_{i+1}$ we have $\varphi(v_{i+1}) - \varphi(v_i) = 1$ and for each negative edge $v_i v_{i+1}$ we have $\varphi(v_{i+1}) - \varphi(v_i) = \frac{r}{2} + 1$. Here the additions on indices of vertices are taken modulo k and the subtractions of the values of φ are taken modulo r using $[0, r)$ as the reference set.

The following theorem then is essential for proving that the circular chromatic number of any (finite) signed graph is a rational number and that it is computable.

THEOREM 1.6. [9] *Given a signed graph (G, σ), we have $\chi_c(G, \sigma) = r$ if and only if the following two conditions are satisfied: (G, σ) admits a circular r-coloring, and in every circular r-coloring φ of (G, σ), there is a tight cycle with respect to φ.*

COROLLARY 1.7. [9] *Any signed graph (G, σ) which is not a forest has a cycle with s positive edges and t negative edges such that*

$$\chi_c(G, \sigma) = \frac{2(s+t)}{2a+t}$$

for some integer a. In particular, $\chi_c(G, \sigma) = \frac{p}{q}$ for some $p \leq 2|V(G)|$.

It follows from the definition that every signed bipartite graph (not necessarily simple) is circular 4-colorable. Simply assign 0 to vertices of one part and 1 to the vertices of the other part. However, this should not mislead to underestimating the study of circular chromatic number of signed bipartite graphs. Since $S(G)$ preserves the homomorphism properties of G, it is natural to expect that it can be used to determine the circular chromatic number of G. This has indeed been proved to be the case in [9].

PROPOSITION 1.8. *Given a simple graph G, we have $\chi_c(S(G)) = 4 - \frac{4}{\chi_c(G)+1}$.*

Observe that if G is a planar graph then so is $S(G)$. Furthermore, $S(G)$ is a bipartite graph in which vertices of one part are all of degree 2. Let $\mathcal{SPB}_2$ be the class of signed bipartite planar simple graphs in which one partite set has maximum degree of at most 2. It is clear that for each planar graph G, $S(G)$ is in $\mathcal{SPB}_2$ and that core of each signed bipartite graph in $\mathcal{SPB}_2$ is a subgraph of $S(G)$ for some planar graph G. We note, furthermore, that $\mathcal{SPB}_2$ is included in the class of signed bipartite planar 2-degenerate graphs.

Combining these observations with Proposition 1.8, we have the following reformulation of the 4-color theorem.

THEOREM 1.9 (4-color theorem restated). *Every signed graph in $\mathcal{SPB}_2$ admits a circular $\frac{16}{5}$-coloring.*

This then naturally leads to two questions, each based on dropping one of the conditions.

PROBLEM 1.1. *What is the best upper bound on the circular chromatic number of signed 2-degenerate simple graphs?*

PROBLEM 1.2. *What is the best upper bound on the circular chromatic number of signed bipartite planar simple graphs?*

In [9] it is shown that the answer for both questions is 4. Furthermore, a sequence of signed bipartite 2-degenerate graphs is built whose circular chromatic number tends to 4. It is then left as open problem whether one can build an example reaching the exact bound of 4.

Let $\mathcal{C}_{<4}$ be the class of signed graphs of circular chromatic number strictly smaller than 4. The questions then are equivalent to ask: 1. Does $\mathcal{C}_{<4}$ contain the class of all signed 2-degenerate simple graphs? 2. Does $\mathcal{C}_{<4}$ contain the class of all signed bipartite planar simple graphs?

In this work we answer these questions. In fact, using the number of vertices as a parameter, we provide an improved upper bound for each of the two problems and we show that our bounds are tight. More precisely, we prove the followings.

THEOREM 1.10. *If (G, σ) is a signed 2-degenerate simple graph on n vertices, then $\chi_c(G, \sigma) \leq 4 - \frac{2}{\lfloor \frac{n+1}{2} \rfloor}$. Moreover, this upper bound is tight for each value of $n \geq 2$.*

THEOREM 1.11. *If (G, σ) is a signed bipartite planar simple graph on n vertices, then $\chi_c(G, \sigma) \leq 4 - \frac{4}{\lfloor \frac{n+2}{2} \rfloor}$. Moreover, this upper bound is tight for each value of $n \geq 2$.*

The paper is organized as follows. In the next section we prove Theorem 1.10. In section 3 we present two graph operations each of which is $\mathcal{C}_{<4}$-closed. Using this, in section 4, we prove Theorem 1.11. Finally in the last section we mention some related problems.

2. Signed 2-degenerate simple graphs. In this section, we first prove the following theorem which, in particular, implies that circular chromatic number of any signed 2-degenerate simple graph is strictly smaller than 4. Then using the notion of tight cycle and Corollary 1.7, we will conclude Theorem 1.10.

THEOREM 2.1. *Let $\hat{G}$ be a signed simple graph with a vertex w of degree 2. If the signed graph $\hat{G} - w$ has circular chromatic number strictly less than 4, then $\hat{G}$ also has circular chromatic number strictly less than 4.*

Proof. Let $\hat{G}$ be a minimum counterexample to the theorem. Then it follows immediately that the underlying graph G is connected and has no vertex of degree 1. Let u and v be the two neighbors of w. Since circular chromatic number is invariant under switching, and without loss of generality, we may assume both uw and vw are positive edges in $\hat{G}$.

Let $\hat{G}' = \hat{G} - w$ and let ϵ be a positive real number smaller than 2, such that $\hat{G}'$ admits a circular $(4 - \epsilon)$-coloring. Let C be the circle of circumference $4 - \epsilon$.

By rotational symmetries of the circle we can assume that $\varphi(u) = 0$. Then considering symmetries along the diameters of the circle, in particular the one that contains 0, we may assume $\varphi(v) \geq 2 - \frac{\epsilon}{2}$. Furthermore, we may assume $\varphi(v) < 2$ as otherwise we can complete φ to a coloring of $\hat{G}$ simply by setting $\varphi(w) = 1$.

Our aim is to present a circular $(4 - \frac{\epsilon}{4})$-coloring ψ of $\hat{G}$. To this end, first we do a uniform scaling of the circle C to a circle C' with the factor of $\gamma = \frac{4-\frac{\epsilon}{2}}{4-\epsilon}$. This leads to a circular $(4 - \frac{\epsilon}{2})$-coloring $\varphi' : V(\hat{G}') \to [0, 4 - \frac{\epsilon}{2})$ where $\varphi'(x) = \gamma \cdot \varphi(x)$.

The mapping φ' has the property that for a positive edge xy the points $\varphi'(x)$ and $\varphi'(y)$ are at distance (on C') at least γ (which is $1 + \frac{\epsilon}{8-2\epsilon}$) and that the same holds for the distance between $\varphi'(x)$ and the antipode of $\varphi'(y)$ whenever xy is a negative edge. Observe that $\varphi'(u) = 0$ and $\varphi'(v) \geq 2 - \frac{\epsilon}{4}$.

Next we introduce a circular $(4 - \frac{\epsilon}{4})$-coloring of $\hat{G}'$ by inserting an interval of length $\frac{\epsilon}{4}$ inside C' to obtain a circle C'' of circumference $4 - \frac{\epsilon}{4}$. Assuming this interval is inserted at point $1 - \frac{\epsilon}{8}$ of C', the new coloring ψ of $\hat{G}'$ is defined as follows.

$$\psi(x) = \begin{cases} \varphi'(x), & \text{if } \varphi(x) < 1 - \dfrac{\epsilon}{8}, \\ \varphi'(x) + \dfrac{\epsilon}{4}, & \text{if } \varphi(x) \geq 1 - \dfrac{\epsilon}{8}. \end{cases}$$

We need to verify that ψ is a circular coloring of $\hat{G}'$. For a positive edge xy, it's immediate to see that the distance of $\psi(x)$ and $\psi(y)$ is at least 1, because in changing C' to C'' the distance between two points does not decrease. For a negative edge xy, we note that since the diameter of the circle is changed, the antipode of each point is shifted by $\frac{\epsilon}{8}$. To be more precise, if a is a point of circle C' with a_1 as its antipode, and a' and a'_1 are the images of these points at C'' after inserting an interval of length $\frac{\epsilon}{4}$, the antipode of a' on C'' is at distance $\frac{\epsilon}{8}$ from a'_1 (see Figure 1a and Figure 1b). Since in C' the distance between $\varphi'(x)$ and the antipode of $\varphi'(y)$ is at least $1 + \frac{\epsilon}{8-2\epsilon}$, even after this shift of $\frac{\epsilon}{8}$ the distance between $\psi(x)$ and the antipode of $\psi(y)$ is at least 1 and, therefore, ψ is a circular $(4 - \frac{\epsilon}{4})$-coloring.

(a) Circle C' with $r' = 4 - \dfrac{\epsilon}{2}$

(b) Circle C'' with $r'' = 4 - \dfrac{\epsilon}{4}$

Fig. 1: Circles with given circumferences

Finally, as $\psi(u) = 0$ and $\psi(v) \geq 2$, we may complete the circular $(4 - \frac{\epsilon}{4})$-coloring ψ of $\hat{G}'$ to $\hat{G}$ simply by setting $\psi(w) = 1$. ∎

We observe that in this proof for two vertices x and y of $\hat{G} - w$ if we have $\varphi(x) = \varphi(y)$, then we have $\psi(x) = \psi(y)$.

From the statement of this theorem, it follows immediately that every signed 2-degenerate simple graph admits a $(4-\epsilon)$-coloring for some positive real number ϵ. Next we use the notion of tight cycle to give a precise upper bound in terms of the number of vertices.

THEOREM 2.2. *For any signed* 2*-degenerate simple graph* (G, σ) *on* n *vertices, we have:*

- *For each odd value of* n, $\chi_c(G, \sigma) \leq 4 - \dfrac{4}{n+1}$,
- *For each even value of* n, $\chi_c(G, \sigma) \leq 4 - \dfrac{4}{n}$.

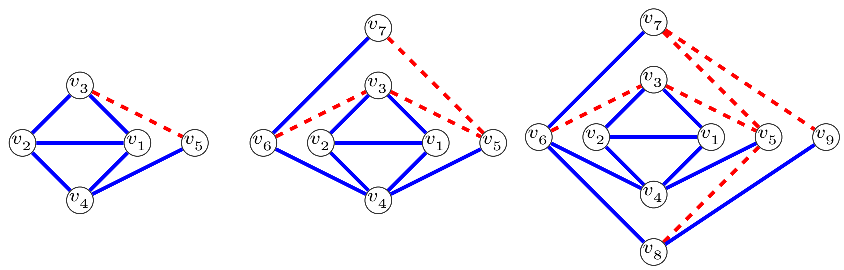
Proof. As stated in Corollary 1.7, we know that $\chi_c(G, \sigma) = \frac{p}{q}$ where p is twice the length of a cycle in G. Thus p is an even integer satisfying $p \leq 2n$. Since $\chi_c(G, \sigma) < 4$ we have $\frac{p}{q} < 4$, in other words, $p < 4q$. As p and q are integers, and moreover p is an even integer, we have $p \leq 4q-2$. Therefore, $\chi_c(G,\sigma) \leq \frac{4q-2}{q} = 4 - \frac{2}{q}$. On the other hand $\chi_c(G,\sigma) \leq \frac{2n}{q}$.

For a fixed n, the sequence $(\frac{2n}{q})_{q\in\mathbb{N}}$ is decreasing, whereas the sequence $(4-\frac{2}{q})_{q\in\mathbb{N}}$ is increasing. It is easy to check that

$$\max_{q\in\mathbb{N}} \min\left\{\frac{2n}{q}, 4-\frac{2}{q}\right\} = \begin{cases} 4-\dfrac{4}{n+1} & \text{for } q=\frac{n+1}{2} \text{ if } n \text{ is odd,} \\ 4-\dfrac{4}{n} & \text{for } q=\frac{n}{2} \text{ if } n \text{ is even.} \end{cases} \qquad \square$$

Next we show that the bound in Theorem 2.2 is tight for each value of $n \geq 2$. We construct a sequence of signed 2-degenerate simple graphs Ω_i reaching the bound for $n = 2i+1$. For even values of n, it is enough to add an isolated vertex to Ω_i.

Let $\Omega_1 = (K_3, +)$, that is the complete graph on three vertices with all edges being positive. Let v_1, v_2, v_3 be its vertices. Starting with Ω_1, we define the sequence Ω_i of signed graphs as follows. Given Ω_i on vertices $v_1, v_2, \ldots, v_{2i+1}$, we first add a vertex v_{2i+2} which a copy of v_{2i+1}, i.e., it sees each of the two neighbors of v_{2i+1} with edges of the same sign. Then we add a new vertex v_{2i+3} which is joined to v_{2i+1} and v_{2i+2}, to one with a negative edge and to the other with a positive edge. Observe that Ω_i has $2i+1$ vertices and is 2-degenerate. The elements Ω_2, Ω_3 and Ω_4 of the sequence are illustrated in Figure 2a, Figure 2b, Figure 2c respectively.



(a) Ω_2 (b) Ω_3 (c) Ω_4

Fig. 2: Signed graphs Ω_i

Proposition 2.3. *Given a signed graph Ω_i as defined above, we have*

$$\chi_c(\Omega_i) = 4 - \frac{4}{|V(\Omega_i)|+1}.$$

Proof. We prove by induction a slightly stronger claim. Let $r_i = 4 - \frac{2}{i+1}$. We claim that $\chi_c(\Omega_i) = r_i$ and, moreover, in any circular r_i-coloring of Ω_i the tight cycle is a Hamiltonian cycle.

The case $i = 1$ of this claim is immediate. That $\chi_c(\Omega_i) \leq r_i$ follows from Theorem 2.2. To show that $\chi_c(\Omega_i) \geq r_i$, it is enough to show that Ω_i is not r_{i-1}-colorable, because there are no rational numbers between r_{i-1} and r_i with a numerator at most $2(2i+1)$. To this end, and toward a contradiction, assume ψ is a circular r_{i-1}-coloring of Ω_i. We claim that $\psi(v_{2i-1}) = \psi(v_{2i})$. That is because ψ is also a circular r_{i-1}-coloring of Ω_{i-1}, and in any such a coloring the tight cycle (of Ω_{i-1}) is a Hamilton cycle. As v_{2i-1} is of degree 2 in Ω_{i-1} and v_{2i} is a copy of v_{2i-1}, we must have $\psi(v_{2i-1}) = \psi(v_{2i})$. But then to complete the circular r_{i-1}-coloring to v_{2i+1} we must have a point on the circle which is at distance at least 1 from both $\psi(v_{2i-1})$ and its antipode. But that is only possible if the circumference of circle used for coloring is at least 4. Thus $\chi_c(\Omega_i) = \frac{4i+2}{i+1}$. We then observe that $gcd(4i+2, i+1) = 1$ when i is even and $gcd(4i+2, i+1) = 2$ when i is odd. Hence any tight cycle of Ω_i in a circular $\frac{4i+2}{i+1}$-coloring is a Hamilton cycle, completing the proof. ∎

3. $\mathcal{C}_{<4}$-closed operations.

Recall that $\mathcal{C}_{<4}$ is the class of signed graphs of circular chromatic number strictly smaller than 4. In this section, we present two graph operations that preserve membership in this class.

We first observe that Theorem 2.1 could also be viewed as an operation that preserves membership in this class: For each $(G, \sigma) \in \mathcal{C}_{<4}$ and any pair of distinct vertices x and y of (G, σ), if we add a vertex u and join it to x and y with edges of arbitrary signs, then the resulting signed graph is also in $\mathcal{C}_{<4}$.

A slight modification and generalization of this one is based on the following notation. Let (G, σ) be a signed graph and let u be a vertex of (G, σ). We define $F_u(G, \sigma)$ to be the signed graph obtained from (G, σ) by contracting all the edges incident with u and keeping signs of all other edges as it is. One could easily observe that for (switching) equivalent signatures σ and σ', the signed graphs $F_u(G, \sigma)$ and $F_u(G, \sigma')$ might not be switching equivalent. The claim of next theorem is that the inverse operation of F_u is a $\mathcal{C}_{<4}$-closed operation.

THEOREM 3.1. *Given a signed graph (G, σ) and a vertex u of (G, σ), if $\chi_c(F_u(G, \sigma)) < 4$, then $\chi_c(G, \sigma) < 4$.*

As $F_u(G, \sigma)$ and $F_u(G, \sigma')$ might not be switching equivalent even if σ and σ' are switching equivalent, in applying this theorem it is important to choose a suitable signature (switching equivalent to σ). In particular, if two neighbors of u, say x and y, are adjacent with a positive edge, then $F_u(G, \sigma)$ will have a positive loop and so its circular chromatic number is ∞. Similarly, if two neighbors of u have another common neighbor v which sees one with a positive edge and the other with a negative edge, then $F_u(G, \sigma)$ has a digon (that is the signed graph on two vertices adjacent with two edges, one positive and the other negative) and hence does not belong to $\mathcal{C}_{<4}$.

The proof of this theorem is quite similar to the proof of Theorem 2.1. We consider a circular $(4-\epsilon)$-coloring of $F_u(G, \sigma)$. Then we consider a corresponding coloring on $(G-u, \sigma)$ noting that all neighbors of u are colored with a same color. We then modify the coloring as in the proof of Theorem 2.1 to find a color for u. We leave the details to the reader.

Next we define an edge-operation which also preserves membership in $\mathcal{C}_{<4}$. Let $\hat{G}$ be a signed graph with a positive edge uv. We define $F_{uv}(\hat{G})$ to be the signed graph obtained from $\hat{G}$ as follows. First we add a copy u' of u, that is to say for every neighbor w of u we join u' to w with an edge which is of the same sign as uw. Similarly, we add a copy v' of v. Then we add two more vertices x and y with the following connections: xu, yv as positive edges and xu', yv', and xy as negative edges. See Figure 3.

With similar techniques to the proof of Theorem 2.1 (and Theorem 3.1), we prove that $\mathcal{C}_{<4}$ is closed under the operation F_{uv}.

THEOREM 3.2. *Given a signed graph $\hat{G}$ and a positive edge uv of $\hat{G}$, if $\chi_c(\hat{G}) < 4$, then $\chi_c(F_{uv}(\hat{G})) < 4$.*

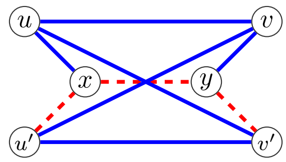


Fig. 3: The operation F_{uv}

Proof. Let φ be a circular $(4-\epsilon)$-coloring of $\hat{G}$ for a positive real number ϵ. We assume, without loss of generality, that $\varphi(u) = 0$ and $1 \leq \varphi(v) \leq 2 - \frac{\epsilon}{2}$. To prove the theorem, we modify φ to a circular $(4-\frac{\epsilon}{4})$-coloring ψ of $\hat{G}$ in such a way that we can extend it to a circular $(4-\frac{\epsilon}{4})$-coloring of $F_{uv}(\hat{G})$. On $\hat{G}$, the coloring ψ is defined as in the proof of Theorem 2.1. With $\gamma = \frac{4-\frac{\epsilon}{2}}{4-\epsilon}$, recall that ψ is defined on $V(G)$ as follows:

$$\psi(z) = \begin{cases} \gamma \cdot \varphi(z), & \text{if } \gamma \cdot \varphi(z) < 1 - \dfrac{\epsilon}{8}, \\ \gamma \cdot \varphi(z) + \dfrac{\epsilon}{4}, & \text{otherwise.} \end{cases}$$

The coloring ψ is extended to a coloring of $F_{uv}(\hat{G})$ as follows:

$$\psi(u') = \frac{\epsilon}{8},\ \psi(v') = \gamma \cdot \varphi(v) + \frac{\epsilon}{8},\ \psi(x) = 1, \text{ and } \psi(y) = \gamma \cdot \varphi(v) + \frac{\epsilon}{4} - 1.$$

What remains to prove is that the conditions of the circular coloring are verified for edges incident with the new vertices: u', v', x, and y. By the definition of ψ, the five edges incident with x or y satisfy the conditions of circular coloring. It remains to verify the condition for edges incident with u' and v' but not incident with x or y.

We first consider the edges incident with u'. Recall that u' is a copy of u. Let w be a neighbor u in $\hat{G}$. Based on the sign of wu' we consider two cases.

- wu' is a positive edge.
  We need to show that the distance between $\psi(w)$ and $\psi(u')$ is at least 1. Using the definition of circular coloring based on the circle, we consider both clockwise and anticlockwise distances on the circle. The anticlockwise path of the circle from u' to w passes through u and since u and w are already proved to be at distance at least 1, the anticlockwise distance from $\psi(u')$ to $\psi(w)$ is larger than 1. For the clockwise direction, since uw is a positive edge we have $\varphi(w) \geq 1$. Thus, by the definition of ψ, we have $\psi(w) = \gamma \cdot \varphi(w) + \frac{\epsilon}{4}$ whereas $\psi(u') = \frac{\epsilon}{8}$. Therefore, the clockwise distance of $\psi(u')$ and $\psi(w)$ is larger than the clockwise distance of $\varphi(u) = 0$ and $\gamma \cdot \varphi(w)$ which is at least 1.
- wu' is a negative edge.
  In circular $(4-\epsilon)$-coloring φ the distance of $\varphi(u)$ and $\varphi(w)$ is at most $1 - \frac{\epsilon}{2}$. Again we consider two possibilities depending on if the distance is obtained in clockwise direction starting from 0 or anticlockwise. For clockwise direction, we observe that $\gamma \cdot \varphi(w) < 1 - \frac{\epsilon}{8}$. Thus in defining ψ the distance of $\psi(u)$ and $\psi(w)$ remains the same as the distance of $\gamma \cdot \varphi(u)$ and $\gamma \cdot \varphi(w)$, and the distance of $\psi(u')$ to $\psi(w)$ is actually shorter. If the distance of $\varphi(w)$ and $\varphi(u)$ is obtained on anticlockwise direction starting from 0, then this distance is at

most $1-\frac{\epsilon}{2}$. Therefore, the distance of $\gamma\cdot\varphi(w)$ and $\gamma\cdot\varphi(u)$ is at most $(1-\frac{\epsilon}{2})\gamma$ which is strictly smaller than $1-\frac{\epsilon}{4}$. As $\psi(u')=\frac{\epsilon}{8}$, the distance between $\psi(w)$ and $\psi(u')$ remains strictly smaller than $1-\frac{\epsilon}{8}$, thus the negative edge wu' satisfies the condition.

We now consider the edges incident with v'. Observe that since $1\le\varphi(v)\le 2-\frac{\epsilon}{2}$, we have $\gamma\le\gamma\cdot\varphi(v)\le\gamma\cdot(2-\frac{\epsilon}{2})$. By the definition of ψ, and because $\gamma\cdot\varphi(v)\ge 1-\frac{\epsilon}{8}$, we have that

$$\gamma+\frac{\epsilon}{8}\le\psi(v')=\gamma\cdot\varphi(v)+\frac{\epsilon}{8}\le\gamma(2-\frac{\epsilon}{2})+\frac{\epsilon}{8}=2-\frac{\epsilon}{8}.$$

As v' is a copy of v, based on the sign of wv we consider two cases.

- wv' is a positive edge.
  We need to show that the distance between $\psi(w)$ and $\psi(v')$ is at least 1. Since wv is a positive edge, it implies two possibilities: (1) $\varphi(w)\in[0,1-\frac{\epsilon}{2}]$, (2) $\varphi(w)\in[2,4-\epsilon]$. For case (1), as $\gamma\cdot\varphi(w)<1-\frac{\epsilon}{8}$, then $\psi(w)=\gamma\cdot\varphi(w)$ and thus the distance between $\psi(w)$ and $\psi(v')$ is larger than $\gamma+\frac{\epsilon}{8}$. For case (2), $\psi(w)=\gamma\cdot\varphi(w)+\frac{\epsilon}{4}$. Thus the distance between $\psi(w)$ and $\psi(v')$ is at least $1+\frac{\epsilon}{8-2\epsilon}+\frac{\epsilon}{8}$, hence strictly larger than 1.
- wv' is a negative edge.
  As wv is a negative edge in $\hat{G}$, in any circular $(4-\epsilon)$-coloring φ, the distance of $\varphi(v)$ and $\varphi(w)$ is at most $1-\frac{\epsilon}{2}$ and then the distance of $\gamma\cdot\varphi(v)$ and $\gamma\cdot\varphi(w)$ is at most $\gamma(1-\frac{\epsilon}{2})<1-\frac{\epsilon}{4}$. Also, we have that $\frac{\epsilon}{2}\le\varphi(w)\le 3-\epsilon$. By the definition of ψ, if $\gamma\cdot\varphi(w)\ge 1-\frac{\epsilon}{8}$, then $\psi(w)=\gamma\cdot\varphi(w)+\frac{\epsilon}{4}$ and thus the distance between $\psi(w)$ and $\psi(v')$ is at most $\gamma(1-\frac{\epsilon}{2})+\frac{\epsilon}{8}<1-\frac{\epsilon}{8}$. It remains to show that if $\gamma\cdot\varphi(w)<1-\frac{\epsilon}{8}$, then the distance between $\psi(w)$ and $\psi(v')$ is smaller than $1-\frac{\epsilon}{8}$. In this case, $\psi(w)=\gamma\cdot\varphi(w)$. Therefore, compared with the distance between $\gamma\cdot\varphi(w)$ and $\gamma\cdot\varphi(v)$, the distance between $\psi(w)$ and $\psi(v')$ is increased by $\frac{\epsilon}{8}$, therefore, it is at most $\gamma(1-\frac{\epsilon}{2})+\frac{\epsilon}{8}<1-\frac{\epsilon}{8}$. □

4. Signed bipartite planar simple graphs. In this section, we would like to prove Theorem 1.11. As in section 2, we will first show that the circular chromatic number of any signed bipartite planar simple graph is strictly smaller than 4. Then we use the notion of tight cycle to get an improved upper bound. Finally, we show that this upper bound is tight.

To this end, we will work with a minimum counterexample. One of properties of a minimum counterexample follows from the following folding lemma of [5]. We recall that a plane graph or a signed plane graph is a (signed) planar graph together an embedding on the plane. For a plane graph, a *separating* l-cycle is an l-cycle which is not a face.

LEMMA 4.1 (Bipartite folding lemma). *Let $\hat{G}$ be a signed bipartite plane graph and let $2k$ be the length of its shortest negative cycle. Let F be a face whose boundary is not a negative cycle of length $2k$. Then there are vertices v_{i-1}, v_i, v_{i+1}, consecutive in the cyclic order of the boundary of F, such that identifying v_{i-1} and v_{i+1}, after a possible switching at one of the two vertices, yields a signed bipartite plane graph whose shortest negative cycle is still of length $2k$.*

We observe that by applying this lemma repeatedly we get a homomorphic image of $\hat{G}$ which is also a signed bipartite plane graph in which every facial cycle is a negative cycle of length exactly $2k$. Furthermore, as an extension of the handshake lemma, one observes in a 2-connected signed plane graph, the sign of a cycle is the product of the signs of all the faces that it bounds. In particular when all faces are negative, the sign of a cycle is determined by the parity of the number of faces it bounds.

THEOREM 4.2. *For any signed bipartite planar simple graph* (G, σ)*, we have* $\chi_c(G, \sigma) < 4$.

Proof. Assume that (G, σ) is a minimum counterexample, i.e., for no $\epsilon > 0$, (G, σ) admits a circular $(4 - \epsilon)$-coloring, and $|V(G)|$ is minimized.

The minimality of (G, σ), together with the bipartite folding lemma, implies that every facial cycle of (G, σ), in any planar embedding of G, is a negative 4-cycle. From here on, we will consider (G, σ) together with a planar embedding. Moreover, since any subgraph of (G, σ) is also a signed bipartite planar simple graph and in $\mathcal{C}_{<4}$ (by the minimality of (G, σ)), it follows from Theorem 2.1 that $\delta(G) \geq 3$.

We proceed by proving some structural properties of (G, σ) in the form of claims.

Claim 1. *Every vertex of even degree in* (G, σ) *must be in a separating* 4*-cycle.*

Assume to the contrary that a vertex u is of even degree and it is in no separating 4-cycle. Let C be the boundary of the face in $(G - u, \sigma)$ which contains u. This cycle C in the embedding of (G, σ) bounds $d(u)$ faces, each of which is a negative 4-cycle. As $d(u)$ is even, C is a positive cycle. Since switching does not affect the circular chromatic number, we may assume σ is a signature in which all the edges of C are positive.

Let (G', σ') be the signed graph obtained from (G, σ) by the following operations. First we contract all the edges incident with u. Then for each set of parallel edges of the same sign resulted from the contraction, we delete all but one. We observe that, as u is in no separating 4-cycle, (G', σ') has no digon. Thus (G', σ') is a signed simple graph. Furthermore, it is a signed bipartite planar simple graph which has less vertices than (G, σ). Thus it admits a circular $(4 - \epsilon)$-coloring for some positive ϵ. But then Theorem 3.1 implies that $\chi_c(G, \sigma) < 4$.

Claim 2. *For every pair of adjacent vertices each of an odd degree in* (G, σ)*, at least one is in a separating* 4*-cycle.*

The proof of this claim is similar to the previous one. Towards the contradiction, let x and y be two adjacent vertices of odd degrees, neither of which is in a separating 4-cycle. We consider the facial cycle C which is obtained after deleting x and y, and once again conclude that C must be a positive cycle as it must bound an even number of (negative) faces in (G, σ). Without loss of generality, we assume that σ assigns positive signs to all the edges of C and that $\sigma(xy) = -$. Let $x_1, x_2, \ldots, x_\ell$ be the neighbors of x distinct from y in the cyclic order of the embedding and, similarly, let $y_k, y_{k-1}, \ldots, y_1$ be the neighbors of y, distinct from x, in the cyclic order (see Figure 4). Thus x_1y_1 and $x_\ell y_k$ are both edges of C and hence both are positive. We have two assertions on the neighborhood of x and y.

The first is that x_1y_1 and $x_\ell y_k$ are the only edges connecting some x_i to some y_j. That is because any other connection x_iy_j would create a separating 4-cycle xx_iy_jy but we have assumed (towards a contradiction) that x and y are in no such a 4-cycle. The second is that $\sigma(xx_1) = \sigma(yy_1)$ and that $\sigma(xx_\ell) = \sigma(yy_k)$. To see that $\sigma(xx_1) = \sigma(yy_1)$, we consider the face xx_1y_1y. We already know that xy is a negative edge and that x_1y_1 is a positive edge. For this face to be a negative 4-cycle then we must have $\sigma(xx_1) = \sigma(yy_1)$. That $\sigma(xx_\ell) = \sigma(yy_k)$ follows from the same argument by considering the face $xx_\ell y_k y$.

To complete the proof of the claim, we consider two signed graphs. One is the signed graph $\hat{G}'$ built from (G, σ) as follows. First we delete the edge xy. Next we

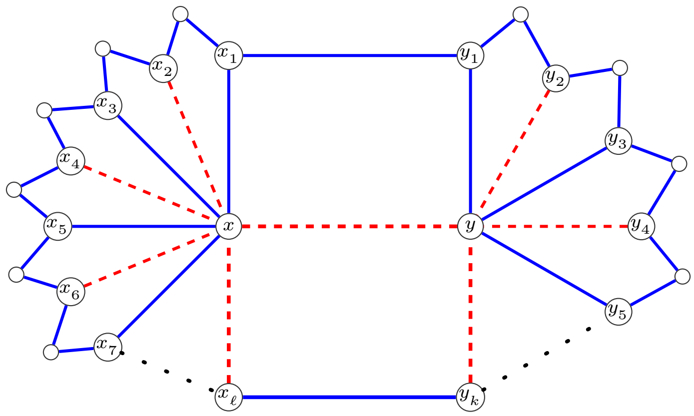


Fig. 4: $\{x, y\}$-neighborhood

contract all the remaining edges incident with x (respectively, y) and denote the new vertex by u (resp. v). Finally, for each set of parallel edges of the same sign, we delete all but one. We observe that $\hat{G}'$ is a signed bipartite planar simple graph with no digon because x and y are in no separating 4-cycle. We note furthermore that in $\hat{G}'$ the vertex u is connected to the vertex v with a positive edge (resulted from x_1y_1 and $x_\ell y_k$). By the minimality of (G, σ), we conclude that $\chi_c(\hat{G}') < 4$.

The other signed graph, $\hat{G}''$, is obtained from (G, σ) as follows. The positive neighbors of x (respectively, y) are identified into a new vertex u (resp. v). The negative neighbors of x distinct from y (respectively, negative neighbors of y distinct from x) are identified into a new vertex u' (resp. v'). As before, among a set of parallel edges of the same sign we delete all but one. We note that $\hat{G}''$ is not necessarily planar anymore. It follows from the discussion on the neighborhood of x and y that in $\hat{G}''$ there is no edge connecting u' to v and, similarly, no edge connecting u to v'. Moreover, u is connected to v only with a positive edge and u' is connected to v' only with a positive edge.

Overall we observe that $\hat{G}''$ is a (proper) subgraph of $F_{uv}(\hat{G}')$. It follows from Theorem 3.2 that $F_{uv}(\hat{G}')$ and, therefore, $\hat{G}''$ is in $\mathcal{C}_{<4}$, but $\hat{G}''$ is a homomorphic image of (G, σ) which implies $\chi_c(G, \sigma) \leq \chi_c(\hat{G}'')$.

Claim 3. *The underlying graph G of $\hat{G}$ has no separating 4-cycle.*

Towards a contradiction, assume that there is a separating 4-cycle and let C be a separating 4-cycle with the minimum number of vertices inside. Let v_1, v_2, v_3, and v_4 be the four vertices of C in this cyclic order. Let u be a vertex inside C. As (G, σ) is bipartite, u can be adjacent to at most two vertices of C. Since (G, σ) has minimum degree at least 3, u must have a neighbor, say v, which is not on C and thus inside C. By Claim 1 and Claim 2, at least one of u or v, say u, is in a separating 4-cycle, denoted C_u. Since C contains the minimum number of vertices inside, C_u cannot be all inside C. Thus u is adjacent to two vertices of C. Noting that G is bipartite, and by symmetry, we may assume v_1 and v_3 are adjacent to u. Then of the two 4-cycles $uv_1v_2v_3$ and $uv_1v_4v_3$ one contains v and thus is a separating 4-cycle with less vertices inside than C. This contradicts the choice of C and, hence, proves the claim.

To complete the proof of the theorem, we observe that, by Claims 1 and 3, all vertices must be of odd degree, and, by Claim 2, no two of them can be adjacent, but then G has no edge and any mapping to the points of any circle is circular coloring, a contradiction with our choice of (G, σ). ∎

Next, using the notion of tight cycle, we improve the bound of Theorem 4.2. We provide a concrete bound in terms of the number of vertices and then show that this improved bound is tight.

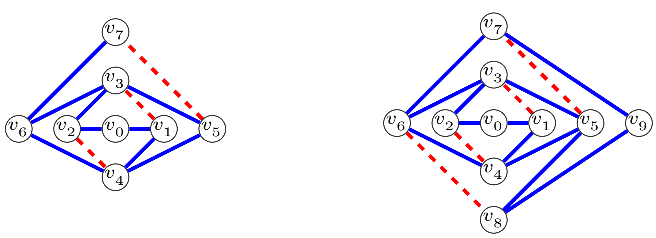
THEOREM 4.3. *For any signed bipartite planar simple graph (G, σ) on n vertices, we have:*

- *For each odd value of n, $\chi_c(G, \sigma) \leq 4 - \dfrac{8}{n+1}$.*
- *For each even value of n, $\chi_c(G, \sigma) \leq 4 - \dfrac{8}{n+2}$.*

Moreover, these bounds are tight for each value of $n \geq 2$.

Proof. As stated in Corollary 1.7, we know that $\chi_c(G, \sigma) = \frac{p}{q}$ where p is twice the length of a cycle in G. As G is a bipartite graph, the length of each cycle is even. Thus $p = 4k$ for some positive integer k such that $2k \leq n$. By Theorem 4.2, we have $\chi_c(G, \sigma) = \frac{p}{q} < 4$, in other words, $4k < 4q$. As k and q are integers, we have $k+1 \leq q$. Hence, $\chi_c(G, \sigma) \leq \frac{4k}{k+1} = 4 - \frac{4}{k+1}$. The upper bounds claimed in the theorem then follows by noting that $n \geq 2k$ and that $n \geq 2k+1$ when n is odd.

To prove that the bounds are tight, for $n = 2i$, we need to build an example Γ_i^*. Then by adding an isolated vertex to Γ_i^*, we get an example that works for $n = 2i+1$. For $i \geq 2$, the signed graph Γ_i^* is built from the signed graph Ω_{i-1} of Figure 2 by subdividing the edge v_1v_2 once, and assigning a positive sign to one of the resulting edges and a negative sign to the other. In Figure 5 switching equivalent versions of Γ_4^* and Γ_5^* are presented. The equivalence of the signatures follows from the fact that in both presentations all facial cycles are negative 4-cycles.



(a) A switching of Γ_4^* (b) A switching of Γ_5^*

Fig. 5: Signed graphs Γ_i^*

In the following we shall prove that $\chi_c(\Gamma_i^*) = 4 - \frac{4}{i+1}$. Since Γ_i^* is a signed bipartite planar simple graph on $2i$ vertices, by the first part of the proof, $\chi_c(\Gamma_i^*) \leq 4 - \frac{4}{i+1}$. It remains to show that $\chi_c(\Gamma_i^*) \geq 4 - \frac{4}{i+1}$. To prove this we use induction on i and a stronger fact that in every circular $\frac{4i}{i+1}$-coloring of Γ_i^* every tight cycle is a Hamilton cycle. The added claim follows from the fact that Γ_i^* has only $2i$ vertices: If i is even, then $2i$ and $i+1$ are relatively prime and thus, by Corollary 1.7, the only possibility for the length of a tight cycle is $2i$; If i is odd, say $i = 2j+1$, then $\chi_c(\Gamma_i^*) = \frac{2(2j+1)}{j+1}$. Noting that $2j+1$ and $j+1$ are relatively prime, it follows from Corollary 1.7 that the length of a tight cycle is a multiple of $2j+1$. However, as the underlying graph is bipartite, there is no odd cycle, and, hence, the length of a tight cycle is $4j+2 = 2i$.

We now apply the induction on i. That $\chi_c(\Gamma_1^*) \geq 4 - \frac{4}{1+1} = 2$ is immediate. Assume that $\chi_c(\Gamma_i^*) \geq 4 - \frac{4}{i+1}$ (thus $\chi_c(\Gamma_i^*) = 4 - \frac{4}{i+1}$). Considering Γ_{i+1}^*, as it contains Γ_i^* as a subgraph, we have $\chi_c(\Gamma_{i+1}^*) \geq 4 - \frac{4}{i+1}$. Applying the tight cycle argument then we know that $\chi_c(\Gamma_{i+1}^*) \in \{\frac{4i}{i+1}, \frac{4i+4}{i+2}\}$. However, $\chi_c(\Gamma_{i+1}^*) = \frac{4i}{i+1}$ is not possible, because, considering the tight Hamilton cycle of Γ_i^* (with respect to a circular $\frac{4i}{i+1}$-coloring), in any circular $\frac{4i}{i+1}$-coloring of Γ_{i+1}^*, the vertex v_{2i} must be mapped to the antipode of the image of v_{2i-1} (possibly after a switching), and then v_{2i+1} must be at distance at least 1 from both of these two images, which is not possible because the circumference of the circle is strictly less than 4. We are done.□

We note that Γ_i^* is the core of the signed graph Γ_i defined in [9]. Moreover, the formula for the circular chromatic number of Γ_{2k}^* is implicit in Corollary 46 of [9].

5. Discussion and Questions. In this work we have observed that bounding the circular chromatic number of a very restricted families of signed graphs can capture some of the most motivating problems in graph theory such as the 4-color theorem.

Then by strengthening some results from [9] we provided improved bounds for two families of signed graphs: signed 2-degenerate simple graphs and signed bipartite planar simple graphs.

We note that some of the well-known problems in circular coloring of graphs fit into this study by viewing a graph G as a signed graph $(G,+)$ where all edges are positive. In particular, providing the best possible bound for the circular chromatic number of planar graphs of a given odd girth is one of main questions in graph theory which captures the 4-color theorem, the Grötzsch theorem, and the Jaeger-Zhang conjecture.

Here we mention a few new questions that are based on the notion of the circular coloring of signed graphs.

QUESTION 1. *Given a signed planar simple graph $\hat{G}$, does there exist an $\epsilon = \epsilon(\hat{G})$ such that $\hat{G}$ admits a circular $(6-\epsilon)$-coloring?*

First example of a signed planar simple graph whose circular chromatic number is larger than 4 is given in [2]. An example of signed planar simple graph whose circular chromatic number is $\frac{14}{3}$ is given in [9]. The upper bound of 6 follows from the fact that planar simple graphs are 5-degenerate. The existence of any signed planar simple graph with circular chromatic number larger than $\frac{14}{3}$ is an open problem.

Restricted on the class of signed bipartite planar graphs and with an added negative girth condition (that is the length of a shortest negative cycle), we have the following question.

QUESTION 2. *Given a signed bipartite planar graph $\hat{G}$ of negative girth 6, does there exist an $\epsilon = \epsilon(\hat{G})$ such that $\hat{G}$ admits a circular $(3-\epsilon)$-coloring?*

That every signed bipartite planar graph of negative girth at least 6 admits a circular 3-coloring is recently proved in [8], noting that this proof uses the 4-color theorem and some extensions of it. On the other hand, the best example of signed bipartite planar graph of negative girth 6 we know has circular chromatic number $\frac{14}{5}$. It remains an open problem to build such signed graphs of circular chromatic number between $\frac{14}{5}$ and 3.

We should mention that a negative answer to Question 2 would imply a negative answer to Question 1. Let $T_2(G,\sigma)$ be a signed graph obtained from (G,σ) by subdividing each edge uv once and then assign a signature in such a way that the sign of the corresponding uv-path is $-\sigma(uv)$. Viewing positive and negative paths of length

2 as $\mathcal{I}_-$ and $\mathcal{I}_+$ (respectively), and applying Lemma 41 of [9] we have

$$\chi_c(T_2(G,\sigma)) = \frac{4\chi_c(G,\sigma)}{2+\chi_c(G,\sigma)}.$$

For signed bipartite planar graphs of negative girth at least 8, the upper bound of $\frac{8}{3}$ for their circular chromatic numbers is proved in [4]. For signed bipartite planar graphs of negative girth $2k$, $k \geq 5$, the best current bound follows from recent results of [3].

Acknowledgments. We would like to thank L. A. Pham for discussion on the sequence Γ_i^*. We would also like to thank the reviewers for careful reading and comments on the first version of the paper.

REFERENCES

[1] Y. Kang and E. Steffen, *Circular coloring of signed graphs*, J. Graph Theory, 87 (2018), pp. 135–148, https://doi.org/10.1002/jgt.22147.
[2] F. Kardoš and J. Narboni, *On the 4-color theorem for signed graphs*, European J. Combin., 91 (2021), p. 103215, https://doi.org/10.1016/j.ejc.2020.103215.
[3] J. Li, R. Naserasr, Z. Wang, and X. Zhu, *Circular flow in mono-directed signed graphs*, Dec. 2022, https://arxiv.org/abs/1403.2226.
[4] R. Naserasr, L. A. Pham, and Z. Wang, *Density of C_{-4}-critical signed graphs*, J. Combin. Theory Ser. B, 153 (2022), pp. 81–104, https://doi.org/10.1016/j.jctb.2021.11.002.
[5] R. Naserasr, E. Rollová, and E. Sopena, *Homomorphisms of planar signed graphs to signed projective cubes*, Discrete Math. Theor. Comput. Sci., 15 (2013), pp. 1–11.
[6] R. Naserasr, E. Rollová, and E. Sopena, *Homomorphisms of signed graphs*, J. Graph Theory, 79 (2015), pp. 178–212, https://doi.org/10.1002/jgt.21817.
[7] R. Naserasr, E. Sopena, and T. Zaslavsky, *Homomorphisms of signed graphs: An update*, European J. Combin., 91 (2021), p. 103222, https://doi.org/10.1016/j.ejc.2020.103222.
[8] R. Naserasr and Z. Wang, *Circular coloring of signed bipartite planar graphs*, in Extended Abstracts EuroComb 2021, Springer, 2021, pp. 208–213.
[9] R. Naserasr, Z. Wang, and X. Zhu, *Circular chromatic number of signed graphs*, Electron. J. Combin., 28 (2021), pp. Paper No. 2.44, 40, https://doi.org/10.37236/9938.
[10] T. Zaslavsky, *Signed graphs*, Discrete Appl. Math., 4 (1982), pp. 47–74, https://doi.org/10.1016/0166-218X(82)90033-6.